"ON THE OPTIMALITY OF CENTRAL PLACES"

by
Martine QUINZII*
Jacques-François THISSE**

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* Martine QUINZII, University of Southern California, Los Angeles, USA

** Jacques-François THISSE, CORE, Université Catholique de Louvain, Belgium
and INSEAD, Fontainebleau, France

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ON THE OPTIMALITY OF CENTRAL PLACES

MARTINE QUINZII
JACQUES-FRANCOIS THISSE

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ABSTRACT

Using the Eaton and Lipsey model, we show that a hierarchical system of central places is socially optimal: firms having less frequent purchases are clustered with firms having more frequent purchases in any configuration minimizing total transport costs.

1. INTRODUCTION

Central Place Theory is a major topic of research in location theory. First developed by Christaller in purely geographic terms and then formulated in economic terms by Lõsch, the object of the theory is to explain the observed spatial distribution of economic activities in a hierarchical system of urban centers (see Mulligan (1984) for a recent and detailed survey).

The standard theory begins with the analysis of the spatial organization of the producers of a given commodity. It is assumed that consumers are evenly distributed over the plane and have identical demand functions for the good which they buy from the firm with the lowest full price (*i.e.*, mill price plus transport cost). The theory asserts that both in a market equilibrium and at a social optimum, firms are located at the vertices of a regular lattice, serve identical market areas and charge the same mill price.

The passage to the multiproduct case rests on geometric considerations. Consider n commodities $i = 1, \ldots, n$ and let R_i denote the radius of the market area served by a producer of commodity i. The commodities can be indexed in such a way that $R_1 \leq R_2 \leq \ldots \leq R_n$. Christaller, Lõsch and their successors argued that the different systems of market areas can be arranged in such a way that a location where commodity i is provided also accommodates firms supplying all commodities of index inferior to i. A market place where the available commodities have index 1 to i is called a **central place of order i**. The central places of different orders form a regular lattice with a nested system of market areas. The **hierarchical principle** simply asserts that all the commodities available in a central place of order i are are also available in all central places of superior order.

The bulk of the research on Central Place Theory has been directed towards identifying geometric conditions under which a superposition of regular structures is possible (the more complete and elaborate study of this kind can be found in Alao *et al.*, (1977)). These considerations are only interesting if they are based on natural microeconomic foundations. There is an infinite number of ways of superposing regular networks of producers of different commodities with different market areas. If there are no economic forces which lead firms of different types to cluster, it is hard to see why the central place system would be more likely to emerge than any other configuration.

Eaton and Lipsey (1982) were the first to propose an economic foundation for the Central Place Theory. In their model, the force which drives the clustering of firms of different types is the occurrence of multipurpose shopping (see Thill and Thomas (1987) for a survey of this vivid topic of economic geography). They assume that consumers can group their purchases of different commodities in order to reduce transport costs. This opportunity creates demand externalities which firms exploit by clustering with firms of other types. In a model in which consumers are uniformly distributed on the line, Eaton and Lipsey identify a set of conditions under which the only equilibria that can emerge verify the hierarchical principle. (Notice that Stahl and Varaiya (1978) have proposed a similar approach in an unpublished paper; see also Stahl (1987)).

The analysis of Eaton and Lipsey can be viewed as a positive foundation for Central Place Theory. The object of this paper is to analyze the normative aspect, namely to prove the optimality of the central place configuration. The main assumptions of the model and our principal result can be summarized as follows. Consumers are identical and uniformly distributed on a circle. There are two commodities ($i = 1, 2$) and two types of firms, n of type 1 producing the first good, m of type 2 producing the second with $m \leq n$. The demand for the second good is less than the demand for the first in the sense that each consumer must visit a firm of type 1 α times ($\alpha > 1$) for each visit to a firm of type 2. Consumers can undertake multipurpose shopping by purchasing one unit of each good on the same trip. Under these assumptions, we show, for all α greater than 1 and for all $m \leq n$, that the configuration which minimizes the total transport cost has the property that firms of type 2 are clustered with firms of type 1. Such a configuration thus exhibits central places of order 1 where good 1 is available and central places of order 2 where goods 1 and 2 are available.

In analyzing the result, it is useful to distinguish two cases. The first is the case where n is an integer multiple of m; the second where n and m are relatively prime. The first case corresponds to that studied in the plane by economic geographers. To minimize total transport cost of consumers to firms of type 1 (type 2), firms of each type should be symmetrically located on the circle. Since n is a multiple of m the symmetric distributions for the two types of firms can be superposed so as to create locations at which there are central places of order 2. This solution to the location problem becomes optimal as soon as the assumption of multipurpose shopping is invoked.

The result is substantially more significant in the more difficult case where m and n are relatively prime. In this case, there are essentially two opposing forces at work: symmetry in location minimizes the total transport cost of consumers on single purpose trips while clustering of firms of type 2 with firms of type 1 maximizes the total savings from multipurpose shopping (but disrupts symmetry). Our result shows that the gains from clustering always outweigh the loss resulting from the lack of symmetry, thereby providing a strong normative justification to the basic tenet of Central Place Theory.

2. The Model

We develop our normative theory of central places within the model of Eaton and Lipsey (1982). The assumptions are as follows.

(H.1) There are two commodities 1 and 2. Commodity 1 is produced and sold at a constant marginal cost c_1 in n A-firms. Commodity 2 is produced and sold at a constant marginal cost c_2 in m B-firms.

(H.2) The number of A-firms is larger than or equal to the number of B-firms.

(H.3) Consumers are evenly distributed with a unit density over a circle C of given length.

(H.4) Consumers have identical consumptions. Each consumer buys one unit of commodity 2 (by normalization of the time period) and $\alpha > 1$ units of commodity 1.

(H.5) Consumers' possibilities of storage per time period are such that they cannot purchase more than one unit of commodity 1 per shopping trip.

(H.6) The cost of a shopping trip is a linear function of the distance covered (the transport rate is normalized to one by an adequate choice of the length unit) and is independent of the quantities (0 or 1) of commodities 1 and 2 purchased.

(H.7) Given the locations of A-firms and B-firms, consumers minimize their transport cost on each shopping trip. When a consumer purchases one unit of each commodity on a single trip, we say that he makes a **multipurpose trip**. Otherwise, the consumer is said to make a **single-purpose trip**.

(H.8) The planner locates the A-firms and B-firms along the circle C in order to minimize consumers' total transport costs.

