# "TWO-FUND SEPARATION, FACTOR STRUCTURE AND ROBUSTNESS"

by

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# Two-Fund Separation, Factor Structure and Robustness

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**Abstract**

This paper studies two-fund separation and factor structures in asset markets. Sharpened necessary and sufficient conditions for distributional two-fund separation are presented. When asset prices are flexible, two-fund separation is equivalent to a two-factor structure in the joint distribution of total returns. When one of the funds is a riskless asset, separation is equivalent to the market model plus the CAPM equation. With flexible prices, separation is equivalent to the market model. The market model is robust to small changes in the asset supplies only if the joint distribution of total returns is spherically generated.

# 1 Introduction

Two important implications of the classical CAPM are two-fund separation and the CAPM equation. Two-fund separation means that it is possible to package the assets into two portfolios or "mutual funds" such that all investors choose to hold a portfolio which is a combination of the funds. The CAPM equation says, in the case where there is a riskless asset, that the excess expected rate of return to each asset (excess over the riskless rate) is proportional to the excess expected rate of return on the market portfolio of risky assets.

Distributional two-fund separation derives these results without relying on the assumption of mean-variance behavior. Some joint distributions of rates of return have the separation property that all risk-averse von Neumann–Morgenstern utility maximizing potential investors want to hold a combination of two funds. Ross (1978) identified these distributions and showed that they lead to the CAPM equation when there is a riskless asset and when the market portfolio of risky assets is a combination of the funds (which it has to be in equilibrium).

This paper develops some new and sharper distributional conditions for two-fund separation, explores the close relation between separation and factor models (including the market model), and identifies the exact conditions under which the market model is robust to small changes in the asset supplies. Section 2 develops the conditions for separation in the case where there is no riskless asset, while in Section 3 there is a riskless asset which is assumed to be one of the funds. Robustness is investigated in Section 4.

Ross's condition for weak two-fund separation says that the return to each asset equals the return to a combination of the funds plus a residual which has zero expectation given the return to each efficient combination of the funds. Theorem 2 of this paper states several stronger results. The residual has zero expectation given the return to each combination of the funds, whether efficient or not. Equivalently, the residual has zero expectation given the returns to the two factors jointly. Another way of saying this is that the returns follow a two-factor model of the kind used by Connor (1984) and that idiosyncratic risk is not priced. One of the equivalent conditions is a strengthening of strong separation. The proof uses a key mathematical result about conditional expectations which is stated in Theorem 1.

Whether there is two-fund separation depends on the joint distribution of rates of return, which in turn depends on the prices of the assets. If the joint distribution of total returns is considered exogenously given and fixed but the prices and hence the distribution of rates of return are allowed to vary, then separation depends on the prices. Theorem 3 characterizes those distribution of total returns for which there exists a system of asset prices that gives rise to two-fund separation. The condition simply says that the total asset returns follow a two-factor model with the returns to the funds as factors. Furthermore, for those distributions for which a separating price system exists, the theorem identifies all such price systems.

Section 3 considers the special case where one of the funds is a riskless asset and the other is the market portfolio of risky assets. Two necessary and sufficient conditions for an asset price system to give rise to separation are given in Theorem 4. One says that the excess returns to the risky assets over the return to a riskless investment with the same value follow a factor model where the factor is the excess return to the risky market portfolio. The other condition says that the total returns to the risky assets follow what we call the market model and that the total-return version of the CAPM equation holds. The market model says that the total returns to the risky assets conform to a one-factor model where the factor is the total return to the risky market portfolio.

This sheds some new light on the CAPM equation. Ross (1978) noted that separation implies the CAPM equation. Chen and Ingersoll (1983) and Stapleton and Subrahmanyam (1983) showed that in the presence of the market model, if some potential investor would choose to hold a combination of the riskless asset and the risky market portfolio, then the CAPM equation holds. Connor (1984) and Kwon (1985) showed that in the presence of the market model, equilibrium implies the CAPM equation. Theorem 4 in this paper says that the CAPM equation and the market model are jointly equivalent to separation.

When prices are allowed to vary, the necessary and sufficient condition for separation is simply the market model. This is shown in Theorem 5, which also characterizes the price systems that yield separation when the market model holds.

The market model is a joint condition on the supplies of risky assets and the distribution of total returns per share. It holds if (and only if) the returns follows a one-factor model and the market portfolio is well-diversified in the sense of Connor (1984). As emphasized by Shanken (1982, 1985), this property of the market portfolio is essential for the testability of asset pricing models based on factor structures. Hence, the market model is the most relevant version of the one-factor model.

Given the distribution of total returns, it is desirable that the market model be robust to small changes in the asset supplies. As noted by Connor (1984) and emphasized by Wei (1988), this is a restrictive requirement. Theorem 7 shows exactly how restrictive: only if the returns are spherically generated does the market model continue to hold after a small perturbation of the asset supplies. The spherically generated distributions of asset returns are exactly the same as those with the property that the distribution of any portfolio is determined by its mean and variance. So, we can conclude that if robustness is required, then the market model implies mean-variance behavior. Consequently, those derivations of the CAPM equation which rely on the market model or imply the market model are not significantly more general than the classical mean-variance CAPM. They include two-fund separation with a riskless asset as well as the other derivations mentioned above. In particular, Connor's (1984) "insurable factor economy," in the case of one factor and finitely many assets, is equivalent to the market model. So, we can establish exactly what is required for insurability to be robust (generic) in a one-factor economy with finitely many assets: the asset returns have to be spherically generated.

All mathematical proofs are collected in Section 5.

# 2 Two-Fund Separation

There are $n$ assets $i = 1, \ldots, n$. The *returns* (total gross returns) per share are given by a random vector $R$ with finite mean. A *portfolio* is an $n$-vector $x$; its $i$'th entry records the number of shares held of asset $i$. The random return to a portfolio $x$ is $x'R$. The vector of per-share *prices* of the assets is $p \neq 0$ and the *cost* of a portfolio $x$ is $p'x$.

Investors have von Neumann–Morgenstern *utility functions* $u$ which are concave, strictly increasing functions defined on the (entire) real line. They choose their portfolios subject to a wealth constraint but face no constraints on short-selling (or borrowing, if there is a riskless asset).

Although it is often preferable to express asset pricing models in rates of return, the present analysis is carried out in terms of total return and asset prices. The total-returns formulation is necessary and essential in parts of the analysis, and for convenience and consistency, total returns are used throughout. However, any statement that involves a fixed asset price vector can equivalently be formulated in terms of rates of return. That is the situation in Theorem 2 (conditions for two-fund separation), which could just as well have been stated in terms of rates of return. Indeed, one may interpret the asset price vector $p$ as a vector of ones, the returns vector $R$ as a vector of rates of return, and the entries in a portfolio $x$ as fractions of wealth invested.

A technical advantage of using total returns and asset prices instead of rates of return is that this makes it easier to deal with hedge portfolios. A *hedge portfolio* is a portfolio $x$ with zero cost, $p'x = 0$. A *budget-feasible portfolio* is a portfolio $x$ with cost one, $p'x = 1$.

