"THE EXPECTED UTILITY OF PORTFOLIOS OF ASSETS"

by

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The Expected Utility of Portfolios of Assets

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Abstract

This paper describes two models of asset markets and portfolio choice: one where the von Neumann-Morgenstern utility function is defined on the non-negative real line and short-selling is not allowed, and one where the von Neumann-Morgenstern utility function is defined on the entire real line and short-selling may be possible. A number of properties of the derived utility function for portfolios, needed in demand and equilibrium analysis, are investigated with particular attention to the possibility that the von Neumann–Morgenstern utility function may be unbounded below and that the derived (expected) utility of some portfolios may be negative infinity.

1 Introduction

This paper analyzes the utility function for portfolios of assets derived from expected utility based on a von Neumann–Morgenstern utility function and a joint distribution of total returns to the assets. This type of utility function has been used in the literature in a large number of models of portfolio choice and asset market equilibrium. The aim is to demonstrate under what conditions it has the properties needed in demand and equilibrium analysis.

Two models are distinguished. In Model 1, the utility function is defined on the non-negative real line, it may take the value minus infinity at zero, the returns to all assets are non-negative, and short-selling is disallowed. In Model 2, the utility function is defined on the entire real line (and unbounded below), returns may well be negative, and short-selling may be allowed.

In order not unduly to restrict the applicability of Model 1, it is necessary not only to allow the utility function to approach negative infinity as total portfolio return goes to zero but actually to let it take the value negative infinity at zero. It is argued that this is acceptable from the point of view of decision theory, even though most axiomatic theories of expected utility imply a finite-valued utility function.

The fact that the utility function is unbounded below in Model 2 implies that some portfolios may have negative infinite expected utility. If the utility function in Model 1 is unbounded below, then some portfolios can have negative infinite expected utility even though they have positive return for sure.

It turns out that negative infinite expected utility can lead to discontinuity of the derived utility function for assets. To avoid discontinuities, an assumption (Assumption 3) is imposed on the joint distribution of asset returns and on the von Neumann-Morgenstern utility function. The assumption has the effect that negative infinite (expected) utility occurs only in Model 1, only if the von Neumann–Morgenstern utility function takes the value negative infinity at zero, and only if the portfolio has positive probability of zero return.

In its general form, the assumption is a joint condition on the distribution of returns and the utility function. However, it is desirable to replace it with separate conditions, one on the distribution of returns and one on the utility function. The condition on the distribution of returns (Condition 1) limits the probability of very low returns. The condition on the utility function (Condition 2) limits the risk aversion at low levels of return. Specifically, in Model 1, the risk aversion is bounded below by the risk aversion of a utility function with constant relative risk aversion, while in Model 2, the risk aversion is bounded below by the risk aversion of a utility function with constant absolute risk aversion.

Properties of utility functions (or preferences) for portfolios or consumption bundles needed in demand and equilibrium analysis are identified by Debreu (1959), Mas-Colell (1985), and Balasko (1988). Apart from continuity, the properties investigated in the present paper include various kinds of differentiability, strict quasi-concavity, and differentiable strict concavity.

Because the derived utility function is concave, directional derivatives exist even if the von Neumann–Morgenstern utility function is not differentiable. In Model 1 (where portfolio returns are never negative), some directional derivatives may be infinite if the derivative of the von Neumann–Morgenstern utility function goes to infinity as portfolio return goes to zero. The directional derivative is continuous at portfolios (and in directions) where it is infinite.

If the von Neumann-Morgenstern utility function is differentiable and takes finite values with probability one, then the derived utility function for portfolios is continuously differentiable. The requirement that the von Neumann-Morgenstern utility function be differentiable can be weakened. In particular, if the joint distribution of returns is absolutely continuous with respect to Lebesgue measure, then no differentiability assumption is needed.

The derived utility function will be twice continuously differentiable if the von Neumann-Morgenstern utility function is twice continuously differentiable with non-decreasing second derivative and if an appropriate integrability requirement is met. The integrability requirement will be met if the returns have finite variances and if the above-mentioned Condition 1 on the returns distribution and Condition 2 on the utility function hold.

The derived utility function will often be strictly concave or strictly quasi-concave even if the von Neumann–Morgenstern utility function is not strictly concave. We find the exact necessary and sufficient conditions. The expected utility will be differentiably strictly concave if the requirements for second order continuous differentiability are met and if the Neumann–Morgenstern utility function has negative second derivative.

2 The Models

There are n *assets* or *securities*, $j = 1, \ldots, n$. A *portfolio* is represented by an n-vector x, where the j'th entry x_j indicates the number of shares or units of the j'th asset in the portfolio. A portfolio x may well have some negative entries x_j, interpreted as *short-selling* or borrowing.

In some versions of the model, it may be relevant to impose restrictions, such as full or partial prohibitions against short-selling or other institutional constraints, on the portfolio choices available to the investor. This is formalized by the investor's *choice set* X, which is a subset of $\Re^n$. The interpretation of X is that when the investor selects a portfolio, he may choose it only among portfolios in X. Typically X has the form $[0, \infty)^n$, $\Re^{n-h} \times [0, \infty)^h$ or $\Re^n$, reflecting short-selling constraints for all assets, for the last h assets, or no constraints. The choice set corresponds to the consumption set in microeconomic demand and equilibrium analysis, cf. Debreu (1959). We assume that the choice set X is closed and convex. This assumption is standard and appears to be indispensable in demand and equilibrium analysis.

The *total returns* per share of the assets are given by a random vector R where the j'th entry R_j indicates the (random) total (gross) return per share of asset j. The total return to a portfolio x is $x'R$. The investor considers the returns R to be distributed in $\Re^n$ according to a *probability distribution* π.

Assumption 1 *Existence of expectations: R has a finite mean.*

The investor has a *von Neumann–Morgenstern utility function* u which is a function of end-of-period wealth or total portfolio return $x'R$. Two situations need to be distinguished. If short-selling is not possible and all individual assets have non-negative total returns for sure, then all portfolios in the choice set X have non-negative total returns for sure, and the von Neumann–Morgenstern utility function needs to be defined only on the non-negative part of the real line. In this case, we permit the value negative infinity at zero in order to allow for utility functions that approach negative infinity as total portfolio return goes to zero. Otherwise, if either short-selling is allowed or some assets have negative returns with positive probability, then there is no limit to how large negative portfolio returns are possible with positive probability, and so it is necessary to have the von Neumann–Morgenstern utility function defined on the entire real line.