Some comments are now in order. The assumption of a one-dimensional space is very common in location-theoretic models and greatly simplifies the analysis. A circular market, instead of a line segment, has been chosen in order to avoid boundary problems. Assumption (H.4) of fixed consumption proportion is also made for mathematical convenience. Allowing for substitution between the two commodities would dramatically complicate the model by making the consumption proportion dependent on the A-firm and B-firm locations and, therefore, consumer-specific. There are more A-firms than B-firms because, by (H.4) and (H.5), there are more trips to A-firms. Finally, assumptions (H.5) and (H.7) imply that each consumer buys $(\alpha - 1)$ units of commodity 1 from the nearest firm, thus making each time a single-purpose trip.

The key-assumption for proving the optimality of central places is given by (H.6). Via the indivisibility in transport conveyed by (H.6), the essence of multipurpose shopping is caught, while keeping the model analytically tractable. The possibility of grouping the purchases of one unit of each commodity opens the door to various shopping options to the consumers. More specifically, in the case depicted in Figure 1 where $\overline{A_1B_1} > \overline{xB_2}$ and $\overline{A_2B_2} > \overline{xA_1}$ ($\overline{AB}$ denotes the distance between A and B), a consumer at x chooses

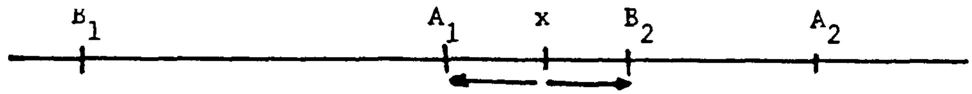

Figure 1

Figure 2

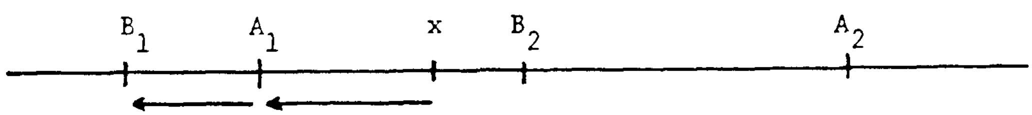

Figure 3

to patronize the nearest B firm, *i.e.* B_2, to purchase good 2 and to buy the last unit of commodity 1 from A_1, the nearest A firm. The consumer thus makes two single-purpose trips. However, if $\overline{A_1B_2} > \overline{xA_2}$, as in Figure 2, the consumer prefers to make a multipurpose trip in which he buys from the nearest B-firm (B_2) and the second-nearest A-firm (A_2). Looking now at Figure 3, we see that another case may still arise in which the consumer at x chooses to purchase from the second-nearest B-firm because this allows him to save on the transport cost of obtaining the last unit of commodity 1. As $\overline{xB_1} < \overline{xA_2}$, the consumer makes a multipurpose trip, visiting A_1 and B_1. (Notice that the same occurs if A_1 and B_1 are permuted.) Ultimately, the option he chooses depends on the consumer's relative position with respect to the nearest and second-nearest A-firms and B-firms. In consequence, the locations of the A-firms and B-firms are interdependent at the optimum.

If there were only single-purpose trips, each consumer would visit α times the nearest A-firm and once the nearest B-firm. The optimal solution would then consist in any superposition of two equidistant patterns with m and n firms respectively. Therefore, the two systems of firms being independent, agglomeration of A-firms and B-firms would not generally occur at the optimum.

On the contrary, when multipurpose trips are allowed, we prove in the next section that B-firms are always clustered with A-firms at the optimum. This is not obvious from the intuitive point of view. For example, when $m = 3$ and $n = 5$, the configuration represented in Figure 4 looks *a priori* as good as that depicted in Figure 5 which is the optimal one.

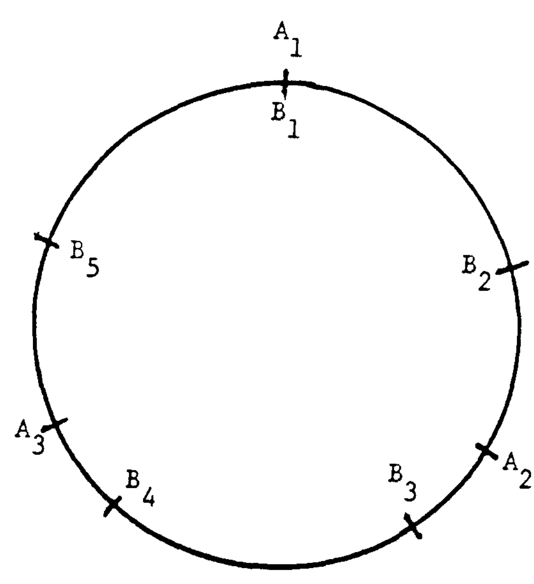

Figure 4

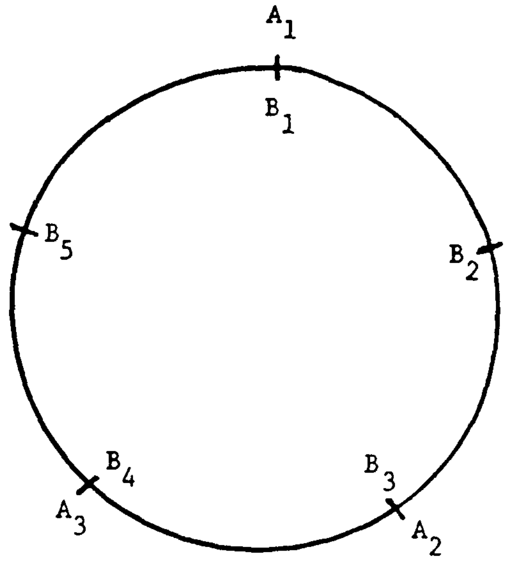

Figure 5

3. The Central Place Theorem

In this section, we prove one of the main conjectures of central place theory, *i.e.*, whatever $m \leq n$ and $\alpha > 1$, B-firms are clustered with A-firms at the optimum. To this end, we establish some intermediate results. Lemma 1 is rather straightforward and shows that there must be at least one A-firm between two consecutive B-firms in an optimal pattern. Lemmas 2 and 3 are more complicated. For the two main types of configuration arising when $m \geq 3$, they show that the second B-firm cannot be isolated at the optimum. Two kinds of arguments are used: (i) the minimization of the cost of $(\alpha - 1)$ single-purpose trips to the nearest A-firm; (ii) the minimization of the transport cost of one unit of commodity 2 and of the last unit of commodity 1 on two single-purpose trips or one multipurpose trip. Our main result, the central place theorem, can then easily be proved by applying these lemmas.