## 2.1 Ross' Conditions

Weak two-fund separation means that there exist two portfolios $\phi$, $\psi$, called *funds*, such that any potential investor can find an optimal portfolio which is a combination of the funds. This can be formally described as follows. Let

$$L = \operatorname{span}\{\phi, \psi\}$$

be the set of linear combinations of the two funds. Let $F$ be the set of linear combinations of the funds that have cost one,

$$F = \{x \in L : p'x = 1\}.$$

The portfolios in $F$ will be referred to as *frontier portfolios* and $F$ will be called the *frontier*, by analogy to the mean-variance model. Under appropriate assumptions, $F$ will indeed be the set of mean-variance frontier portfolios, but so far, the terminology should not be taken to mean anything other than what is immediately implied by the definition. *Weak separation* requires that any investor (with wealth one) is just as happy choosing a portfolio in $F$ as any other portfolio he can afford:

> For every concave, strictly increasing utility function $u$ and every budget-feasible portfolio $x$, there is a frontier portfolio $y$ such that $Eu(x'R) \leq Eu(y'R)$.

Again by analogy to the mean-variance model, call a frontier portfolio $x$ *efficient* if it is optimal for some investor. Formally, $x$ is efficient if $x$ is a frontier portfolio and there is a concave strictly increasing utility function $u$ such that $Eu(x'R) \geq Eu(y'R)$ for all portfolios $y$ with $p'y = 1$. Let $A$ denote the set of efficient portfolios. *Ross' (1978) necessary and sufficient condition for weak two-fund separation* can be stated as follows:

> For every budget-feasible portfolio $x$, there is a frontier portfolio $v$ such that
>
> $$E(\delta|\eta'R) = 0$$
>
> for all efficient portfolios $\eta$, where $\delta$ is the residual given by
>
> $$x'R = v'R + \delta.$$

The condition does not say that $E(\delta|\eta'R) = 0$ for all frontier portfolios $\eta$ but only for all efficient portfolios $\eta$. It will be strengthened below in Theorem 2, where it will be shown that weak separation does indeed imply $E(\delta|\eta'R) = 0$ for all frontier portfolios $\eta$. Also, Ross' condition does not say that $E(\delta|\phi'R, \psi'R) = 0$, although it has sometimes been represented that way. The latter is a stronger statement. It will be shown in Theorem 2 that those statements are in fact equivalent.

Weak separation is known to be equivalent to strong separation. *Strong separation* means that for every budget feasible portfolio $x$, there is a frontier portfolio $y$ such that all investors prefer $y$ to $x$:

> For every budget-feasible portfolio $x$ there is a frontier portfolio $y$ such that $y'R$ second-order stochastically dominates $x'R$, *i. e.*, $E(y'R) \geq Eu(x'R)$ for all concave, strictly increasing utility functions $u$.

Strong separation does not say that $y'R$ is less risky that $x'R$ in the Rothschild-Stiglitz (1970) sense. To be less risky means to second order dominate and have the same mean. In the definition of strong separation, it is quite possible that the mean $y'\bar{R}$ is larger than $x'\bar{R}$. However, it will be shown in Theorem 2 that $y$ can be chosen in such a way that $y'R$ is indeed less risky than $x'R$. In fact, it will be shown that $y$ can be chosen such that $E(\delta|y'R) = 0$, where $\delta$ is the residual given by $x'R = y'R + \delta$. This is even stronger than the statement that $y'R$ is less risky than $x'R$, since the latter is equivalent to the existence of two random variables $X$ and $Y$ with the same marginal distributions as $x'R$ and $y'R$—but possibly with a *different joint distribution*—such that $E(d|Y) = 0$, where $d$ is the residual given by $X = Y + d$.

These various strengthened implications of weak separation rely on a mathematical result about conditional expectations to be developed below.

## 2.2 A Theorem About Conditional Expectations

**Theorem 1** *Let $X$ be a random variable and $Y = (Y_1, \ldots, Y_k)'$ a $k$-vector of random variables such that $E[exp(sX + t'Y)] < \infty$ for all real numbers $s$ and all $t = (t_1, \ldots, t_k)'$ in $\mathbb{R}^k$. Let $N$ be a subset of $\mathbb{R}^k$ such that the closed cone generated by $N$ has non-empty interior. The following statements are equivalent:*

1. $E(X|Y_1, \ldots, Y_k) = 0$.
2. $E(X|t_1Y_1 + \ldots + t_kY_k) = 0$ *for all $t$ in $\mathbb{R}^k$.*
3. $E(X|t_1Y_1 + \ldots + t_kY_k) = 0$ *for all $t$ in $N$.*

The proof of Theorem 1, as well as the proofs of the other results, is found in Section 5.

The condition on the set $N$ in the theorem is satisfied if, for example, $N$ is (non-empty and) open or dense in some open set. So, (3) says that for "generic" $t$ (in a weak, local sense), the expectation of $X$ given $t_1Y_1 + \ldots + t_kY_k = t'Y$ is zero.

Statement (1) involves conditioning on $k$ random variables $Y_1, \ldots, Y_k$ jointly. Statements (2) and (3), by contrast, involve conditioning on each of infinitely many random variables of the form $t_1Y_1 + \ldots + t_kY_k$, one at a time. The equivalence should not be confused with the following trivial observation: If $v^1, \ldots, v^k$ is a set of $k$ linearly independent vectors in $\mathbb{R}^k$, then $E(X|Y_1, \ldots, Y_k) = 0$ if and only if $E(X|v^1Y, \ldots, v^kY) = 0$. Obviously, if $Y_1, \ldots, Y_k$ are known then $v^1Y, \ldots, v^kY$ can be computed, and vice versa, so conditioning on $Y_1, \ldots, Y_k$ is equivalent to conditioning on $v^1Y, \ldots, v^kY$. The equivalence in Theorem 1 is more subtle. In particular, it is not enough in (2) or (3) to condition on $t'Y = t_1Y_1 + \ldots + t_kY_k$ for only a finite number of $t$'s. Also, the exponential integrability assumption is necessary; without it, the theorem does not appear to be true.

It is obvious that (1) implies (2) and (2) implies (3). The hard part of the proof of Theorem 1 is to show that (3) implies (1). Here is the intuition. Consider the signed finite measure on $\mathbb{R}^k$ which has "density" $X$ with respect to the distribution of $Y$. It follows from (3) that its characteristic function is zero on the cone generated by $N$ and hence on the closed cone generated by $N$. Consequently, all derivatives of the characteristic function at zero are zero. This implies that the characteristic function is zero, since it is well determined by its derivatives at zero under the exponential integrability assumption. So, the measure is zero, and (1) follows.

## 2.3 Characterizations of Two-Fund Separation

We shall need the following conditions in order to rule out degenerate cases.

**Condition 1** *Exponential integrability:* $E(exp(x'R)) < \infty$ *for all portfolios* $x$.

Condition 1 is needed because use is made of Theorem 1 where a similar assumption appears. It follows from Condition 1 that variances and covariances exist.

Possibly, one of the funds is a hedge portfolio (has zero cost), but we exclude the case where both are hedge portfolios.

**Condition 2** *The funds $\phi$ and $\psi$ are not both hedge portfolios.*

It follows from Condition 2 that the frontier $F$ is a one-dimensional affine space (a line).

**Condition 3** *Not all frontier portfolios have the same expected return.*

Since the frontier $F$ is one-dimensional, Condition 3 implies that for every number $\mu$ there is a unique frontier portfolio $y$ with expected return $y'\bar{R} = \mu$. Ross (1978) implicitly made an assumption similar to Condition 3. Conditions 2 and 3 are jointly equivalent to

$$(\phi'\bar{R})(\psi'p) \neq (\psi'\bar{R})(\phi'p).$$

If neither $\phi$ nor $\psi$ is a hedge portfolio, then this inequality says that they do not have the same expected rate of return.