Model 1 *Limited domain, non-negative returns and no short-selling:*

1. *Limited domain: u is defined on $[0, +\infty)$ and takes finite real values except possibly the value $u(0) = -\infty$.*
2. *Non-negative returns: For all assets j, $R_j \geq 0$ with probability one.*
3. *No short-selling: X is contained in $[0, +\infty)^n$.*

Model 2 *Unlimited domain: u is defined on the entire real line (and takes finite real values).*

The assets (contracts) available in Model 2 are claims to (or liabilities for, or both) random future amounts of wealth, possibly in the form of a consumption good. The claims may be thought of as contracts (rather than assets) because we do not rule out the possibility that the returns may turn out to be negative; thus, there is no telling in advance whether a given contract will be an asset or a liability. This specification allows for the classical case of normal distributions.

Model 1 is a special case of Model 2 except for the possibility that $u(0) = -\infty$. One of these two models and its corresponding assumptions will always be in force. If neither is explicitly specified, either will do.

Assumption 2 *u is strictly increasing, concave and continuous (also at zero).*

Assumption 2 rules out quadratic utility functions. Concavity of u implies continuity when u is defined on the entire real line and continuity except at 0 when u is defined only on the non-negative real line. Assumption 2 forces u to be unbounded below when it is defined on the entire real line.

Assumptions 1 and 2 are maintained throughout the rest of this paper.

Define the *extended choice set* $\bar{X}$ as $\bar{X} = [0, +\infty)^n$ in Model 1 and $\bar{X} = \Re^n$ in Model 2. Then X is contained in $\bar{X}$. Define the *extended utility function* $\bar{V}$ on the extended choice set $\bar{X}$ by

$$\bar{V}(x) = Eu(x'R).$$

Strictly speaking, the utility function V for portfolios is the restriction of the extended utility function $\bar{V}$ to the choice set X.

The expectations symbol "E" refers to the mathematical expectation with respect to the probability distribution π. Write $u(x'R)^+$ for the positive part of $u(x'R)$. More generally, the positive part of a variable Y is $Y^+ = \max\{0, Y\}$ and the negative part is $Y^- = -\min\{0, Y\} = Y^+ - Y$. Since u is concave (Assumption 2) and expectations exist (Assumption 1), it is easily seen that $Eu(x'R)^+ < 0$ for all x in $\bar{X}$, so that $\bar{V}$ is well defined and $\bar{V}(x) < +\infty$. It also follows that $\bar{V}$ is concave.

3 Negative Infinite Expected Utility

Model 1 allows the von Neumann-Morgenstern utility function u to take the value $u(0) = -\infty$. We shall see that this possibility is not too difficult to handle from a technical point of view. I shall argue here that it is acceptable from the point of view of decision theory, and it cannot be avoided without a severe loss of applicability of the model.

Most axiomatic theories of expected utility imply that the von Neumann–Morgenstern utility function is finite-valued, and that the expected utility of all admissible random prospects is finite also. That is true even of those theories that allow for an unbounded utility function, cf. DeGroot (1970), Fishburn (1976, 1982), Ledyard (1971), and Nielsen (1984, 1987a). It would, however, be possible to change the axiom systems underlying these theories so as to allow the von Neumann–Morgenstern utility function to take the value $-\infty$ and/or to allow the expected utility of admissible random prospects to be $-\infty$. This will violate the so-called "Archimedian axiom," but the violation is acceptable from the standpoint of rational behavior. Nielsen (1989) exhibits an example of an axiom system which allows the von Neumann–Morgestern utility function to take the value $+\infty$ and also allows the expected utility of admissible random prospects to be $+\infty$

The Archimedian axiom says that if p, q and r are admissible random prospects (probability distributions) such that $p \succ q \succ r$, then there exist numbers s and t with $0 < s < 1$ and $0 < t < 1$ such that $sp+(1-s)r \succ q \succ tp+(1-t)r$. Here, $\succ$ denotes the (strict) preference relation over admissible random prospects, and $sp + (1 - s)r$ and $tp + (1 - t)r$ denote "compound lotteries" (convex combinations of probability distributions). If u is the von Neumann–Morgenstern utility function and if $-\infty = E_p u < E_q u < E_r u < +\infty$, then $E_p u = -\infty = E_{sp+(1-s)r} u$ for all s with $0 < s < 1$, in violation of the axiom. So, in order to allow the value $-\infty$, the Archimedian axiom (or the corresponding axiom in any particular axiom system) must be modified. The modified axiom would assume the existence of the number s only if the prospect p is not minimal with respect to the preference relation on the set of admissible prospects.

An axiom system and a decision theory modified in this manner recognizes the possible existence of one or more "least preferred" prospects which are so bad that any compound lottery involving a positive probability of one of them is also least preferred. The utility function derived from the theory would assign (expected) utility $-\infty$ to these least preferred prospects. In the one-period portfolio-choice model, a total portfolio return of zero might be treated as being this bad. Such an attitude on the part of the investor may be termed extreme, but it is hardly irrational. Thus, negative infinite utility is acceptable from the point of view of decision theory.

Furthermore, negative infinite utility cannot be avoided without a severe loss of applicability of the model. If u were required to be bounded below, then utility functions such as $u = \log$ would be ruled out. An alternative way to avoid negative infinity might be to have u defined only on the open interval $(0, +\infty)$ and to require $x'R > 0$ with probability one for all portfolios x in X. However, that would not work well, primarily because of the requirement that the choice set X be closed. The choice set X will typically equal $[0, +\infty)^n$. Even if $R_j > 0$ with probability one for all assets j, the choice set $X = [0, +\infty)^n$ contains the portfolio $x = 0$ which has zero return for sure. The zero portfolio cannot be excluded from X without arbitrarily bounding X away from zero, since X has to be closed.

4 Discontinuous Expected Utility

In demand and equilibrium theory, it is virtually always necessary or desirable to assume that the preference relation or the utility function is continuous. The following examples show that without a further integrability assumption, the utility function V derived from expected utility may be discontinuous.