LEMMA 1: *Let B_1 and B_2 be any two consecutive B-firms. Then, at the optimum, the segment $[B_1, B_2]$ of C cannot be a proper subset of the segment $[A_1, A_2]$ where $A_1(A_2)$ is the nearest A-firm on the left (right) of B_1.*

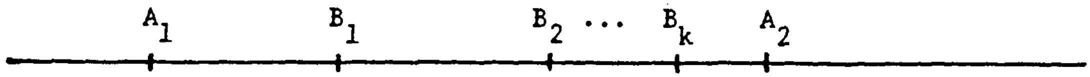

Figure 6a

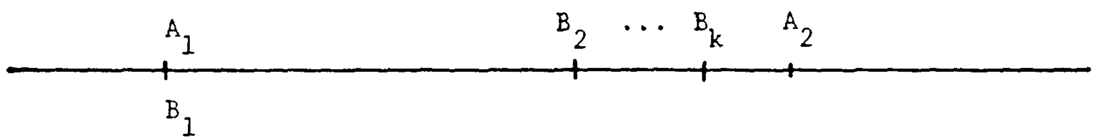

Figure 6b

Figure 6c

PROOF: Assume, on the contrary, that we have one of the configurations depicted in Figure 6a, b and c. Consider the cases of Figures 6a and 6b. Then, locating B_2 with A_2 and moving $B_3 \ldots B_k$ into some segments between A-firms initially without B firms (remember that $m \leq n$)

- does not affect the transport cost incurred in the single-purpose trips to the A-

firms,

- leaves the consumers between A_1 and A_2 indifferent or strictly better-off since they must go to A_1 or A_2 to buy the last unit of commodity 2,
- leaves the consumers situated on the left of A_1 indifferent since they do not purchase from B_2,
- makes some consumers established on the right of A_2 and purchasing from A_2 strictly better-off since they now save their transport cost to a B-firm, and leaves the others indifferent.

Hence, total transport costs can be reduced, a contradiction. Similarly, in the case of Figure 6c, locating B_1 with A_1 reduces total transport cost. Q.E.D.

LEMMA 2: *Let B_1, B_2 and B_3 be three consecutive B-firms. If there are at least two A-firms in the segments $[B_1, B_2[$ and $]B_2, B_3]$, and if B_2 is isolated, then the corresponding configuration is not optimal.*

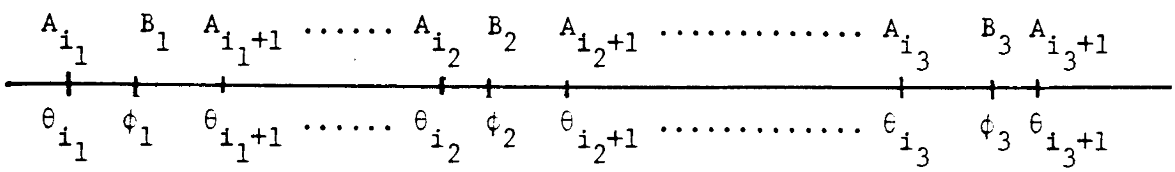

Figure 7

PROOF: Consider the configuration represented in Figure 7 and show that total transport costs can be lowered by locating B_2 with A_{i_2} or A_{i_2+1}. There are two steps in the proof.

Step 1. Let us show that the following four conditions are necessary for the optimum:

$$\frac{\theta_{i_1} + \theta_{i_2}}{2} \geq \theta_{i_1+1} \quad \text{or} \quad \theta_{i_1+1} = \phi_1 \tag{1}$$

$$\frac{\theta_{i_1+1} + \theta_{i_2+1}}{2} \leq \theta_{i_2} \quad \text{or} \quad \theta_{i_2} = \phi_2 \tag{2}$$

$$\frac{\theta_{i_2} + \theta_{i_3}}{2} \geq \theta_{i_2+1} \quad \text{or} \quad \theta_{i_2+1} = \phi_2 \tag{3}$$

$$\frac{\theta_{i_2+1} + \theta_{i_3+1}}{2} \leq \theta_{i_3} \quad \text{or} \quad \theta_{i_3} = \phi_3 \tag{4}$$

Suppose that (1) does not hold and let us show that total transport cost can be reduced by decreasing θ_{i_1+1}. Consider, first, the trips associated with the purchase of good 2 and of the last unit of good 1. The consumers who buy from A_{i_1+1} and B_2 are not made worse off, for they can buy 1 from A_{i_2} on their way to B_2. Furthermore, those who purchase from B_1 and A_{i_1+1} are indifferent or strictly better off when A_{i_1+1} comes closer to B_1. Consequently, the transport cost of these trips decreases with θ_{i_1+1}. Second, the total transport cost of the single purpose trips to A firms is proportional to $[(\theta_{i_1} - \theta_{i_1+1})^2 + (\theta_{i_1+1} - \theta_{i_1+2})^2 + \ldots]$ where the missing terms are independent of θ_{i_1+1}. Taking the first-order derivative of this expresion w.r.t θ_{i_1+1} yields

$$2\theta_{i_1+1} - \theta_{i_1} - \theta_{i_1+2}.$$

If $\theta_{i_1+1} > \frac{\theta_{i_1}+\theta_{i_2}}{2}$, then $\theta_{i_1+1} > \frac{\theta_{i_1}+\theta_{i_1+2}}{2}$ so that the above derivative is strictly positive. This implies that a decrease in θ_{i_1+1} leads to a decrease in the transport cost of commodity 1. Hence, (1) must hold at the optimum. A similar argument shows that (2), (3) and (4) must also be satisfied at the optimum.

Step 2. We now suppose that the firms represented in Figure 7 are located such that (1)-(4) are satisfied, and show that $\theta_{i_2} < \phi_2 < \theta_{i_2+1}$ cannot hold at the optimum. To this end, we consider varying ϕ_2 between θ_{i_2} and θ_{i_2+1}, while keeping the A-firms locations unchanged. Accordingly, the total transport cost of the $(\alpha - 1)$ units of commodity 1 is not affected by our change in ϕ_2.