**Condition 4** *No free lunch: If $e$ is a portfolio in $L$ with $e'R \geq 0$ a. s. and $p'e = 0$ then $e = 0$.*

Condition 4 says that there is no free lunch in $L = \text{span}\{\phi, \psi\}$: If it is violated, then $e'R \geq 0$ a. s., $p'e = 0$, and $e'\bar{R} > 0$ because of Condition 3. Investors will want unlimited amounts of the lunch $e$, and they can find no optimal portfolio. If Condition 4 is violated, then clearly there is weak separation, no matter what other properties the return distributions, funds and prices have. Ross made an assumption resembling Condition 4, that the joint distribution of rates of return be such that there exists a strictly increasing and strictly concave utility function for which the expected utility of portfolios exists and attains a maximum at some portfolio.

Here are the sharpened necessary and sufficient conditions for two-fund separation.

**Theorem 2** *Assume Conditions 1–4. The following statements are equivalent:*

1. *$(R, \phi, \psi, p)$ is (weakly) separating.*

2. *For all hedge portfolios $z$ with zero expected return,*

$$E(z'R|\phi'R, \psi'R) = 0.$$

3. *There exist vectors $a$ and $b$ of factor loadings such that the asset returns follow the factor model*

$$R_i = a_i\phi'R + b_i\psi'R + \epsilon_i$$

*with*

$$E(\epsilon_i|\phi'R, \psi'R) = 0,$$

*and such that for every portfolio $x$,*

$$p'x = (x'a)(p'\phi) + (x'b)(p'\psi).$$

4. *For every budget-feasible portfolio $x$, there exists a frontier portfolio $v$ such that*

$$E(\delta|\eta'R) = 0$$

*for all frontier portfolios $\eta$, where $\delta$ is the residual given by*

$$x'R = v'R + \delta.$$

5. *For every budget-feasible portfolio $x$, there is a frontier portfolio $v$ such that*

$$E(\delta|v'R) = 0,$$

*where $\delta$ is the residual given by*

$$x'R = v'R + \delta.$$

Note that (3) is a two-factor model with factors $\phi'R$ and $\psi'R$ and factor loads $a_i$ and $b_i$. In the various versions of arbitrage pricing theory, different assumptions are made about the residuals $\epsilon_i$ in the underlying factor model. For example, the $\epsilon_i$'s may be uncorrelated across assets or uncorrelated with the factors. In "exact" arbitrage pricing theory, as developed by Connor (1984), the assumption is that the residuals have zero conditional expectation given the factors jointly. That is exactly what is required in

(3), so (3) is the kind of factor model used by Connor. The factor models used in arbitrage pricing theory usually include a constant term and normalize the factors to have zero mean. An equivalent formulation of (3) in terms of a normalized factor model would say that $\epsilon_i$ is given by

$$R_i = \bar{R}_i + a_i\phi'(R - \bar{R}) + b_i\psi'(R - \bar{R}) + \epsilon_i$$

and would impose the additional constraint that

$$\bar{R}_i = a_i(\phi'\bar{R}) + b_i(\psi'\bar{R}).$$

The vectors of factor loadings $a$ and $b$ and prices $p$ are constrained in (3) by the requirement of *exact arbitrage pricing*: the cost of a portfolio $x$ is a linear function of its factor loads $x'a$ and $x'b$, with the coefficients being the costs $p'\phi$ and $p'\psi$ of the factor portfolios. This implies that factor risk is priced while idiosyncratic risk is not priced.

In (4) and in (5), $v$ has to be the unique frontier portfolio with $v'\bar{R} = x'\bar{R}$.

Statement (4) sharpens Ross' condition for weak separation, since it says that $E(\delta|\eta'R) = 0$ for all frontier portfolios $\eta$ and not just for efficient $\eta$'s, c.f. the discussion above.

It follows from Theorem 1 that in both (3) and (4), conditioning on the funds jointly is equivalent to conditioning on each frontier portfolio separately. So, in (3) we might just as well say $E(\epsilon_i|\eta'R) = 0$ for all frontier portfolios $\eta$, and in (5) we might say $E(\delta|\phi'R, \psi'R) = 0$.

(5) implies that for any portfolio there is a combination of the funds which has the same value and the same mean return but is less risky in the Rothschild-Stiglitz (1970) sense (*i.e.*, second-order dominates and has the same mean). This is even stronger than strong two-fund separation, which says that every budget-feasible portfolio is second-order dominated by some budget-feasible combination of the funds. See the discussion of strong separation above.

If the statements in the theorem hold, then the frontier $F$ coincides with the mean-variance frontier: Suppose $y$ is in $F$ and $x$ is another budget-feasible portfolio with $x'\bar{R} = y'\bar{R}$. Then it follows from (4) that $E(x'R|y'R) = y'R$, so that $\mathrm{var}(x'R) \geq \mathrm{var}(y'R)$. So, $y$ minimizes variance among portfolios with cost one and with the same mean as $y$.

Litzenberger and Ramaswamy (1979) provide a simple proof that two-fund separation implies a condition similar to (5). They use a very strong concept of separation: all investors with concave utility functions, not only those with increasing and concave utility functions, are assumed to prefer combinations of the funds to other budget-feasible portfolios. This seems to be the key to the simplification. Also note that (5) is a somewhat weaker implication of separation than (2), (3), or (4).

The hard part of the proof of Theorem 2 is to show that weak separation implies (2). It is shown like this. Define a subset $A_H$ of $F$ consisting of portfolios that are efficient with respect to a set of particularly well-behaved utility functions. By arguments similar to those of Ross (1978), show that $E(x'R|\eta'R) = 0$ for all $\eta$ in $A_H$, and that $A_H$ is convex and contains at least two distinct portfolios. Since the cone in $L = \text{span}\{\phi, \psi\}$ spanned by $A_H$ has non-empty interior relative to $L$, it follows from Theorem 1 that $E(x'R|\phi'R, \psi'R) = 0$.

Theorem 2 and its proof do not seem to carry over to the case of more than two funds. When there are many funds, it can still be shown that portfolios with zero cost and zero expected return have zero conditional expectation given the return to any efficient portfolio, c.f. Ross (1978). However, not enough is known about the structure of the set of efficient portfolios to apply Theorem 1. Possibly, Theorem 2 can be generalized to the case of three funds if one of the funds consists of a riskless asset. This question will be deferred to future work.

## 2.4 Prices Versus Distributions

Theorem 2 takes the asset prices as given as well as the distributions of total returns. However, in a well specified equilibrium model, prices will be endogenously determined by supply and demand on the basis of an exogenously given distribution of total returns. Whether there is or is not a weakly separating price system depends on the joint distribution of total returns. This leads to the question, What distributions of total returns allow two-fund separation for some appropriately chosen price system? The next theorem characterizes those distributions.

**Condition 5** *The returns to the funds are linearly independent: if* $\alpha\phi' R + \beta\psi' R = 0$ *a. s. then* $\alpha = \beta = 0$.