Example 1 In Model 2, assume that the (marginal) distribution of the total return R_1 to the first asset is normal with mean zero and variance one. Suppose that

$$u(t) = -\exp(t^2 + t) \text{ for } t \leq -1$$

and that u is extended to the entire real line in a manner such that it is strictly increasing and concave. If x is a portfolio consisting of σ shares of the first asset, then the total return $x'R$ to x is normally distributed with mean zero and variance σ^2, and

$$\begin{aligned} V(x) &= \int_{-\infty}^{+\infty} u(t)\frac{1}{2\pi\sigma^2}\exp(-t^2/(2\sigma^2))dt \\ &= -\frac{1}{2\pi\sigma^2}\int_{-\infty}^{-1}\exp(t^2+t)\exp(-t^2/(2\sigma^2))dt \\ &\quad -\frac{1}{2\pi\sigma^2}\int_{-1}^{+\infty} u(t)\exp(-t^2/(2\sigma^2))dt. \end{aligned}$$

The second term above is finite because of the concavity of u, so $V(x)$ is finite if and only if the first term is finite. Since

$$\int_{-\infty}^{-1}\exp(t^2+t)\exp(-t^2/(2\sigma^2))dt = \int_{-\infty}^{-1}\exp(t^2(1-1/(2\sigma^2))+t)dt,$$

$V(x)$ is finite if $|\sigma| \leq 1/\sqrt{2}$, and $V(x) = -\infty$ if $|\sigma| > 1/\sqrt{2}$. This shows that V is discontinuous at the portfolio x consisting of $\sigma = 1/\sqrt{2}$ shares of the first asset. □

Example 2 In Model 1, assume that the (marginal) distribution of the total return R_1 to the first asset is lognormal: it is distributed as $\exp(Y)$, where Y is a normally distributed random variable with mean zero and variance one. Let $a \leq 0$ be a number such that the function

$$f(t) = -\exp(t^2/2 + t - \sqrt{-t}) \text{ for } t < a$$

is strictly increasing and concave, and extend f to a strictly increasing and concave function on the entire real line. Suppose that the von Neumann–Morgenstern utility function is $u(t) = f(\log t)$. If x is a portfolio consisting of $s > 0$ shares of the first asset, then the total returns $x'R$ to x is distributed as

$$s\exp(Y) = \exp(Y + \log s) = \exp(Y + c),$$

where $c = \log s$. Consequently,

$$\begin{aligned}
V(x) &= Eu(x'R) \\
&= Ef(\log x'R) \\
&= Ef(\log(\exp(Y + c))) \\
&= Ef(Y + c) \\
&= \int_{-\infty}^{+\infty} f(t)\frac{1}{2\pi}\exp(-(t - c)^2/2)dt \\
&= -\frac{1}{2\pi}\int_{-\infty}^{a} \exp(t^2/2 + t - \sqrt{-t})\exp(-(t - c)^2/2)dt \\
&\quad -\frac{1}{2\pi}\int_{a}^{+\infty} f(t)\exp(-(t - c)^2/2)dt.
\end{aligned}$$

The second term above is finite because f is concave, so $V(x)$ is finite if and only if the first term is finite. Since

$$\begin{aligned}
\int_{-\infty}^{a} \exp(t^2/2 + t - \sqrt{-t})\exp(-(t - c)^2/2)dt &= \\
\int_{-\infty}^{a} \exp((1 + c)t - \sqrt{-t} - c^2/2)dt,
\end{aligned}$$

$V(x)$ is finite if $c \geq -1$ and infinite if $c < -1$. In terms of s, the number of shares, $V(x)$ is finite if $s \geq \exp(-1)$ and infinite if $s < \exp(-1)$. This shows that V is discontinuous at the portfolio consisting of $s = \exp(-1)$ shares of the first asset. □

In Example 2, the problem is not one of total returns that are negative, or even total returns that are zero. The return to the asset is positive with probability one. The discontinuity occurs at a portfolio which consists of a positive number s of shares of the asset. That portfolio, as well as all other portfolios consisting of a positive number of shares (possibly smaller than s), has positive return with probability one. For all these portfolios, there is zero probability that the utility of the return will be negative infinity. However, those portfolios that consist of fewer than s shares of the asset have negative infinite expected utility. The discontinuity occurs not because of zero return or negative infinite utility, but because of negative infinite *expected* utility.

5 Continuous Expected Utility

In order to ensure that $\bar{V}$ is continuous, we will assume that u is integrable below:

Assumption 3 *Integrability (below): $Eu(x'R)^- < +\infty$ for all x in $\bar{X}$ such that $\pi(u(x'R) = -\infty) = 0$.*

Even though this assumption is exactly what is needed from a technical point of view, it is desirable to split it into two seperate, jointly sufficient conditions on the returns distribution π and the utility function u. The condition on π will say that there is not too high probability of very low portfolio returns. The condition on u will say that risk aversion is bounded below at low wealth levels by the risk aversion of a utility function with constant relative risk aversion in Model 1, and by the risk aversion of a utility function with constant absolute risk aversion in Model 2.

Condition 1 *Limited probability of low returns:*

1. *In Model 1, $E[(x'R)^\alpha] < +\infty$ for all $x \in [0, +\infty)^n$, $x \neq 0$, all $\alpha < 0$.*
2. *In Model 2, $E\exp(x'R) < +\infty$ for all x in $\Re^n$.*

Example 3 In Model 1, if there is one asset, and if its total return follows a lognormal distribution, then Condition 1 holds. In Model 2, if R is jointly normally distributed, then Condition 1 holds. □

In order to state the condition on u, a few definitions will be needed.

Suppose u and v are two utility functions, and suppose that I is an interval which is contained in the domain of definition of both u and v. Say that u is less risk averse than v if any lottery accepted by v is also accepted by u, provided that both initial and final wealth are within I. Formally, u is *less risk averse* than v on I if the following holds:

- If Y is a bounded random variable with values in I, if $c \in I$, and if $v(c) \leq Ev(Y)$, then $u(c) \leq Eu(Y)$.

If u and v are twice differentiable, then u is less risk averse than v on I if and only if the measure of absolute risk aversion for u is less than the corresponding measure for v at all wealth levels in I. The present definition does not require differentiability.