Given (2) and (3), $\theta_{i_2} < \phi_2 < \theta_{i_2+1}$ implies that $\frac{\theta_{i_1+1}+\theta_{i_2+1}}{2} \leq \theta_{i_2}$ and $\frac{\theta_{i_2}+\theta_{i_3}}{2} \geq \theta_{i_2+1}$ must be satisfied. Observe that the value of ϕ_2 has no impact on the consumers located outside $[\phi_1, \phi_3]$. Because of (1), the consumers located between ϕ_1 and θ_{i_1+1} shop at B_1 for good 2 and at A_{i_1} or A_{i_1+1} for the last unit of good 1. This latter choice is independent of ϕ_2. Between θ_{i_1+1} and θ_{i_2}, the consumers go to the nearest B-firms and find a A-firm along the way. Indeed, because of (1) and (2), we have $\theta_{i_1+1} \leq \frac{\phi_1+\phi_2}{2}$ and $\theta_{i_2} \geq \frac{\phi_1+\phi_2}{2}$. Between θ_{i_2} and θ_{i_2+1}, (2) and (3) imply that all consumers buy 2 from B_2. But the choice to go to A_{i_2} on A_{i_2+1} for purchasing the last unit of commodity 1 depends on the position of ϕ_2 w.r.t. the middle of $[\theta_{i_2}, \theta_{i_2+1}]$. Two cases must therefore be considered:

(i) if $\phi_2 \leq \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$, then the consumers buying from A_{i_2} and B_2 are located between θ_{i_2} and the point $\bar{x}_1$ for which $\theta_{i_2+1} - \phi_2 = \bar{x}_1 - \theta_{i_2}$, *i.e.*, $\bar{x}_1 = \theta_{i_2} + \theta_{i_2+1} - \phi_2 \geq \phi_2$. Between $\bar{x}_1$ and θ_{i_2+1}, all consumers purchase from B_2 and A_{i_2+1};

(ii) if $\phi_2 > \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$, then the consumers patronizing A_{i_2} and B_2 are situated between θ_{i_2} and the point $\bar{x}_2$ which satisfies $\phi_2 - \theta_{i_2} = \theta_{i_2+1} - \bar{x}_2$, *i.e.*, $\bar{x}_2 = \theta_{i_2} + \theta_{i_2+1} - \phi_2 < \phi_2$. From $\bar{x}_2$ to θ_{i_2+1}, consumers visit B_2 and A_{i_2+1}.

Finally, between θ_{i_2+1} and ϕ_3, the situation is symmetric to that of consumers located between ϕ_1 and θ_{i_2}. To summarize, when $\phi_2 \leq \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$, the cost of tranporting one unit of 1 and the last unit of 2 for the consumers in $[\phi_1, \phi_3]$, considered as a function of ϕ_2, is given by

$$c(\phi_2) = a_1 + \int_{\theta_{i_1+1}}^{\frac{\phi_1+\phi_2}{2}} (x-\phi_1)dx + \int_{\frac{\phi_1+\phi_2}{2}}^{\theta_{i_2}} (\phi_2 - x)dx + \int_{\theta_{i_2}}^{\phi_2} (\phi_2 - \theta_{i_2})dx$$

$$+ \int_{\phi_2}^{\theta_{i_2}+\theta_{i_2+1}-\phi_2} (x-\theta_{i_2})dx + \int_{\theta_{i_2}+\theta_{i_2+1}-\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1} - x)dx$$

$$+ \int_{\theta_{i_2}+1}^{\frac{\phi_2+\phi_3}{2}} (x-\phi_2)dx + \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x)dx + a_3,$$

a_1 and a_3 being the corresponding costs of the consumers located between ϕ_1 and θ_{i_1}, and between θ_{i_3} and ϕ_3, which do not depend on ϕ_2. A simple calculation shows that

$$c'(\phi_2) = \theta_{i_2} + \theta_{i_2+1} - \phi_2 - \frac{1}{2}\phi_1 - \frac{1}{2}\phi_3 \tag{5}$$

and $c''(\phi_2) = -1 < 0$.

When $\phi_2 > \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$, the role of θ_{i_2} and θ_{i_2+1} must be reversed in the above expression of $c(\phi_2)$. As (5) is symmetric in θ_{i_2} and θ_{i_2+1}, the derivative $c'(\phi_2)$ is the same. Consequently, the function $c(\phi_2)$ is strictly concave on $]\theta_{i_2}, \theta_{i_2+1}[$ and, therefore, cannot reach its minimum at an interior point of $[\theta_{i_2}, \theta_{i_2+1}]$. Thus, at the optimum, we must have $\phi_2 \in \{\theta_{i_2}, \theta_{i_2+1}\}$. Q.E.D.

LEMMA 3: *Let B_1, B_2 and B_3 be three consecutive B-firms. If there is one A-firm in the segment $[B_1, B_2[$ and at least two A-firms in $]B_2, B_3]$, and if B_2 is isolated, then the corresponding configuration is not optimal.*

PROOF: Consider the configuration represented in Figure 8 and show that it cannot occur at the optimum. Let us first define the relevant shopping options for the consumers located in $[\phi_1, \theta_{i_2+1}]$ when buying one unit of 2 and the last unit of good 1:

(1) visiting A_{i_1} and B_1,

(2) visiting A_{i_1+1} and B_1,

(3) visiting $A_{i_1+1} = A_{i_2}$ and B_2,

(4) visiting A_{i_2+1} and B_2.

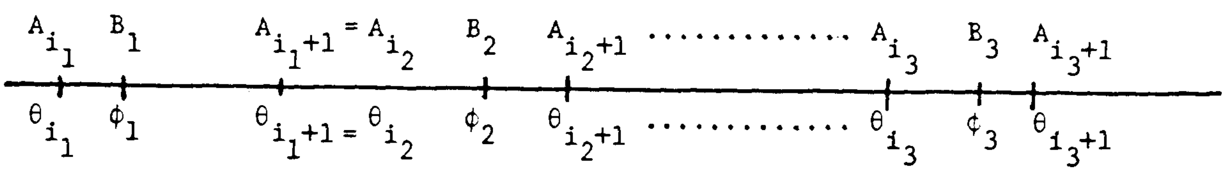

Figure 8

Two cases may arise in regard to the parameters θ_{i_1}, ϕ_1, θ_{i_2}.

Case 1: $\phi_1 \geq \frac{\theta_{i_1}+\theta_{i_2}}{2}$. (6)

The above inequality implies that consumers located in $[\phi_1, \theta_{i_2}]$ buy all the units of commodity 1 from A_{i_1+1}. In consequence, a consumer at $x \in [\phi_1, \theta_{i_2}]$ chooses between option (2) (with a distance to be covered equal to $\theta_{i_2} - \phi_1$) and option (3) (with distance $\phi_2 - x$). The consumer indifferent between these options is then located at $\bar{x} = \phi_1 + \phi_2 - \theta_{i_2}$. Similarly, a consumer at $x \in [\theta_{i_2}, \phi_2]$ chooses among the options (2) (with distance $x - \phi_1$), (3) (with distance $\phi_2 - \theta_{i_2}$) and (4) (with distance $\theta_{i_2+1} - x$). The consumer indifferent between (2) and (3) is located at $\bar{x} = \phi_1 + \phi_2 - \theta_{i_2}$, whereas the one indifferent between (3) and (4) is located at $\bar{\bar{x}} = \theta_{i_2} + \theta_{i_2+1} - \phi_2$.