**Condition 6** *The funds* $\phi$ *and* $\psi$ *do not both have zero expected return.*

**Theorem 3** *Assume Conditions 1, 5 and 6. The following two statements are equivalent.*

1. *There is some price system* $p \neq 0$ *such that* $(R, \phi, \psi, p)$ *is weakly separating (and such that Conditions 2, 3 and 4 hold).*
2. *There exist vectors* $a$ *and* $b$ *of factor loadings such that the asset returns follow the factor model*

$$R_i = a_i \phi' R + b_i \psi' R + \epsilon_i$$

*with*

$$E(\epsilon_i | \phi' R, \psi' R) = 0.$$

*If these statements hold, then the price systems* $p$ *that satisfy (1) are those that admit no free lunch (Condition 4), are not colinear with* $\bar{R}$, *and are such that for all portfolios* $x$,

$$p' x = (x' a)(p' \phi) + (x' b)(p' \psi).$$

(2) says that the total returns follow a factor model (of the kind used by Connor (1984)) with the funds acting as factors. Of course, (2) is identical to (3) of Theorem 2 except that the restriction that idiosyncratic risk must not be priced is not present in (2).

As noted before, except for technicalities involving hedge portfolios and assets with price zero, the conditions for separation in Theorem 2 are essentially conditions on the rates of return. Even so, it is necessary to use the total-returns formulation in Theorem 3. The joint distribution of rates of return depends on the joint distribution of total returns and the vector of prices. If we keep the joint distribution of total returns fixed (we think of it as exogenously given), then the joint distribution of rates of return varies with the price vector. It does not range over all possible joint distributions of rates of return, only over a certain family of joint distributions parametrized by the price vector. Whether the joint distribution of rates of return satisfies the conditions for separation depends on the price vector. It may be that there is no price vector such that the conditions are satisfied. Alternatively, it may be that there are some such price vectors, in which case the conditions will be satisfied for some prices but not for others. Whether there is or is not a price vector such that the conditions for separation are satisfied depends on the joint distribution of total returns. Theorem 3 characterizes those joint distributions of total returns for which there exists at least one possible price vector such that the induced joint distribution of rates of return is weakly separating. The characterization is given in (2): what is required is that the total returns conform to a two-factor model with the funds mimicking the factors. This characterization singles out a class of joint distributions of total returns, and it cannot be rephrased as a statement about rates of return. The question raised and answered in Theorem 3 is one which cannot reasonably be asked in the rates-of-return formulation.

Theorem 3 does something more than characterize those joint distributions of total returns that give rise to separation when combined with appropriate prices. Given one of those joint distributions of total returns, the theorem also characterizes those price vectors that are consistent with separation.

Theorem 3 is potentially useful in equilibrium analysis. Equilibrium analysis distinguishes between exogenous and endogenous variables and derives the equilibrium properties of endogenous variables from the equilibrium-independent properties of the exogenous variables. In the present model, the joint distribution of total returns will be exogenous together with investors' utility functions and initial endowments of assets. Endogenous will be the equilibrium prices and by implication the equilibrium joint distribution of rates of return. For some configurations of exogenous data, separation will hold in equilibrium, for others they will not. In particular, given the joint distribution of total returns, it may be that separation does not hold no matter what the utility functions and endowments are. Or it may be that separation holds in equilibrium for some endowments and utility functions but not for others. A detailed study of these issues would require a second paper, but it should be clear that Theorem 3 provides the key.

Connor (1984) undertook a part of this investigation. He showed that if there is a factor structure in total returns, and if certain other conditions hold, then each investor's equilibrium portfolio is a combination of funds (that mimick the factors and the riskless asset). That is Connor's Theorem 2 and Corollary 2.1. In short, factor structure implies separation. That is the sufficiency part. Theorem 3 of the present paper shows necessity. Theorem 3 implies that two-fund separation can hold in equilibrium only if there is a factor structure in total returns. In short, separation implies a factor structure.

# 3 Separation with a Riskless Asset

Assume that there are $n+1$ assets, the first being riskless with total return per share $R_f > 0$. The other $n$ assets are risky; their total returns per share are given by a random $n$-vector $R$. Portfolios are written as $(x, y)$, where $x$ is the number of shares of the riskless asset held, and $y$ is an $n$-vector which records the number of shares held of the risky asset. The total return to a portfolio $(x, y)$ is $xR_f + y'R$. Normalize the price per share of the riskless asset to one, and let $q$ denote the vector of prices per share of the risky assets.

## 3.1 Conditions for Separation

Let $\omega \neq 0$ be an $n$-vector of share holdings of risky assets (a "risky portfolio"). Think of $\omega$ as the market portfolio of risky assets. We want to investigate the conditions for separation between $\omega$ and the riskless asset, *i. e.*, between the funds $(0, \omega)$ and $(1, 0)$.

As before, let $L$ be the set of linear combinations of the funds,

$$L = \{(x, t\omega) : x, t \in I\!R\};$$

and let $F$ be those combinations of the funds that have cost one,

$$F = \{(x, y) \in L : x + q'y = 1\}.$$

When one of the funds is riskless, the exponential integrability condition (Condition 1) is unnecessary, but the other assumptions of Theorem 2 will be needed. Condition 2 is satisfied since the riskless asset has already been assumed to have cost one. The following condition substitutes for Conditions 3 and 4.

**Condition 7** *$\omega'\bar{R} > (q'\omega)R_f$ but not $\omega'R \geq (q'\omega)R_f$ a. s.*

Condition 7 means that $\omega'R$ does not first-order dominate $(q'\omega)R_f$ and $(q'\omega)R_f$ does not second-order dominate $\omega'R$.

The following theorem states the most informative conditions for separation when one of the funds is riskless.

**Theorem 4** *Assume Condition 7. The following are equivalent:*

1. $((R_f, R), (1,0), (0,\omega), (1,q))$ *is (weakly) separating.*
2. *There exists a vector $b$ of factor loadings such that the returns to the risky assets follow the factor model*

$$R_i = R_f q_i + b_i[\omega' R - (q'\omega)R_f] + \epsilon_i$$

*with*

$$E(\epsilon_i | \omega' R) = 0.$$

3. *There exists a vector $b$ of factor loadings such that the returns to the risky assets follow the factor model*

$$R_i = \bar{R}_i + b_i[\omega'(R - \bar{R})] + \epsilon_i$$

*with*

$$E(\epsilon_i | \omega' R) = 0,$$

*and such that*

$$\bar{R}_i - R_f q_i = b_i[\omega' \bar{R} - (q'\omega)R_f].$$

If variances exist, if $\omega'\Omega\omega \neq 0$, and if $b$ satisfies (2) or the first equation in (3), then $b = \Omega\omega/(\omega'\Omega\omega)$.

(2) says that the excess return to the risky assets (excess over the return to the same amount invested in the riskless asset) follows a one-factor model where the factor is the excess return to the risky market portfolio. The factor does not necessarily have mean zero.

(3) is the normalized version of (2). It says that the total returns to the risky assets follow a one-factor model with factor $\omega'(R - \bar{R})$ (which has mean zero) and with a vector $b$ of factor loads given by the final equation in (3), which is the total-return version of the *CAPM equation.* Note that the CAPM equation comes directly out of (2) by taking expectations on both sides.

The total-return version of the *market model* says that there exist vectors $a, b$ such that for all $i$,

$$R_i = a_i + b_i \omega' R + \epsilon_i,$$

where some restriction or the other is imposed on the residuals $\epsilon_i$. Here we take that restriction to be that

$$E(\epsilon_i|\omega' R) = 0$$

for all $i$. Then the market model is equivalent to the factor model in (3), and in fact the vector $b$ has to be the same in the two formulations (if $\omega' R$ is genuinely risky).