For $\gamma > 1$, let u_γ denote a utility function with constant relative risk aversion (CRRA) γ, defined on $[0, +\infty)$. Specifically,

$$u_\gamma(w) = \begin{cases} -w^{1-\gamma} & \text{if } w > 0 \\ -\infty & \text{if } w = 0 \end{cases}$$

For $\lambda > 0$, let v_λ denote a utility function with constant absolute risk aversion (CARA) λ, defined on the entire real line. Specifically,

$$v_\lambda(w) = -\exp(-\lambda w)$$

Condition 2 *Limited risk aversion at low levels of wealth:*

1. *In Model 1, there exist $w_0 > 0$ and $\gamma > 1$ such that u is less risk averse than the CRRA utility function u_γ on $(0, w_0)$.*
2. *In Model 2, there exists w_0 and $\lambda > 0$ such that u is less risk averse than the CARA utility function v_λ on $(-\infty, w_0)$.*

Condition 2 implies that u is bounded below by a CRRA (respectively, CARA) utility function at low levels of wealth:

Lemma 1 *Suppose u and v are two utility functions. Suppose I is an interval contained in the domain of definition of both u and v. If u is less risk averse than v on I, then there exist numbers $a > 0$ and b such that*

$$u(w) \geq av(w) + b \text{ for all } w \in I$$

PROOF: Let $g : v(I) \to \Re$ be the function $g = u \circ v^{-1}$. Then g is strictly increasing and convex. Choose a and b such that the graph of the affine function $t \mapsto at+b$ is tangent to the graph of g. Then $a > 0$, and $g(t) \geq at+b$ for $t \in v(I)$. For for $w \in I$, $u(w) = g(v(w)) \geq av(w) + b$. □

Proposition 1 *Conditions 1 and 2 jointly imply Assumption 3.*

PROOF: It follows from Condition 1 that the variables $u_\gamma(x'R)$ (in Model 1) and $v_\lambda(x'R)$ (in Model 2) have finite means. The rest follows from Lemma 1. □

Assumption 3 is maintained throughout the rest of this paper, except where Conditions 1 and 2 are explicitly imposed.

It follows from Assumption 3 that $\bar{V}(x) = -\infty$ if and only if $\pi(u(x'R) = -\infty) > 0$. A portfolio has utility negative infinity only if it has positive probability of zero return and the utility of zero return is negative infinity.

It follows from the concavity of u that $\bar{V}$ is concave. It also has the property that if $\bar{V}(y) > -\infty$ and $\bar{V}(x) = -\infty$, then $\bar{V}(tx + (1-t)y) > -\infty$ for all t with $0 < t < 1$. This can be seen as follows. Since $\bar{V}(x) = -\infty$, we are in Model 1, $u(0) = -\infty$, and $x'R$ is non-negative with probability one. Since $\bar{V}(y) > -\infty$, $y'R$ is positive with probability one. Hence, $\pi[(tx + (1-t)y)'R = 0] = 0$, and so $\bar{V}(tx + (1-t)y) > -\infty = \bar{V}(x)$.

Since $\bar{V}$ is concave, it is continuous in Model 2, where it takes finite values only and where its domain of definition is all of $\Re^n$. It turns out that $\bar{V}$ is in fact continuous on all of $\bar{X}$ in both models, so that V is in any case continuous on X.

Proposition 2 *$\bar{V}$ is continuous.*

PROOF: There is nothing to prove in Model 2. In Model 1, let $(x(k))$ be a sequence in $[0, +\infty)^n$ converging to some portfolio x in $[0, +\infty)^n$.

Consider first the case where $\pi(u(x'R) = -\infty) = 0$. For large k,

$$\frac{1}{2}x_j < x(k)_j$$

for those assets j such that $x_j > 0$, and

$$x(k)_j < x_j + 1$$

for all assets j, so that

$$(x'R)/2 \leq x(k)'R \leq (x+\iota)'R$$

and

$$u((x'R)/2) \leq u(x(k)'R) \leq u((x+\iota)'R)$$

almost surely (where ι is the portfolio consisting of one share of each asset). By Assumption 3, the functions $u((x'R)/2)$ and $u((x+\iota)'R)$ are integrable. Since $u(x(k)'R)$ converges to $u(x'R)$ almost surely, $\bar{V}(x(k)) = Eu(x(k)'R)$ converges to $\bar{V}(x) = Eu(x'R)$ by the Dominated Convergence Theorem.

Next, consider the case where $\pi(u(x'R) = -\infty) > 0$. For large k, $x(k)_j < x_j + 1$, so that

$$0 \leq u(x(k)'R)^+ \leq u((x+\iota)'R)^+$$

almost surely. Since $u(x(k)'R)$ converges to $u(x'R)$ almost surely and $u((x+\iota)'R)^+$ is integrable because u is concave, $Eu(x(k)'R)^+$ converges to $Eu(x'R)^+$, which is finite. Since $u(0) = -\infty$ and u is continuous, there is a compact subset C of $[0, +\infty)^n$ such that $\pi(C) > 0$ and $u(x'R) = -\infty$ for all R in C. Given $t > 0$ there is $s > 0$ such that $u(s)^- > t$. For large k, $x(k)'R < s$ for all R in C, so that

$$Eu(x(k)'R)^- \geq \pi(C)u(s)^- \geq \pi(C)t$$

and, hence, $Eu(x(k)'R)^-$ converges to $+\infty$ and $\bar{V}(x(k)) = Eu(x(k)'R)$ converges to $-\infty = Eu(x'R) = V(x)$. □

6 Directional Derivatives

In this section, we compute the directional derivatives of $\bar{V}$.

Recall that the derivatives u'^- and u'^+ from the left and the right exist because the utility function u is concave. In Model 1, if $u(0) = -\infty$, set $u'^+(0) = +\infty$.

Proposition 3 *Directional derivatives of $\bar{V}$: Let x and e be portfolios such that x and $x+e$ belong to $\bar{X}$ and such that $\bar{V}(x+e) > -\infty$. Then $\bar{V}$ has the one-sided directional derivative*

$$\bar{V}'(x;e) = E[1_{\{e'R>0\}}(e'R)u'^+(x'R)] + E[1_{\{e'R<0\}}(e'R)u'^-(x'R)]$$

at x in the direction e. The second term is finite, and the first term is finite or $+\infty$. It is finite if $x-e$ belongs to $\bar{X}$. It is infinite if $\bar{V}(x) = -\infty$.