Finally, a consumer located at $x \in [\phi_2, \theta_{i_2+1}]$ chooses between option (3) (with distance $x - \theta_{i_2}$) and option (4) (with distance $\theta_{i_2+1} - \phi_2$). The consumer indifferent between (3) and (4) is established at $\bar{\bar{x}} = \theta_{i_2} + \theta_{i_2+1} - \phi_2$. It is readily verified that $\bar{x} \leq \bar{\bar{x}}$ iff $\phi_1 + 2\phi_2 \leq 2\theta_{i_2} + \theta_{i_2+1}$. As the maximum value of ϕ_2 is θ_{i_2+1}, the above inequality is trivially satisfied if $\theta_{i_2} \geq \frac{\phi_1+\theta_{i_2+1}}{2}$. Let us, therefore, consider the following two sub-cases.

(a) $$\theta_{i_2} \geq \frac{\phi_1+\theta_{i_2+1}}{2} \tag{7}$$

Hence we have the following partition of $[\phi_1, \theta_{i_2+1}]$:

- $\phi_1 \leq x < \bar{x}$: consumers choose (2),
- $\bar{x} \leq x < \bar{\bar{x}}$: consumers choose (3),
- $\bar{\bar{x}} \leq x \leq \theta_{i_1+1}$: consumers choose (4),

Furthermore, (7) implies that $\bar{x} \leq \theta_{i_2}$. As in lemma 2, the cost of transporting one unit of 2 and the last unit of 1 for the consumers in $[\phi_1, \phi_3]$, as a function $c(\phi_2)$ of ϕ_2, depend on the position of B_2 w.r.t. the middle of the segment $[A_{i_2}, A_{i_2+1}]$. However, as seen in Step 2 of this lemma, this does not affect the properties of $c(\phi_2)$. For this reason, we focus only upon the case

$$\phi_2 \leq \frac{\theta_{i_2} + \theta_{i_2+1}}{2} \tag{8}$$

Notice that (8) implies that $\phi_2 \leq \bar{\bar{x}}$. We then have:

$$c(\phi_2) = \int_{\phi_1}^{\phi_1+\phi_2-\theta_{i_2}} (\theta_{i_2} - \phi_1)dx + \int_{\phi_1+\phi_2-\theta_{i_2}}^{\theta_{i_2}} (\phi_2 - x)dx$$

$$+ \int_{\theta_{i_2}}^{\phi_2} (\phi_2 - \theta_{i_2})dx + \int_{\phi_2}^{\theta_{i_2}+\theta_{i_2+1}-\phi_2} (x - \theta_{i_2})dx$$

$$+ \int_{\theta_{i_2}+\theta_{i_2+1}-\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1} - \phi_2)dx + \int_{\theta_{i_2+1}}^{\frac{\phi_2+\phi_3}{2}} (x - \phi_2)dx$$

$$+ \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x)dx + a_3,$$

where a_3 stands for the transport cost of one unit of 2 and the last unit of 1 for the consumers in $[\theta_{i_3}, \phi_3]$, which does not depend on ϕ_2. A standard calculation shows that

$$c'(\phi_2) = 2\theta_{i_2} + \theta_{i_2+1} - \phi_1 - \frac{3}{2}\phi_2 - \frac{1}{2}\phi_3$$

and $c''(\phi_2) = -\frac{3}{2} < 0$. Therefore, $c(\phi_2)$ is strictly concave on $]\theta_{i_2}, \theta_{i_2+1}[$ which, in turn, implies that $c(\phi_2)$ cannot be minimized at an interior point of the interval.

(b) $$\theta_{i_2} < \frac{\phi_1+\theta_{i_2+1}}{2} \qquad (9)$$

First, (9) implies that $\theta_{i_2} < \bar{x}$. In addition, we know that $\bar{x} \leq \bar{\bar{x}}$ iff $\phi_2 \leq \bar{\phi}_2 \underset{\text{def}}{=} \theta_{i_2} + \frac{\theta_{i_2+1}-\phi_1}{2}$. If $\phi_2 \geq \bar{\phi}_2$, we have the partition shown on Figure 9 of $[\theta_{i_2}, \theta_{i_2+1}]$ where the consumers' preferences between the different options are also indicated.

Accordingly, in $[\bar{\bar{x}}, \bar{x}]$, consumers prefer (2) to (3), and (4) to (3). It remains, therefore, to compare (2) and (4). Clearly, the separation point is given by the middle of the segment $[B_1, A_{i_2+1}]$ which, because of $\phi_2 \geq \bar{\phi}_2$, lies between $\bar{\bar{x}}$ and $\bar{x}$. Thus

- for $\phi_1 \leq x \leq \frac{\phi_1+\theta_{i_2+1}}{2}$, the consumers choose (2)
- for $\frac{\phi_1+\theta_{i_2+1}}{2} < x \leq \theta_{i_2+1}$, they adopt (4).

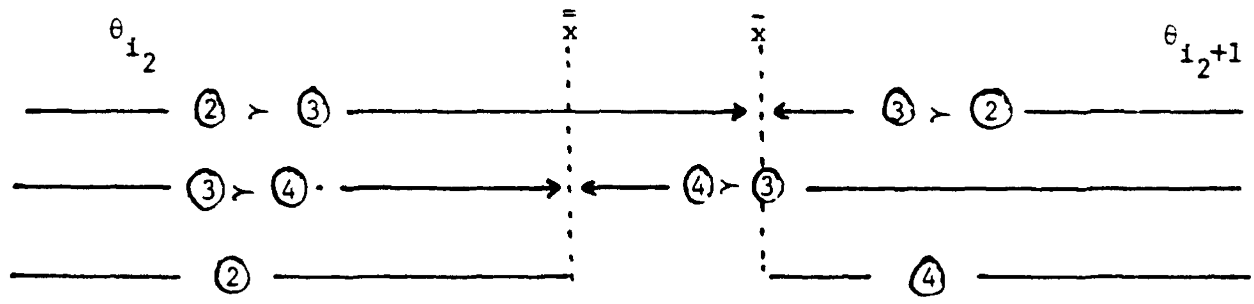

Figure 9

hence, $c(\phi_2)$ can be written as

$$c(\phi_2) = \int_{\phi_1}^{\theta_{i_2}} (\theta_{i_2} - \phi_1)dx + \int_{\theta_{i_2}}^{\frac{\phi_1+\theta_{i_2+1}}{2}} (x - \phi_1)dx + \int_{\frac{\phi_1+\theta_{i_2+1}}{2}}^{\phi_2} (\theta_{i_2+1} - x)dx$$

$$+ \int_{\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1} - \phi_2)dx + \int_{\theta_{i_2+1}}^{\frac{\phi_2+\phi_3}{2}} (x - \phi_3)dx + \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x)dx + a_3.$$

Taking the first-order derivative yields

$$c'(\phi_2) = \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3. \tag{10}$$