The equivalence of (1) and (3) in Theorem 4 means that two-fund separation with a riskless asset is equivalent to the market model plus the CAPM equation.

Linear pricing means that asset prices are linear in mean return and factor load. Since the riskless asset has zero factor load and price one, the coefficient to the mean must be $1/R_f$. So, *linear pricing* says that there exists a constant $\gamma$ such that $q = (\bar{R} - \gamma b)/R_f$. In the presence of the market model, the CAPM equation is equivalent to linear pricing (if $\omega' R$ is genuinely risky).

To avoid some potential confusion, a few comments on the literature are in order here. Chen and Ingersoll (1983) use a rate-of-return formulation and a $k$-factor model, but restated in terms of total returns in the case of one factor their result says the following: If returns conform to the market model and if $q$ is a price system such that some utility function (sufficiently well-behaved to justify differentiation under the expectations operator) has an optimum which is separated (is a combination of the riskless asset and a risky market portfolio), then the CAPM equation holds. Stapleton and Subrahmanyam (1983) show a result which can be interpreted to say the same thing. Connor's (1984) Theorem 3 implies that if returns conform to the market model and if $q$ is an equilibrium price system, then prices are linear, *i. e.*, the CAPM equation holds. That is Kwon's (1985) result: If the market model holds and $q$ is an equilibrium price system, then the CAPM equation holds. Both Connor and Kwon implicitly assume that at least one of the investors has a well-behaved utility function. All these results identify conditions which, combined with the market model, imply the CAPM equation. By contrast, the result reported in Theorem 4 is that the market model and the CAPM equation are jointly equivalent to separation.

## 3.2 Prices Versus Distributions

Theorem 4, like Theorem 2, takes the asset prices as given. Since, in an equilibrium model, prices will be determined endogenously, the question arises, For what exogenous distributions of total returns is it possible to find a separating price system (when the risky fund $\omega$ is thought of as the exogenously given total supply of risky assets)? This is addressed in the next theorem, which is analogous to Theorem 3.

**Theorem 5** *Assume that $\omega' R$ is genuinely risky (not $= \omega' \bar{R}$ a. s.). The following statements are equivalent.*

1. *There is a price system $q$ such that Condition 7 holds and there is (weak) separation between the riskless asset and $\omega$.*
2. *There exists a vector $b$ of factor loadings such that the returns to the risky assets follow the factor model*

$$R_i = \bar{R}_i + b_i[\omega'(R - \bar{R})] + \epsilon_i$$

*with*

$$E(\epsilon_i | \omega' R) = 0.$$

*If (2) holds, then the $q$'s that satisfy (1) are those of the form*

$$q = (\bar{R} - \gamma b)/R_f,$$

*where* $0 < \gamma < -\mathrm{essinf}[\omega'(R - \bar{R})]$.

If $b$ satisfies (2) and if variances exist, then $b = \Omega\omega/(\omega'\Omega\omega)$.

(2) says that the total returns to the risky assets follow a one-factor model with $\omega'(R - \bar{R})$ as factor. As noted above in connection with the first equation in (3) of Theorem 4, (2) is equivalent to the the market model (as the term is used here). So, the equivalence in Theorem 5 says that existence of a separating price system is equivalent to the market model. Unlike (3) of Theorem 4, (2) does not specify what the factor loads are, but they can be computed as $b = \Omega\omega/(\omega'\Omega\omega)$ (if variances exist), and then the separating prices can be computed from the second part of the theorem.

The second part of the theorem essentially says that the price systems that satisfy (1) are those that satisfy linear pricing and Condition 7.

# 4 Robustness

We have seen in Theorem 5 that whether or not separation is possible depends on whether the market model holds. The market model holds if and only if the returns follow a one-factor model and the market portfolio is well-diversified, which means that it has no idiosyncratic risk. Connor (1984) defines what he calls an *insurable* factor economy with $k$ factors and finitely or countably many assets. In the case of one factor and finitely many assets, the definition amounts to the requirement that the market portfolio be well-diversified. Shanken (1982, 1985) has emphasized that this property of the market portfolio is essential for the testability of pricing models based on factor structures. Thus, the market model is the most relevant version of the one-factor model.

It would be desirable that insurability and separation not depend too strongly on the exact configuration of the market portfolio—in other words, that it be *robust* to small changes (or mismeasurements or misspecifications) in the asset supplies. However, as noted by Connor (1984) and emphasized by Wei (1988), the requirement that the market portfolio be well diversified is restrictive. In the case of one factor, we can show exactly how restrictive. If the market model holds and is robust, *i. e.*, it holds not only for the given asset supplies but also for asset supplies close to it, then the distribution of total returns has to be spherically generated. The market model is robust only in that special case. This is shown in Theorem 7. For clarity, the underlying result about spherically generated random variables is stated separately in Theorem 6.

## 4.1 A Theorem About Spherical Vectors

A random $m$-vector $Y$ is *spherical* if $MY$ has the same distribution as $Y$ for all orthogonal $m$ by $m$ matrices $M$. A spherical random vector must have zero mean (if it has a mean at all) since zero is the only point that stays put under all orthogonal transformations. A random $n$-vector $X$ is *spherically generated* if there exists a spherical $m$-vector $Y$ and an $n$ by $m$ matrix $G$ such that $X$ is distributed as $GY$. If so, then it is possible to choose $m = n$.

**Theorem 6** *Let $X$ be a random $n$-vector with $EX = 0$ and such that $E[exp(t'X)] < \infty$ for all $t$ in $\mathbb{R}^n$. Suppose the components of $X$ are not all perfectly correlated. Let $N$ be a subset of $\mathbb{R}^n$ such that the closed cone generated by $N$ has non-empty interior. The following statements are equivalent:*

1. *For each $\omega$ in $N$ there exists a vector $b$ such that $E(X|\omega'X) = (\omega'X)b$ a. s.*
2. *For each $\omega$ in $\mathbb{R}^n$ there exists a vector $b$ such that $E(X|\omega'X) = (\omega'X)b$ a. s.*
3. *$X$ is spherically generated.*

Like in Theorem 1, the condition on the set $N$ in Theorem 6 is satisfied if, for example, $N$ is (non-empty and) open or dense in some open set. So, condition (1) says that for generic $\omega$ (in a weak, local sense), the conditional expectation of each $X_i$ given $\omega'X$ is proportional to $\omega'X$.

The equivalence of (2) and (3) is shown by Hardin (1982), without the exponential integrability assumption - only the existence of variances is needed, or, if the components of $X$ have linear rank at least three, the existence of means. However, what we are interested in here is the equivalence of (1) and (3), and that appears to require the stronger integrability assumption.

The tough part of the proof is to show that (1) implies (3). It can be assumed that the components of $X$ are uncorrelated with unit variance. Using (1), one can show that the gradient of the characteristic function of $X$ points radially outwards at points in $N$. The same must be true at points in some open convex cone $C$, which implies that the characteristic function is constant on the intersection of $C$ with each sphere centered at the origin. Consider the unique function $g$ which is constant on each of the spheres and equals the characteristic function on $C$. It has the same derivatives of all orders at zero as the characteristic function of $X$. If $g$ were itself a characteristic function, it would therefore have to equal the characteristic function of $X$ under the exponential integrability assumption. The characteristic function of $X$ would then be constant on spheres, so that $X$ would be spherical. The actual argument has to be twisted a little differently because I cannot show directly that $g$ is a characteristic function.