PROOF: Observe that

$$-1_{\{e'R<0\}}\frac{u((x+te)'R) - u(x'R)}{t} \rightarrow -1_{\{e'R<0\}}(e'R)u'^-(x'R)$$

almost surely as $t \rightarrow 0$, $t > 0$. The convergence is dominated:

$$\begin{aligned} 0 &\leq -1_{\{e'R<0\}}\frac{u((x+te)'R) - u(x'R)}{t} \\ &\leq -1_{\{e'R<0\}}[u((x+e)'R) - u(x'R)] \\ &\leq u(x'R)^+ - u((x+e)'R). \end{aligned}$$

The latter term has finite expectation because u is concave and because $\bar{V}(x+e) > -\infty$. By the Dominated Convergence Theorem,

$$E\left[1_{\{e'R<0\}}\frac{u((x+te)'R) - u(x'R)}{t}\right] \rightarrow E[1_{\{e'R<0\}}(e'R)u'^-(x'R)],$$

as $t \rightarrow 0$, $t > 0$, and the limit is finite. Observe furthermore that

$$0 \leq 1_{\{e'R>0\}}\frac{u((x+te)'R) - u(x'R)}{t} \rightarrow 1_{\{e'R>0\}}(e'R)u'^+(x'R)$$

almost surely and monotonically as $t \to 0$, $t > 0$. Consequently,

$$E[1_{\{e'R>0\}} \frac{u((x+te)'R) - u(x'R)}{t}] \to E[1_{\{e'R>0\}}(e'R)u'^{+}(x'R)]$$

as $t \to 0$, $t > 0$. It follows that

$$\frac{\bar{V}(x+te) - \bar{V}(x)}{t} \to E[1_{\{e'R>0\}}(e'R)u'^{+}(x'R)] + E[1_{\{e'R<0\}}(e'R)u'^{-}(x'R)]$$

as $t \to 0$, $t > 0$. If $x - e$ belongs to $\bar{X}$, then the directional derivative is finite because $\bar{V}$ is concave and finite on the line segment $[x - e/2, x + e/2]$ (if $e = 0$, then the directional derivative is zero). Finally, if $\bar{V}(x) = -\infty$, then

$$\frac{\bar{V}(x+te) - \bar{V}(x)}{t} = +\infty$$

for all $t > 0$, so that $\bar{V}'(x; e) = +\infty$. □

The proof of Proposition 3 relies on dominated convergence to derive the limit involving u'^{-} but relies on monotone convergence to derive the limit involving u'^{+}. Monotone convergence could also be invoked in the former case, but the domination is needed in order to establish that the limit is finite.

In Model 1, we will say that u is differentiable at zero even though it is defined only on $[0, +\infty)$ and even if $u'^{+}(0) = +\infty$, and we will write $u'(0) = u'^{+}(0)$.

If x and e are portfolios such that x and $x+e$ belong to $\bar{X}$ and $\bar{V}(x+e) > -\infty$, and if u is π-almost surely differentiable at $x'R$, then the directional derivative takes the more familiar form

$$\bar{V}'(x; e) = E[(e'R)u'(x'R)].$$

If, in addition, $x - e$ belongs to $\bar{X}$ and $\bar{V}(x) > -\infty$, then the directional derivative is finite, and it is two-sided:

$$\bar{V}'(x; e) = -\bar{V}'(x; -e) = E[(e'R)u'(x'R)].$$

7 Continuous Differentiability

This section explores the continuity properties of the directional derivatives of $\bar{V}$.

The following Proposition relies only on the fact that $\bar{V}$ is concave, and continuous and has the property that $\bar{V}(tx + (1-t)y) > -\infty$ when $\bar{V}(y) > -\infty$ and $0 < t < 1$.

Proposition 4 *Model 1. Continuous directional derivatives when $\bar{V}'(x;e) = +\infty$: The (one-sided) directional derivative $\bar{V}'(x;e)$ is continuous on (relatively to) the set*

$$B = \{(x,e) : x \in \bar{X},\ x + e \in \bar{X},\ \bar{V}(x+e) > -\infty\}$$

at points (x,e) where $\bar{V}'(x;e) = +\infty$.

PROOF: Suppose $(x(k), e(k))$ is a sequence in B converging to $(x,e) \in B$. Given any positive integer N, there exists $s > 0$ with $s \leq 1$ and such that

$$\frac{\bar{V}(x+se) - \bar{V}(x)}{s} \geq N + 1.$$

For each k and for all t with $0 < t \leq s$, $x + te$ and $x(k) + te(k)$ belong to X, $\bar{V}(x+te) > -\infty$, $\bar{V}(x(k) + te(k)) > -\infty$, and

$$\frac{\bar{V}(x(k)+te(k)) - \bar{V}(x(k))}{t} \geq \frac{\bar{V}(x(k)+se(k)) - \bar{V}(x(k))}{s}.$$

Consequently,

$$\bar{V}'(x(k);e(k)) \geq \frac{\bar{V}(x(k)+se(k)) - \bar{V}(x(k))}{s}.$$

Since $\bar{V}$ is continuous,

$$\frac{\bar{V}(x(k)+se(k)) - \bar{V}(x(k))}{s} \geq \frac{\bar{V}(x+se) - \bar{V}(x(k))}{s} - 1$$

for sufficiently large k. For such k,

$$\begin{aligned}
\bar{V}'(x(k);e(k)) &\geq \frac{\bar{V}(x(k)+se(k)) - \bar{V}(x(k))}{s} \\
&\geq \frac{\bar{V}(x+se) - \bar{V}(x(k))}{s} - 1 \\
&\geq N.
\end{aligned}$$

So,

$$\bar{V}'(x(k); e(k)) \to +\infty = \bar{V}'(x; e)$$

□

Let e^j denote the portfolio that consists of one share of asset j and nothing else. Then $\bar{V}'(x; e^j)$ is the (possibly only one-sided) partial derivative in the direction of asset j.

Proposition 5 *Continuous differentiability. Assume that for all x in an open set G contained in $\bar{X}$, u is π-almost surely differentiable at $x'R$. In Model 1, assume that either $u(0) > -\infty$ or $R \neq 0$ π-almost surely. Then $\bar{V}$ is continuously differentiable on G with $\bar{V}'(x) = E[Ru'(x'R)]$.*

PROOF: First,

$$\bar{V}'(x; e^j) = E[R_j u'(x'R)] < +\infty$$

for $x \in G$. This follows from Proposition 3 by setting $e = \epsilon e^j$, where $\epsilon > 0$ is chosen so small that $x_j - \epsilon > 0$. It follows from the remark made after the proposition that the derivative is a two-sided partial derivative. It remains to be shown that it is a continuous function of $x \in G$.