Assume now that $\phi_2 < \bar{\phi}_2$. As $\bar{x} < \bar{\bar{x}}$, the situation is identical to that analyzed in sub-case (a) so that we have

$$c'(\phi_2) = 2\theta_{i_2} + \theta_{i_2+1} - \phi_1 - \frac{3}{2}\phi_2 - \frac{1}{2}\phi_3. \tag{11}$$

As (10) and (11) take the same value at $\bar{\phi}_2$, $c'(\phi_2)$ is continuous on $[\theta_{i_2}, \theta_{i_2+1}]$. Between θ_{i_2} and $\bar{\phi}_2$, $c'(\phi_2)$ is decreasing. Beyond $\bar{\phi}_2$, $c'(\phi_2)$ is increasing but $c'(\theta_{i_2+1}) < 0$. Thus there does not exist a point of the interval $]\theta_{i_2}, \theta_{i_2+1}[$ where $c'(\phi_2) = 0$ and where $c'(\phi_2)$ is increasing. This implies that $c(\phi_2)$ cannot be minimized at an interior point of the interval and the lemma is proved.

Case 2: $$\phi_1 < \frac{\theta_{i_1}+\theta_{i_2}}{2} \tag{12}$$

This case is more complex than the previous one because some consumers located in $[\phi_1, \theta_{i_2}]$ adopt option (1). This implies that a consumer at $x \in [\phi_1, \theta_{i_2}]$ has to choose among (1) (with distance $x - \theta_{i_1}$), (2) (with distance $\theta_{i_2} - \phi_1$) and (3) (with distance $\phi_2 - x$). The consumer indifferent between (1) and (2) is at $\tilde{x} = \theta_{i_1} + \theta_{i_2} - \phi_1$, while the consumer indifferent between (2) and (3) is still at $\bar{x} = \phi_1 + \phi_2 - \theta_{i_2}$.

Observe now that $\tilde{x} \geq \bar{x}$ iff

$$\phi_2 - \theta_{i_2} \leq \theta_{i_1} + \theta_{i_2} - 2\phi_1. \tag{13}$$

If (13) holds, then $\bar{x} < \theta_{i_2}$ and (2) will never be adopted by a consumer in $[\phi_1, \theta_{i_2}]$. As ϕ_2 is contained in $[\theta_{i_2}, \theta_{i_2+1}]$, we have the following sub-cases (c), (d) and (e).

(c) $$\theta_{i_2+1} - \theta_{i_2} \leq \theta_{i_1} + \theta_{i_2} - 2\phi_1 \tag{14}$$

so that (13) is satisfied for all ϕ_2. Then, a consumer at $x \in \left[\phi_1, \frac{\theta_{i_1}+\phi_2}{2}\right[$ chooses (1) ($\frac{\theta_{i_1}+\phi_2}{2}$ is the location of the consumer indifferent between (1) and (3)), a consumer at $x \in \left[\frac{\theta_{i_2}+\phi_2}{2}, \bar{\bar{x}}\right[$ chooses (3) and a consumer at $x \in [\bar{\bar{x}}, \theta_{i_2+1}]$ chooses (4). Hence, $c(\phi_2)$ can be written as

$$\begin{aligned} c(\phi_2) = &\int_{\phi_1}^{\frac{\theta_{i_1}+\phi_2}{2}} (x - \theta_{i_1})dx + \int_{\frac{\theta_{i_1}+\phi_2}{2}}^{\theta_{i_2}} (\phi_2 - x)dx + \int_{\theta_{i_2}}^{\phi_2} (\phi_2 - \theta_{i_2})dx \\ &+ \int_{\phi_2}^{\theta_{i_2}+\theta_{i_2+1}-\phi_2} (x - \theta_{i_2})dx + \int_{\theta_{i_2}+\theta_{i_2+1}-\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1} - \phi_2)dx \\ &+ \int_{\theta_{i_2+1}}^{\frac{\phi_2+\phi_3}{2}} (x - \phi_2)dx + \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x) + a_3 \end{aligned}$$

when $\phi_2 \leq \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$, or the symmetric expression w.r.t. θ_{i_2} and θ_{i_2+1} when $\phi_2 > \frac{\theta_{i_2}+\theta_{i_2+1}}{2}$. Taking the first-order derivative yields in both cases

$$c'(\phi_2) = \theta_{i_2} + \theta_{i_2+1} - \frac{1}{2}\theta_{i_1} - \phi_2 - \frac{1}{2}\phi_3 \tag{15}$$

and $c''(\phi_2) = -1 < 0$ so that the minimum of $c(\phi_2)$ must be reached at a boundary point of the interval $[\theta_{i_2}, \theta_{i_2+1}]$.

(d) $$\theta_{i_2+1} - \theta_{i_2} > \theta_{i_1} + \theta_{i_2} - 2\phi_1 \text{ and } \theta_{i_2} \geq \tfrac{\phi_1 + \theta_{i_2+1}}{2} \tag{16}$$

Let $\tilde{\phi}_2 = \theta_{i_1} + 2(\theta_{i_2} - \phi_1)$.

(i) If $\phi_2 \leq \tilde{\phi}_2$, then (13) is satisfied. Therefore, we are in a situation identical to that analyzed is sub-case (c), which implies that $c'(\phi_2)$ is given by (15).

(ii) If $\phi_2 > \tilde{\phi}_2$, then we have $\tilde{x} < \bar{x}$. Furthermore, by the second inequality of (16), we obtain $\bar{x} \leq \theta_{i_2}$. Accordingly,

- for $\phi_1 \leq x < \tilde{x}$, consumers adopt (1)
- for $\tilde{x} \leq x < \bar{x}$, they adopt (2)
- for $\bar{x} \leq x < \bar{\bar{x}}$, they choose (3)
- for $\bar{\bar{x}} \leq x \leq \theta_{i_2+1}$, they choose (4).

In consequence, $c(\phi_2)$ becomes

$$c(\phi_2) = \int_{\phi_1}^{\theta_{i_1}+\theta_{i_2}-\phi_1} (x - \theta_{i_1})dx + \int_{\theta_{i_1}+\theta_{i_2}-\phi_1}^{\phi_1+\phi_2-\theta_{i_2}} (\theta_{i_2} - \phi_1)dx + \int_{\phi_1+\phi_2-\theta_{i_2}}^{\theta_{i_2}} (\phi_2 - x)dx$$

$$+ \int_{\theta_{i_2}}^{\phi_2} (\phi_2 - \theta_{i_2})dx + \int_{\phi_2}^{\theta_{i_2}+\theta_{i_2+1}-\phi_2} (x - \theta_{i_2})dx$$

$$+ \int_{\theta_{i_2}+\theta_{i_2+1}-\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1} - \phi_2) + \int_{\theta_{i_2}+1}^{\frac{\phi_2+\phi_3}{2}} (x - \phi_2) + \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x)dx + a_3.$$

Again a straightforward calculation shows that

$$c'(\phi_2) = 2\theta_{i_2} + \theta_{i_2+1} - \frac{3}{2}\phi_2 - \phi_1 - \frac{1}{2}\phi_3. \tag{17}$$

As (15) and (16) take the same value at $\tilde{\phi}_2$, $c'(\phi_2)$ is continuous and strictly decreasing. Thus, $c(\phi_2)$ reaches its minimum at one boundary point of the interval $[\theta_{i_2}, \theta_{i_2+1}]$.