## 4.2 Robustness of the Market Model

The following theorem says that the market model is robust if and only if $R - \bar{R}$ is spherically generated.

**Theorem 7** *Assume Condition 1. Assume that the components of $R$ are not all perfectly correlated. Let $N$ be a subset of $\mathbb{R}^n$ such that the closed cone generated by $N$ has non-empty interior. The following statements are equivalent.*

1. *The market model holds for all $\omega$ in $N$.*
2. *The market model holds for all $\omega$ in $\mathbb{R}^n$.*
3. *$R - \bar{R}$ is spherically generated.*

Theorem 7 follows directly form Theorem 6.

Theorem 7 has two important consequences. One is that distributional two-fund separation with a riskless asset is not more general than the mean-variance CAPM when robustness is required. The other consequence is that in a one-factor economy, insurability is generic only in the special case where unanticipated returns are spherically generated.

Chamberlain (1983), in a rate-of-return formulation, shows that when there is a riskless asset, the distribution of returns to all portfolios is determined by its mean and variance if and only if the vector of returns to the risky assets is spherically generated. See also Owen and Rabinovitch (1983). Chamberlain's result can be restated in the present total-returns framework as follows.

**Theorem 8** *Assume that the components of $R$ have finite variances. The following statements are equivalent.*

1. *$R - \bar{R}$ is spherically generated.*
2. *If $(x, y)$ and $(z, v)$ are two portfolios, then $xR_f + y'R$ and $zR_f + v'R$ have the same distribution if and only if they have the same mean and the same variance.*

When the distribution of returns to all portfolios is determined by its mean and variance, as in (2), then for any von Neumann–Morgenstern utility function, the expected utility of the return to a portfolio is a function of the mean and standard deviation of return. When the von Neumann–Morgenstern utility function is concave and strictly increasing, this derived utility function has the properties usually assumed in mean-variance portfolio analysis: It is concave, an increasing function of the mean, and a decreasing function of the standard deviation. So, we can conclude from Theorems 7 and 8 that mean-variance analysis is appropriate and adequate if the market model holds and is robust. Consequently, distributional two-fund separation with a riskless asset, as well as any other theory that implies or assumes the market model, is not more general than the classical CAPM when robustness to small changes in the total supplies of risky assets is required.

According to Connor (1984), insurability arises naturally in an economy with infinitely many assets but is restrictive and non-generic when there are only finitely many assets. Our Theorem 7 substantiates the latter claim: In the case of one factor, insurability is generic (robust) only if the returns to the risky assets are spherically generated. I conjecture that insurability is also highly restrictive in the case of $k$ factors and infinitely many assets.

# 5 Proofs

PROOF OF THEOREM 1:

(1) implies (2): $E(X|t'Y) = E[E(X|Y)|t'Y] = E(0|t'Y) = 0$.

(2) implies (3): Obvious.

(3) implies (1): Define a finite (signed) measure $\nu$ on Borel sets $A$ in $\mathbb{R}^k$ by $\nu(A) = E[X1_A(Y)]$, and let $f$ be its Fourier transform. For $t$ in the cone generated by $N$,

$$\begin{aligned} f(t) &= \int \exp(it'r)\nu(dr) \\ &= E[X\exp(it'Y)] \\ &= E[\exp(it'Y)E(X|t'Y)] \\ &= E[\exp(it'Y)0] \\ &= 0, \end{aligned}$$

and hence, by continuity, $f(t) = 0$ for $t$ in the closed cone $C$ generated by $N$. Consequently, all derivatives of $f$ at 0 are zero. The exponential-integrability assumption implies that $f$ is an entire analytic function. It follows that $f = 0$, $\nu = 0$ and $E(X|Y) = 0$. □

A simpler proof[1] that (3) implies (1) in Theorem 1 is possible along the following lines if $Y$ follows a discrete distribution. First, (3) holds for all $t$ in the cone generated by $N$. A limit argument will show that it holds for all $t$ in its clusure, which by assumption contains a non-empty open set $U$. Because $Y$ takes only a finite or countable number of values and $U$ is open, there is some $t$ in $U$ such that $t'y \neq t'z$ whenever $y$ and $z$ are two different values of $Y$. For this $t$, knowledge of $t'Y$ implies knowledge of $Y$, and so (1) follows.

PROOF OF THEOREM 2:

(4) implies (5) implies (1): Obvious.

(1) implies (2): Let $H$ be the set of bounded, positive, continuous, decreasing functions on the real line differentiable from the right with a derivative that is bounded away from zero on bounded intervals. Let $U_H$ be the set of differentiable functions with derivative in $H$. They are concave

[1]This proof is due to Karl Vind

and strictly increasing. Let $A_H$ denote the set of frontier portfolios that are efficient with respect to $U_H$. Formally, a frontier portfolio $\eta$ is in $A_H$ if there is $u$ in $U_H$ such that $Eu(\eta' R) \geq Eu(y'R)$ for all budget-feasible portfolios $y$.

The arguments of Ross (1978) show that $E(z'R|\eta' R) = 0$ for all portfolios $\eta$ in $A_H$, and that $A_H$ is convex and contains at least two distinct portfolios. It follows that $E(z'R|\eta' R) = 0$ for all $\eta$ in the cone in $L$ generated by $A_H$; and by Theorem 1 the same holds for all $\eta$ in $L$, since the cone has non-empty interior relative to $L$. Another application of Theorem 1 shows that $E(z'R|\phi' R, \psi' R) = 0$.

(2) implies (3): The matrix

$$\begin{bmatrix} \phi & \psi \end{bmatrix}' \begin{bmatrix} \bar{R} & p \end{bmatrix}$$

is regular because of Conditions 2 and 3. Let $a$ and $b$ be given by the matrix equation

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} \bar{R} & p \end{bmatrix} \left( \begin{bmatrix} \phi & \psi \end{bmatrix}' \begin{bmatrix} \bar{R} & p \end{bmatrix} \right)^{-1}$$

Then

$$\begin{bmatrix} \phi & \psi \end{bmatrix}' \begin{bmatrix} a & b \end{bmatrix}$$

is an identity matrix, so that $\phi' a = 1$, $\phi' b = 0$, $\psi' a = 0$, and $\psi' b = 1$. Clearly $\text{span}\{a, b\} = \text{span}\{\bar{R}, p\}$, so that every portfolio orthogonal to $a$ and $b$ is also orthogonal to $p$. Let $V$ be the linear space of portfolios $z$ with $E(z'\epsilon|\phi' R, \psi' R) = 0$. Then $V$ contains the orthogonal complement of $\text{span}\{a, b\}$, and it contains $L$ because $\phi'\epsilon = \psi'\epsilon = 0$. Since $\text{span}\{a, b\}^{\perp} \cap L = \{0\}$, V contains all of $I\!R^n$. For any portfolio $x$, the portfolio

$$z = x - (x'a)\phi - (x'b)\psi$$

is orthogonal to $a$ and $b$. So, it is orthogonal to $p$, and it follows that

$$p'x = (x'a)(p'\phi) + (x'b)(p'\psi).$$

(3) implies (4): Since every portfolio with zero factor loadings is a hedge portfolio, $\text{span}\{a, b\} = \text{span}\{\bar{R}, p\}$. Choose $v$ in $F$ with $x'\bar{R} = v'\bar{R}$. Then $x - v$ is orthogonal to $p$ and $\bar{R}$, hence to $a$ and $b$, so

$$\delta = (x - v)' R = (x - v)'\epsilon,$$

and

$$
\begin{aligned}
E(\delta|\phi'R,\psi'R) &= E((x-v)'R|\phi'R,\psi'R) \\
&= E((x-v)'\epsilon|\phi'R,\psi'R) \\
&= 0.
\end{aligned}
$$

It follows that $E(\delta|\eta'R)=0$ for all $\eta$ in $L$ and in particular for all $\eta$ in $F$. □

PROOF OF THEOREM 3:

(1) implies (2): Follows directly from Theorem 2.