Consider an asset j and a portfolio x in G. Pick a finite set C of portfolios whose convex hull $\mathrm{conv}(C)$ contains x and is contained in G. Let B be the set of R such that u is differentiable at $x'R$, and at $c'R$ for all c in C. Then $\pi(B) = 1$, and $R_j u'^-(y'R)$, as a function of $y \in \mathrm{conv}(C)$, is continuous at $y = x$ when $R \in B$. Furthermore, if $y \in \mathrm{conv}(C)$, then

$$\min\{c'R : c \in C\} = \min\{z'R : z \in \mathrm{conv}(C)\} \leq y'R$$

so that

$$\begin{aligned} 0 &\leq |R_j u'^-(y'R)| \\ &\leq |R_j u'(\min\{c'R : c \in C\})| \\ &\leq \max\{|R_j u'(c'R)| : c \in C\} \end{aligned}$$

for all R in B. Since C is contained in G, the bound on the right hand side of the last inequality above is integrable. It follows from the standard result about continuity of an integral that $E[R_j u'^-(y'R)] = E[R_j u'(y'R)]$ is continuous at $y = x$. □

Proposition 5 relies on the assumption that u is almost surely differentiable at $x'R$ for all x in G. That assumption will of course be satisfied if u is differentiable. However, it will often be satisfied even if u is not differentiable.

Example 4 If π is absolutely continuous with respect to Lebesgue measure, then the mere concavity of u implies that u is π-almost surely differentiable at $x'R$ for all $x \neq 0$ in $\bar{X}$. This is so because the set of points in the domain of u where u is not differentiable is at most countable. For any portfolio $x \neq 0$ in $\bar{X}$, the set of returns vectors R such that u is not differentiable at $x'R$ is a union of at most a countable number of hyperplanes, and hence it has probability zero. □

Example 5 Suppose the first asset is riskless. Let R_f be its return per share (subscript "f" for risk-*free*), that is, $R_1 = R_f$ with probability one. Write the vector of total returns per share as (R_f, R), where R is now the vector of total returns per share to the last $n-1$ assets. Write portfolios in the form (x, y), where x is the number of shares of the first (riskless) asset in the portfolio, and where the entries in the vector y are the numbers of shares of the last $n-1$ assets. The total return to a portfolio (x, y) is then $xR_f + y'R$. In this case, the distribution π cannot be absolutely continuous with respect to Lebesgue measure. However, if the marginal distribution of returns to assets $2, \ldots, n$ is absolutely continuous with respect to Lebesgue measure in $\Re^{n-1}$, then u is almost surely differentiable at $xR_f + y'R$ unless the portfolio consists only of riskless shares, *i.e.*, unless $y = 0$. □

Example 6 If there exists a return vector $\hat{R} \neq 0$ such that $\pi(\hat{R}) > 0$, then u has to be differentiable everywhere in order to be differentiable at $x'R$ almost surely for all x in the interior of $\bar{X}$. □

8 Second Order Derivatives

Proposition 6 *Second order continuous differentiability. Assume that u is twice continuously differentiable with u'' non-decreasing. If $E|R_iR_ju''(x'R)| < +\infty$ for all i, j and all x in some open set G contained in $\bar{X}$, then $\bar{V}$ is twice continuously differentiable on G with*

$$\bar{V}''_{ij}(x) = E[R_iR_ju''(x'R)]$$

for each pair i, j of assets.

PROOF: By Proposition 5, $\bar{V}$ is continuously differentiable with $\bar{V}'(x) = E[Ru'(x'R)]$ (which is finite) for x in the interior of $\bar{X}$. Consider assets i and j and a portfolio $x \in G$. Pick a finite set C of portfolios whose convex hull conv(C) contains x and is contained in G. Then $R_iR_ju''(y'R)$, as a function of $y \in \text{conv}(C)$, is continuous at $y = x$. Furthermore, for $y \in \text{conv}(C)$,

$$\min\{c'R : c \in C\} = \min\{z'R : z \in \text{conv}(C)\} \leq y'R$$

so that

$$\begin{aligned} 0 &\leq |R_iR_ju''(y'R)| \\ &\leq |R_iR_ju''(\min\{c'R : c \in C\})| \\ &\leq \max\{|R_iR_ju''(c'R)| : c \in C\} \end{aligned}$$

for all R in B. Since C is contained in G, the bound on the right hand side of the last inequality above is integrable.

It follows from the standard result about continuity of an integral that $E[R_iR_ju''(y'R)]$, as a function of y, is continuous at $y = x$.

It follows from the standard result about differentiation of an integral that $E[R_iR_ju''(x'R)]$ is the derivative at $y = x$ of the function $E[R_ju'(y'R)]$. □

If u is twice differentiable, let u_A denote the absolute risk aversion function. Note that if u exhibits decreasing absolute risk aversion, then u'' is non-decreasing.

Lemma 2 *Make the following assumptions:*

1. *u is twice continuously differentiable (except at zero in Model 1).*
2. *u'' is non-decreasing.*
3. *Conditions 1 and 2 hold.*
4. *R_j has finite variance for each j in Model 1.*

Then $E|R_iR_ju''(x'R)| < +\infty$ for all i, j and all x in the interior of $\bar{X}$.

PROOF: If $t \leq w_0$ then

$$u'(t) = u'(w_0) \exp \left[- \int_{w_0}^{t} u_A(w) dw \right]$$

so that

$$u''(t) = -u_A(t) u'(w_0) \exp \left[\int_{t}^{w_0} u_A(w) dw \right]$$

In Model 2, Condition 2 implies that $u_A(t) \leq \lambda$ for $t \leq w_0$. Hence,

$$\begin{aligned} u''(t) &\geq -\lambda u'(w_0) \exp(\lambda(w_0 - t)) \\ &= -a \exp(-\lambda t) \end{aligned}$$

where

$$a = \lambda u'(w_0) \exp(\lambda w_0) > 0.$$

When $x'R \leq w_0$, $|u''(x'R)| \leq a \exp(-\lambda x'R)$. When $x'R \geq w_0$, $|u''(x'R)| \leq |u''(w_0)|$. Consequently, for all R,

$$\begin{aligned} &|R_i R_j u''(x'R)| \leq \\ &\quad (\exp(R_i) + \exp(-R_i))(\exp(R_j) + \exp(-R_j)) \left[a \exp(-\lambda x'R) + |u''(w_0)| \right] \end{aligned}$$

Given Condition 1, this implies that $E|R_i R_j u''(x'R)| < +\infty$.