(e) $$\theta_{i_2+1} - \theta_{i_2} > \theta_{i_1} + \theta_{i_2} - 2\phi_1 \text{ and } \theta_{i_2} < \tfrac{\phi_1 + \theta_{i_2+1}}{2} \tag{18}$$

Then (9), (13) and (18) imply that $\tilde{\phi}_2 = \theta_{i_1} + 2(\theta_{i_2} - \phi_1)$ and $\bar{\phi}_2 = \theta_{i_2} + \frac{\theta_{i_2+1} - \phi_1}{2}$ are both between θ_{i_2} and θ_{i_2+1}. It follows immediately from the first inequality of (18) that $\tilde{\phi}_2 < \bar{\phi}_2$. As a result, we have the following partition:

(i) if $\phi_2 \le \tilde{\bar{\phi}}_2$, then the situation is the same as in sub-case (c) so that $c'(\phi_2)$ is given by (15),

(ii) if $\tilde{\bar{\phi}}_2 < \phi_2 \le \bar{\bar{\phi}}_2$, then the situation is identical to that analyzed in (d-ii) which implies that $c'(\phi_2)$ is now given by (17),

(iii) if $\phi_2 > \bar{\bar{\phi}}_2$, then

- for $\phi_1 \le x < \tilde{x}$, consumers choose (1)
- for $\tilde{x} \le \phi_1 < \frac{\phi_1+\theta_{i_2+1}}{2}$ (the location of the consumer indifferent between (2) and (4)), they choose (2)
- for $\frac{\phi_1+\theta_{i_2+1}}{2} \le x \le \theta_{i_2+1}$, they choose (4).

Hence,

$$c(\phi_2) = \int_{\phi_1}^{\theta_{i_1}+\theta_{i_2}-\phi_1} (x-\theta_{i_1})dx + \int_{\theta_{i_1}+\theta_{i_2}-\phi_1}^{\theta_{i_2}} (\theta_{i_2}-\phi_2)dx + \int_{\theta_{i_2}}^{\frac{\phi_1+\theta_{i_2+1}}{2}} (x-\phi_1)dx$$

$$+ \int_{\frac{\phi_1+\theta_{i_2+1}}{2}}^{\phi_2} (\theta_{i_2+1}-x)dx + \int_{\phi_2}^{\theta_{i_2+1}} (\theta_{i_2+1}-\phi_2)dx$$

$$+ \int_{\theta_{i_2+1}}^{\frac{\phi_2+\phi_3}{2}} (x-\phi_2)dx + \int_{\frac{\phi_2+\phi_3}{2}}^{\theta_{i_3}} (\phi_3 - x) + a_3$$

with

$$c'(\phi_2) = \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3 < 0.$$

An argument similar to that developed for sub-case (b) shows that $c'(\phi_2)$ is strictly negative when it is increasing so that there cannot be a minimum in $]\theta_{i_2}\theta_{i_2+1}[$. Here also, $c(\phi_2)$ is minimized at one of the endpoints of the interval and the lemma is proved. Q.E.D.

Equipped with the above lemmas, we can now prove our main result.

THEOREM 1: *There are n A-firms and m B-firms with $m \le n$. Then, at any optimal configuration, each B-firm is clustered with an A-firm.*

PROOF:

(i) If $m = n$, then it is obvious that the optimal configuration is given by the superposition of the two equally equidistant patterns.

(ii) If $m = 1$ and $n > 1$, then the equidistant pattern of A-firms with the single B-firms clustered with any A-firm is optimal.

(iii) If $m = 2$ and $n > 2$, then the situation is similar to that where $m = 3$ and $\phi_3 = \phi_1 + 2\pi$.

(iv) Let $m \geq 3$ and $m < n$. Lemma 1 implies that any segment defined by two consecutive B-firms contains at least one A-firm. As $m < n$, there exists at least two consecutive firms, denoted B_1 and B_2, such that $[B_1, B_2]$ contains at least two A-firms. By Lemma 2 or 3, B_1 and B_2 must each be clustered with one A-firm. Given Lemma 1, the segment $]B_2, B_3]$ must contain at least one A-firm so that $[B_2 B_3]$ contains at least two A-firms and lemmas 2 or 3 applies to B_3. Repeating the argument for each subsequent B_i, $i = 4, \ldots, m$ yields the desired result. Q.E.D.

Thus at the optimum, we obtain a hierarchical system of centers of orders 2 and 1. Every center of order 2 contains one A-firm and one B-firm, whereas every center of order 1 always contains one A-firm. Consequently, a consumer buys $\alpha - 1$ units of commodity 1 from the nearest A-firm and one unit of commodity 2 from the nearest B-firm; he buys the last unit of commodity 1 from the A-firm clustered with the nearest B-firm without incurring any additional cost.

Theorem 2 allows us to determine precisely the optimal configuration of A- and B-firms as well as the minimum total transport cost.

THEOREM 2: *Assume $m < n$ and let q and r verifying $n = mq + r$ with $0 \leq r < m$. Then, the optimal configuration is such that (i) there are r segments $]B_i, B_{i+1}]$ containing $q+1$ A-firms equally spaced at distance $\frac{2\pi}{n+\frac{(m-r)q}{\alpha-1+q}}$; (ii) there are $m-r$ segments $]B_i, B_{i_1+1}]$ containing q A-firms equally spaced at distance $\frac{2\pi}{n-\frac{r(q+1)}{\alpha+q}}$. Furthermore, the minimum total* *transport cost is given by* $c^* = \frac{\pi^2}{r\frac{q+1}{\alpha+q}+(m-r)\frac{q}{\alpha+q+1}}$.

PROOF: To ease the burden of notation, set $\theta_1 = 0$, $z_i = \theta_{i+1} - \theta_i$ for $i = 1 \ldots n-1$ and $z_n = 2\pi - \theta_n$, where the θ_i are the radial coordinates of the A-firms along C. Let also k_i be the number of A-firms in the segment $]B_i, B_{i+1}]$ (see Figure 10). We have $k_1 + k_{2+} \ldots k_m = n$.