(1) and (2) imply that $p$ is as described: Condition 4 is assumed in (1). Conditions 2 and 3 imply that $\bar{R}$ and $p$ are not colinear. Since

$$\phi'R=(\phi'a)\phi'R+(\phi'b)\psi'R$$

and

$$\psi'R=(\psi'a)\phi'R+(\psi'b)\psi'R,$$

Condition 5 implies that $\phi'a=1$, $\phi'b=0$, $\psi'a=0$ and $\psi'b=1$. For any portfolio $x$, the portfolio

$$z=x-(x'a)\phi-(x'b)\psi$$

has zero factor loadings and hence zero cost. It follows that

$$p'x=(x'a)(p'\phi)+(x'b)(p'\psi).$$

If (2) holds and $p$ is as described, then $p$ satisfies (1): Since span$\{p,\bar{R}\}$ has dimension two and is orthogonal to all portfolios with zero factor loads, span$\{p,\bar{R}\}$ = span$\{a,b\}$. If $e$ is in $L$ and $e$ is orthogonal to $a$ and $b$, then $e'R=e'\epsilon$, and

$$e'R=E(e'R|\phi'R,\psi'R)=E(e'\epsilon|\phi'R,\psi'R)=0.$$

This implies that $e=0$. Now the portfolio

$$e=(\phi'\bar{R})\psi-(\psi'\bar{R})\phi$$

is in $L$, is non-zero, and is orthogonal to $\bar{R}$. Hence it is not orthogonal to $p$, so

$$(\phi'\bar{R})(\psi'p) - (\psi'\bar{R})(\phi'p) \neq 0.$$

This shows Conditions 2 and 3, and Condition 4 is assumed. Weak separation follows from (3) of Theorem 2.

(2) implies (1): It has to be shown that there exists $p$ as described. Let $f$ be the orthogonal projection mapping from span$\{a,b\}$ to $L$. As above, if $e$ is in $L$ and $e$ is orthogonal to $a$ and $b$, then $e = 0$. So $f$ is onto (and $L$ has dimension two). The set

$$Q = \{e \in L : e'R \geq 0 \text{ a. s.}\}$$

is a closed convex cone containing no line. Since $\bar{R} = (\phi'\bar{R})a + (\psi'\bar{R})b$, $\bar{R}$ is in span$\{a,b\}$. Choose $p$ in span$\{a,b\}$ such that $f(p)$ is not colinear with $f(\bar{R})$ and such that $f(p)'e \neq 0$ for all $e \neq 0$ in $Q$. Then $p$ is not colinear with $\bar{R}$, Condition 4 holds because $p'e = f(p)'e$ for $e$ in $L$, and $p$ assigns zero value to portfolios with zero factor loadings because $p$ is in span$\{a,b\}$. □

PROOF OF THEOREM 4:

(2) implies (3): The second equation in (3) follows by taking expectations on both sides of the equation in (2). The first equation in (3) follows by deducting the second equation from the equation in (2).

(1) implies (2): See also Huang and Litzenberger (1988, 4.12). Theorem 2 cannot be used directly since we do not make the exponential-integrability assumption. However, the proof of Theorem 2 can be adapted. For $k$ in $H$, let $u_k$ be a utility function in $U_H$ with derivative $k$. It follows from Condition 7 that there is some $k$ in $H$ such that $u_k$ has an optimum $(1 - tq'\omega, t\omega)$ in $F$ with $t \neq 0$. The first-order condition says that

$$E[k((1 - tq'\omega)R_f + t\omega'R)(\omega'R - (q'\omega)R_f)] = 0.$$

Consider a fixed $i = 1, ..., n$. If for some $h$ in $H$,

$$E[h((1 - tq'\omega)R_f + t\omega'R)(\omega'R - (q'\omega)R_f)] = 0,$$

then $(1 - tq'\omega, t\omega)$ is optimal for $u_h$ on $F$ and hence among all portfolios with cost one, so $E[h((1 - tq'\omega)R_f + t\omega'R)(R_i - q_iR_f)] = 0$ and

$$E[h((1 - tq'\omega)R_f + t\omega'R)(E(R_i|\omega'R) - q_iR_f)] = 0.$$

By Ross (1978) Theorem A.1., there is a constant $b_i$ such that $E(R_i|\omega' R) - q_i R_f = b_i(\omega' R - (q'\omega)R_f)$.

(3) implies (2): Add the equations in (2).

(2) implies (1): A portfolio with cost one has the form $(1-q'y, y)$. The portfolio

$$(1-(b'y)(q'\omega), (b'y)\omega)$$

is in $F$ and is less risky than $(1-q'y, y)$ in the Rothschild-Stiglitz (1970) sense: If $b'y = 0$, then this portfolio consists of one share of the riskless asset. From (2), $y'\bar{R} = (q'y)R_f P$, so that

$$E[(1-q'y)R_f + y'R] = R_f.$$

If $b'y \neq 0$, then

$$\begin{aligned} & E[(1-q'y)R_f + y'R|(1-(b'y)(q'\omega))R_f + (b'y)\omega' R] \\ = \quad & E(y'R|\omega' R) - (q'y)R_f + R_f \\ = \quad & (b'y)(\omega' R - (q'\omega)R_)f) + R_f \\ = \quad & (1-(b'y)(q'\omega))R_f f + (b'y)\omega' R. \end{aligned}$$

□

PROOF OF THEOREM 5: Conclude from (2) that $\omega' b = 1$, since

$$\omega'(R-\bar{R}) = E(\omega'(R-\bar{R})|\omega' R) = (\omega' b)[\omega'(R-\bar{R})].$$

Suppose (2) holds and $q$ has the form indicated. Then $\bar{R} - R_f q = \gamma b$ and

$$\omega'\bar{R} - R_f(q'\omega) = \gamma > 0.$$

If $\omega' R \geq R_f(q'\omega)$ a. s., then

$$\omega' R \geq R_f(\bar{R} - \gamma b)'\omega / R_f = \omega'\bar{R} - \gamma,$$

or

$$-\gamma \leq \text{essinf}[\omega'(R-\bar{R})],$$

a contradiction. This shows Condition 7. Furthermore,

$$\bar{R} - R_f q = \gamma b = [\omega'\bar{R} - R_f(q'\omega)]b,$$

so (3) of Theorem 4 holds and hence (1) holds. Conversely, suppose $q$ satisfies (1). Then (3) of Theorem 4 holds. Set $\gamma = \omega'\bar{R} - (q'\omega)R_f$. Then $\gamma > 0$ and $q = (\bar{R} - \gamma b)/R_f$. If $\gamma \geq -\text{essinf}[\omega'(R - \bar{R})]$ then $-\gamma \leq \omega' R - \omega'\bar{R}$ a. s. and

$$\omega' R \geq \omega'\bar{R} - \gamma = (q'\omega)R_f,$$

a contradiction. So, $q$ has the form indicated.