In Model 1, Condition 2 implies that $u_A(t) \leq \gamma / t$ for $t \leq w_0$. Hence,

$$\begin{aligned} u''(t) &\geq -(\gamma/t) u'(w_0) \exp(\gamma(\log w_0 - \log t)) \\ &= -u'(w_0) w_0^{\gamma} \gamma t^{-\gamma-1} \end{aligned}$$

When $x'R \leq w_0$, $|u''(x'R)| \leq u'(w_0) w_0^{\gamma} \gamma (x'R)^{-\gamma-1}$. When $x'R \geq w_0$, $|u''(x'R)| \leq |u''(w_0)|$. Consequently, for all R,

$$\begin{aligned} |R_i R_j u''(x'R)| &\leq x_i^{-1}(x'R) x_j^{-1}(x'R) \left[u'(w_0) w_0^{\gamma} \gamma (x'R)^{-\gamma-1} + |u''(w_0)| \right] \\ &= (x_i x_j)^{-1} u'(w_0) w_0^{\gamma} \gamma (x'R)^{1-\gamma} + (x_i x_j)^{-1} (x'R)^2 |u''(w_0)| \end{aligned}$$

Given Condition 1 and the assumption of finite variances, this implies that $E|R_i R_j u''(x'R)| < +\infty$. □

Note that in Model 2, the assumption of finite variances follows from Condition 1.

Proposition 7 *Second order continuous differentiability. Under the assumptions of Lemma 2, $\bar{V}$ is twice continuously differentiable on the interior of $\bar{X}$ with*

$$\bar{V}''_{ij}(x) = E[R_i R_j u''(x'R)]$$

for each pair i, j of assets.

PROOF: Follows from Lemma 2 and Proposition 6. □

9 Strict Quasi-Concavity and Concavity

The derived utility function $\bar{V}$ may be strictly concave or strictly quasi-concave even if u is not strictly concave. That depends not only on u but also on the distribution π. This section identifies the exact conditions for $\bar{V}$ to be strictly concave or strictly quasi-concave. We also find a set of sufficient conditions for $\bar{V}$ to be differentiably strictly concave.

Since $\bar{V}$ is concave, it is strictly concave if and only if it is not affine on any line segment, and it is strictly quasi-concave if and only if it is not constant on any line segment.

Let J be the set of maximal non-degenerate intervals in the domain of u where u is affine. The set J may be empty. Its elements are closed intervals, and there are at most countably many of them. The intersection of any two elements of J either is empty or consists of a common endpoint.

Proposition 8 *$\bar{V}$ is strictly concave if and only if there do not exist portfolios x and $e \neq 0$ such that*

1. *x and $x + e$ belong to $\bar{X}$.*
2. *With probability one, either $e'R = 0$ or $x'R$ and $x'R + e'R$ both belong to the same element of J.*

PROOF: Suppose portfolios x and e as described do exist. Then

$$\pi(e'R > 0 \text{ and } u'^{+}(x'R) \neq u'^{+}(x'R + e'R/2)) = 0$$

and

$$\pi(e'R < 0 \text{ and } u'^{-}(x'R) \neq u'^{-}(x'R + e'R/2)) = 0,$$

so that $\bar{V}'(x+e/2;e) = \bar{V}'(x;e)$, implying that $\bar{V}$ is not strictly concave. Conversely, if $\bar{V}$ is not strictly concave, then there exist portfolios x and $e \neq 0$ such that x and $x+e$ belong to $\bar{X}$ and $\bar{V}'(x+e;e) = \bar{V}'(x;e)$. From the formula for the directional derivative, it follows that

$$\begin{aligned} & E[1_{\{e'R>0\}}(e'R)u'^{+}(x'R)] + E[1_{\{e'R<0\}}(e'R)u'^{-}(x'R)] \\ = \quad & E[1_{\{e'R>0\}}(e'R)u'^{+}(x+e)'R)] + E[1_{\{e'R<0\}}(e'R)u'^{-}(x+e)'R)]. \end{aligned}$$

Since

$$1_{\{e'R>0\}}(e'R)u'^{+}(x'R) \geq 1_{\{e'R>0\}}(e'R)u'^{+}(x+e)'R)$$

and

$$1_{\{e'R<0\}}(e'R)u'^{-}(x'R) \geq 1_{\{e'R<0\}}(e'R)u'^{-}(x+e)'R)$$

almost surely, it follows that

$$1_{\{e'R>0\}}u'^{+}(x'R) = 1_{\{e'R>0\}}u'^{+}(x+e)'R)$$

and

$$1_{\{e'R<0\}}u'^{-}(x'R) = 1_{\{e'R<0\}}u'^{-}(x+e)'R)$$

almost surely. This implies that almost surely, either $e'R = 0$ or else $x'R$ and $x'R + e'R$ belong to the same element of J. $\square$

Example 7 Model 2, normal distributions. Suppose R is normally distributed with mean $\bar{R}$ and a positive semidefinite covariance matrix Ω. Then $\bar{V}$ fails to be strictly concave if and only if u is affine on both $[0,+\infty)$ and $(-\infty,0]$. If so, then the set J consists either of the real line (if u is risk neutral) or of the two intervals $[0,+\infty)$ and $(-\infty,0]$. In the latter case, if x and $e \neq 0$ are two portfolios, then $x'R$ and $x'R + e'R$ both belong to the same element of J if and only if either $x = 0$ or there is $t > 0$ such that $e = tx$. On the other hand, if u is given by, for example,

$$u(w) = \begin{cases} t-1 & \text{if } t \geq 1 \\ 2(t-1) & \text{if } t \leq 1 \end{cases}$$

then $\bar{V}$ is strictly concave even though u is not. $\square$

Example 8 Model 1. Complete state-contingent markets with n states. For each $j = 1, \ldots, n$, let e^j be the j'th unit vector, so that $e^j_j = 1$ while $e^j_k = 0$ for $k \neq j$. Suppose the probability distribution π is concentrated on the points e^j, *i.e.*, $\pi(e^j) > 0$ for all j and $\sum_j \pi(e^j) = 1$. This corresponds to a state-preference model with n states $j = 1, \ldots, n$. Asset j has return one in state j and return zero in all other states. The markets are complete in the sense that every (non-negative) vector of state-contingent returns can be achieved through some portfolio. The derived utility function $\bar{V}$ is strictly concave only if the von Neumann-Morgenstern utility function u is strictly concave. To see this, suppose that u is not strictly concave. Then J contains some non-degenerate interval I (where u is affine). Choose t and $\epsilon > 0$ such that t and $t + \epsilon$ belong to I. Let $x = te^1$ and $e = \epsilon e^1$. In state 1, $x'R = t$ and $x'R + e'R = t + \epsilon$, and in all other states, $x'R = x'R + e'R = 0$. Hence, with probability one, $x'R$ and $x'R + e'R$ both belong to the same element of J, and $\bar{V}$ fails to be strictly concave. □

For I in J, let s_I denote the slope of the graph of u on I.