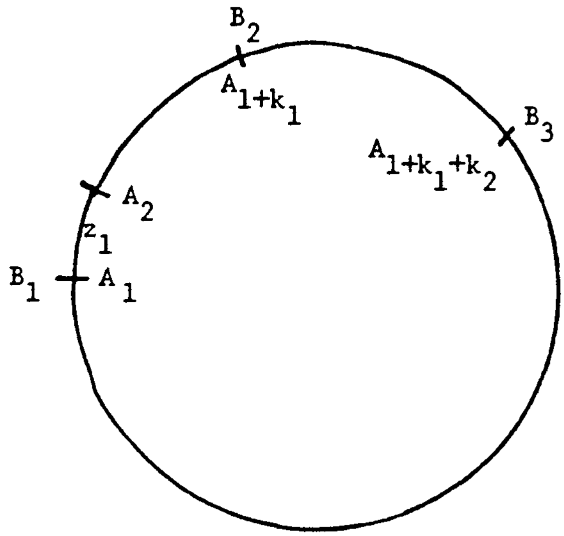

Figure 10

It is readily verified that the total transport cost is equal to

$$c = \frac{\alpha-1}{4}(z_1^2 + z_2^2 + \ldots z_n^2) + \frac{1}{4}[(z_1 + \ldots + z_{k_1})^2 + (z_{k_{1+1}} + \ldots + z_{k_1+k_2})^2$$

$$+ \ldots (z_{k_1+\ldots+k_{m-1}+1} + \ldots + z_n)^2]. \tag{19}$$

We solve the minimization problem of (19) w.r.t. $z_1 \ldots z_n$ and $k_1 \ldots k_m$ in two steps. In the first one, we keep $k_1 \ldots, k_m$ fixed and minimize (19) w.r.t. $z_1 \ldots z_n$. In the second step, we determine the optimal values of $k_1 \ldots, k_m$.

Step 1. We want to minimize c subject to $\sum_{i=1}^{n} z_i = 2\pi$. As c is convex in $z_1 \ldots z_n$, the

FOC are sufficient and yield

$$\begin{aligned}
(\alpha-1)z_1 + (z_1 + \ldots + z_{k_1}) &= 2\lambda \\
&\vdots \\
(\alpha-1)z_{k_1} + (z_1 + \ldots + z_{k_1}) &= 2\lambda \\
(\alpha-1)z_{k_1+1} + (z_{k_1+1} + \ldots + z_{k_1+k_2}) &= 2\lambda \\
&\vdots \\
(\alpha-1)z_n + (z_{k_1+\ldots+k_{m-1}+1} + \ldots + z_n) &= 2\lambda.
\end{aligned}$$

where λ is the Lagrange multiplier of the constraint. Clearly, we have

$$\begin{gathered}
z_1 = \ldots = z_{k_1} \stackrel{\text{def}}{=} d_1 \\
\vdots \\
z_{k_1+\ldots+k_{m-1}+1} = \ldots = z_n \stackrel{\text{def}}{=} d_m
\end{gathered} \tag{20}$$

and

$$(\alpha-1)d_1 + k_1 d_1 = \ldots = (\alpha-1)d_m + k_m d_m. \tag{21}$$

Using (20), the constraint can be written as

$$\sum_{j=1}^{m} k_j d_j = 2\pi. \tag{22}$$

Furthermore, (21) and (22) imply

$$d_1 = \frac{2\pi}{(\alpha-1)+k_1} \cdot \frac{1}{\frac{k_1}{(\alpha-1)+k_1} + \ldots + \frac{k_m}{(\alpha-1)+k_m}}$$

$$d_m = \frac{2\pi}{(\alpha-1)+k_m} \cdot \frac{1}{\frac{k_1}{(\alpha-1)+k_1} + \ldots + \frac{k_m}{(\alpha-1)+k_m}}. \tag{23}$$

Replacing in c yields

$$c = \frac{\pi^2}{\frac{k_1}{(\alpha-1)+k_1} + \ldots + \frac{k_m}{(\alpha-1)+k_m}}. \tag{24}$$

Step 2. Minimizing (24) w.r.t. $k_1 \ldots k_m$ amounts to maximizing

$$\frac{k_1}{(\alpha-1)+k_1} + \ldots + \frac{k_m}{(\alpha-1)+k_m} \tag{25}$$

subject to $k_1 \dots k_m \in N$ and $\sum_{j=1}^{m} k_j = n$. Relaxing the integer constraints $k_j \in N$, we easily verify that (25) is a concave function which is maximized at $k_1 = \dots = k_m = \frac{n}{m}$. Therefore, the integer solution is given by one of the nearest points with integer coordinates belonging to the hyperplane $\sum_{j=1}^{m} k_j = n$. Dividing n by m, we get the quotient q and the rest r, *i.e.*, $n = mq + r$ with $0 \leq r < m$ (or equivalently, $q \leq \frac{n}{m} < q + 1$). Given that $r(q+1) + (m-r)q = n$, the integer solutions are such that r values of k equal $q+1$, while $(m-r)$ values of k equal q.

In consequence, there are r segments containing $q+1$ equidistant A-firms and $m-r$ segments containing q equidistant A-firms at the optimum. Replacing in (23) and (24) we obtain the distances and the minimum cost c^* given in the statement of the theorem. Q.E.D.

So the optimal pattern of A-firms departs from the symmetric configuration as a function of α. When α increases, this pattern comes closer to the symmetric configuration that we find as a limit for $\alpha \longrightarrow \infty$.

4. CONCLUSIONS

For simplicity, we have supposed here a fixed number of firms $(m+n)$. Clearly, it would be preferable to allow for a variable number of firms, taking into account the fact that an A-firm has to bear a fixed cost F_1 and a B-firm a fixed cost F_2, with $F_1 \leq F_2$. The objective of the planner is now to find the configuration minimizing $nF_1 + mF_2 + c^*(m,n)$. This is a nonlinear integer mathematical program whose solution is hard to obtain. However, as $F_1 \leq F_2$, we must have $m^* \leq n^*$ at the optimal solution so that the clustering property remains valid for this problem.

Much work remains to be done. First, the model should be extended to the case of several commodities. Given that our proof is combinatorial, the analysis of the n-commodity case could become cumbersome. Another significant generalization of the model is to work within a two-dimensional space. The analysis undertaken by Bollobas (1973) for the one-commodity case shows already that the extension will not be easy to perform. Finally, one would like to relax assumption (H.4) stating that consumers consume the same proportion of the two commodities. A preliminary study of the spatial demand with utility-maximizing consumers, made by Thill (1985), casts some doubt on the analytical tractability of such a model.

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