(1) implies (2): Follows from Theorem 4.

(2) implies (1): Note that $0 < -\text{essinf}[\omega'(R - \bar{R})]$ since $\omega' R$ is risky. Pick any $q$ as described.

PROOF OF THEOREM 6: After $L_2$-orthonormalization, assume without loss of generality that the components of $X$ are uncorrelated with unit variance and that $n \geq 2$. We shall show that (1), (2) and (3) are also equivalent to the following statement, where $\theta$ is the characteristic function of $X$, and $S(k)$ is the sphere in $\mathbb{R}^n$ centered at zero with radius $k$.

> There exists a closed convex cone $C$ in $\mathbb{R}^n$ with non-empty interior such that $\theta$ is constant on each of the sets $S(k) \cap C$, $k \geq 0$.

(3) implies (2): See *e. g.*, Hardin (1982).

(2) implies (1): Obvious.

(1) implies the statement above: Clearly (1) holds for every $\omega$ in the cone generated by $N$, and when $\omega \neq 0$,

$$b_i = \text{cov}(X_i, \omega' X)/\text{var}(\omega' X) = \omega_i/(\omega'\omega).$$

The characteristic function of $(X_i, \omega' X)$ is

$$\rho(\lambda, \mu) = \theta(\lambda e^i + \mu\omega).$$

It follows from Lukacs and Laha (1964, Theorem 6.1.1) (or from a simple calculation) that $E(X_i|\omega' X) = (\omega' X)b_i$ implies $\rho_1'(0, \mu) = b_i \rho_2'(0, \mu)$ for all $\mu$, *i. e.*, $\theta_i'(\mu\omega) = b_i \omega' D\theta(\mu\omega)$ for all $\mu$. So,

$$D\theta(\mu\omega) = [\omega' D\theta(\mu\omega)/(\omega'\omega)]\omega.$$

This holds for all $\omega$ in the cone generated by $N$ and by continuity for all $\omega$ in the closed cone generated by $N$. That cone has non-empty interior and therefore contains a closed convex cone $C$ with non-empty interior. Since $D\theta(\mu\omega)$ is proportional to $\omega$ for all $\omega$ in $C$ and all $\mu$, $\theta$ is constant on $S(k) \cap C$ for each $k \geq 0$.

The statement above implies (3): This requires some preparation.

Let $B$ be a set in $\mathbb{R}^n$ with the property that $B$ is contained in the closure of its own interior. A complex-valued function $g$ defined on $B$ is said to be $C^s$ ($s = 0, ..., \infty$) if for every $x$ in $B$ there is an open neighborhood $V$ of $x$ and an $s$ times continuously differentiable function $\hat{g}$ on $V$ such that $g(y) = \hat{g}(y)$ for all $y$ in $B \cap V$. If so, then each derivative of $g$ of order up to $s$ extends uniquely from $\mathrm{int}(B)$ to a continuous function on $B$. A convex set $B$ with non-empty interior has the required property that $B$ be contained in the closure of its own interior.

A *cone cover* of $\mathbb{R}^n$ is a finite family $(K_k)_{k=1}^m$ of closed convex cones in $\mathbb{R}^n$ with non-empty interior, such that every $x \neq 0$ in $\mathbb{R}^n$ lies in $\mathrm{int}(K_k)$ (the interior of $K_k$) for some $k$.

**Lemma 1** *Suppose $g$ is a (complex-valued) function on $\mathbb{R}^n$ and $(K_k)$ is a cone cover. If $g$ is $C^s$ ($s = 0, ..., \infty$) on each $K_k$, then $g$ is $C^s$.*

PROOF: By induction. The case $s = 0$ is clear. Let's look at $s = 1$. We need to show that $g_i'$ exists and is continuous on all of $\mathbb{R}^n$ for each $i = 1, \ldots, n$. For $k = 1, \ldots, m$, let $g^k$ be the unique continuous extension of $g_i'$ from $\mathrm{int}(K_k)$ to $K_k$. If $x \neq 0$, then obviously $g^k(x) = g_i'(x)$. If $h, k$ are such that there is some $x \neq 0$ in $\mathrm{int}(K_h) \cap \mathrm{int}(K_k)$, then $g^h(tx) = g^k(tx)$ for all $t > 0$, so by continuity, $g^h(0) = g^k(0)$. By a chain argument, $g^h(0) = g^k(0)$ for all $h, k$. So, $g^h$ and $g^k$ agree on $K_h \cap K_k$. Consequently, there is a unique function $\bar{g}$ on $\mathbb{R}^n$ such that $\bar{g}(x) = g^h(x)$ when $x$ belongs to $K_k$. It is continuous because it is continuous on each $K_h$. Choose $k$ such that $e^i$ (the $i$'th unit vector in $\mathbb{R}^n$) belongs to $K_k$. Then

$$\bar{g}(0) = g^k(0) = \lim_{t \to 0+} [g(te^i) - g(0)]/|t|.$$

A similar argument shows that

$$\bar{g}(0) = \lim_{t \to 0-} [g(te^i) - g(0)]/|t|.$$

So $g_i'(0)$ exists and is equal to $\bar{g}(0)$, $\bar{g}$ is the $i$'th derivative of $g$, and it is continuous. Now the induction argument: if $g$ is $C^r$, $r < s$, then $g$ is $C^{r+1}$.

To see this let $\alpha$ be a multi-index of order $|\alpha| = r$. Then $D^\alpha g$ is $C^1$ because it is defined on all of $\mathbb{R}^n$ (by the induction assumption) and is $C^1$ on each $K_k$. □

Now back to the proof of Theorem 6.

Let $g$ be the unique (complex valued) function on $\mathbb{R}^n$ which is constant on each of the spheres $S(k)$, $k \geq 0$, and which is identical to $\theta$ on $C$. First, let's show that $g$ is $C^\infty$ and has the same derivatives as $\theta$ at zero. Since $E[\exp(t'X)] < \infty$ for all $t$ in $\mathbb{R}^n$, all moments of $X$ exist and $\theta$ is $C^\infty$ (see for example Cuppens (1975, Section 3.2)). Also, the distribution of $t'X$ is determined by its moments for all $t$, c.f. Billingsley (1986, Chapter 26). Choose a finite family of orthogonal matrices $M_k$, $k = 1, \ldots, m$, such that $M_1 = I$ and such that the cones $(M_k C)$ form a cone cover. On $M_k C$,

$$g(y) = \theta(M_k^{-1} y),$$

so $g$ is $C^\infty$ on $\cap_k C$ for all $k$, hence $g$ is $C^\infty$ by Lemma 1. By continuity, $g$ and $\theta$ have the same derivatives at zero since they coincide on $C$. Let $M$ be any orthogonal matrix, and let $f$ be the characteristic function of $MX$. Then

$$f(t) = E[\exp(it'MX)] = \theta(M't).$$

Since $\theta$ and $g$ have the same derivatives at zero, so do $f$ and the function $t \mapsto g(M't) = g(t)$. So, $f$ and $\theta$ have the same derivatives at zero. The exponential-integrability assumption implies that $f$ and $\theta$ are entire analytic functions. So, $f = \theta$, and the distribution of $X$ is equal to the distribution of $MX$. Hence, $X$ is spherical. □

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