Proposition 9 *$\bar{V}$ is strictly quasi-concave if and only if there do not exist portfolios x and $e \neq 0$ such that*

1. *x and $x + e$ belong to $\bar{X}$.*
2. *With probability one, either $e'R = 0$ or $x'R$ and $x'R + e'R$ both belong to the same element of J.*
3. $\sum_{I \in J} s_I \int_{\{x'R \in I,\, x'R + e'R \in I\}} e'R\pi(dR) = 0.$

PROOF: If x and e are portfolios satisfying 1 and 2, then

$$\begin{aligned} \bar{V}'(x;e) &= E[1_{\{e'R>0\}}(e'R)u'^{+}(x'R)] + E[1_{\{e'R<0\}}(e'R)u'^{-}(x'R)] \\ &= \sum_{I \in J} s_I \int_{\{x'R \in I,\, x'R+e'R \in I,\, e'R>0\}} e'R\, \pi(dR) + \\ &\quad \sum_{I \in J} s_I \int_{\{x'R \in I,\, x'R+e'R \in I,\, e'R<0\}} e'R\, \pi(dR) \\ &= \sum_{I \in J} s_I \int_{\{x'R \in I,\, x'R+e'R \in I\}} e'R\, \pi(dR). \end{aligned}$$

Suppose portfolios x and e satisfying 1–3 do exist. As shown in the proof of Proposition 8, $\bar{V}'(x + e/2; e) = \bar{V}'(x; e)$, so that Statement 3 implies $\bar{V}'(x+e/2; e) = \bar{V}'(x; e) = 0$. It follows that $\bar{V}$ is not strictly quasi-concave. Conversely, if $\bar{V}$ is not strictly quasi-concave, then there exist portfolios x and e such that x and $x + e$ belong to $\bar{X}$ and $\bar{V}'(x + e; e) = \bar{V}'(x; e) = 0$. Statements 1 and 2 follow as in the proof of Proposition 8, and Statement 3 follows from the computation above. □

In Model 1, $\bar{V}$ is always strictly quasi-concave, except in the degenerate case where there exists a portfolio $e \in \bar{X}$, $e \neq 0$, such that $e'R = 0$ almost surely. This follows easily from the third property of x and e in Proposition 9.

Example 9 Model 2, normal distributions. Suppose R is normally distributed with mean $\bar{R}$ and a positive semidefinite covariance matrix Ω. Then $\bar{V}$ fails to be strictly quasi-concave if and only if u is affine on both $[0, +\infty)$ and $(-\infty, 0]$ and there exists $e \neq 0$ such that

$$s^+ \int_{e'R \geq 0} e'R\pi(dR) + s^- \int_{e'R \leq 0} e'R\pi(dR) = 0$$

where s^+ and s^- are the slopes of the graph of u on $[0, +\infty)$ and $(-\infty, 0]$, respectively. Set $m(\alpha) = E[(\alpha + Y)1_{(\alpha+Y)>0}]$, where Y is some standard normal variate. Then $m(\alpha)$ is the non-negative incomplete mean of a normal distribution with mean α and unit variance. For a portfolio $e \neq 0$, let

$$\alpha(e) = \frac{e'\bar{R}}{\sqrt{e'\Omega e}}$$

denote the ratio of mean return to standard deviation of return to e. Then the equation above can be rewritten as

$$\frac{s^+}{s^-} = 1 - \frac{\alpha(e)}{m(\alpha(e))}$$

Is there a portfolio e that satisfies this equation? The maximal ratio of mean to standard deviation available from any portfolio is $(\bar{R}'\Omega\bar{R})^{1/2}$, which is achieved by the portfolio $\Omega^{-1}\bar{R}$. Let $\bar{\alpha}$ denote the unique solution to[1]

$$\frac{s^+}{s^-} = 1 - \frac{\alpha}{m(\alpha)}.$$

[1]The equation in statement 3 of the proposition in Nielsen (1987b) should also read like this (but contains a misprint).

Nielsen (1987b) observes that $\bar{\alpha}$ is the limiting slope of the indifference curves of the preference relation for standard deviation and mean of return, and that there exists a global satiation portfolio if and only if

$$(\bar{R}'\Omega\bar{R})^{1/2} < \bar{\alpha}.$$

We can state here that there exists a portfolio e as described above if and only if there exists a portfolio e with $\alpha(e) = \bar{\alpha}$, if and only if

$$(\bar{R}'\Omega\bar{R})^{1/2} \geq \bar{\alpha},$$

and if and only if there is no global satiation portfolio.

In conclusion, $\bar{V}$ fails to be strictly quasi-concave if and only if u is affine on both $[0, +\infty)$ and $(-\infty, 0]$ and

$$(\bar{R}'\Omega\bar{R})^{1/2} \geq \bar{\alpha},$$

where $\bar{\alpha}$ solves

$$\frac{s^+}{s^-} = 1 - \frac{\alpha}{m(\alpha)}.$$

□

The utility function $\bar{V}$ is said to be *differentiably strictly concave* at x if it is twice continuously differentiable in a neighborhood of x and it has the property that $e'D^2\bar{V}(x)e < 0$ whenever $e \neq 0$. In other words, the second derivative is negative definite at x.

Proposition 10 *Differentiable strict concavity. Under the assumptions of Proposition 6 or Lemma 2, if $u'' < 0$, then $\bar{V}$ is differentiably strictly concave on G or on the interior of $\bar{X}$, respectively.*

PROOF: It follows from Proposition 6 or Lemma 2 that $\bar{V}$ is twice continuously differentiable on G or on the interior of $\bar{X}$ with

$$\bar{V}''_{ij}(x) = E[R_i R_j u''(x'R)].$$

In particular, when $e \neq 0$,

$$e'D^2\bar{V}(x)e = E[(e'R)^2 u''(x'R)] < 0.$$

□

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