

**"IDIOSYNCRATIC RISK, SHARING RULES AND
THE THEORY OF RISK BEARING"**

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Abstract

It has been recognized in the literature that the choice of an agent between risky and riskless assets is complicated by the existence of other unavoidable risks [Ross (1981), Kihlstrom, Romer and Williams (1981), Nachman (1982) and Pratt and Zeckhauser (1987)]. For instance, the purchase of assets by an individual investor may be made in the context of uncertain wage income.

In this paper, we are concerned with the *response* of the agent to the existence of additional non-insurable income risk. In particular, the agent chooses state-dependent shares of aggregate marketable income (a sharing rule) to provide a partial hedge against the idiosyncratic risk. We focus on the form of the sharing rule in order to determine whether and when the agent is a buyer / seller of insurance. Since our emphasis is on the sensitivity of the optimal allocation decision, we use the concepts of absolute prudence and the precautionary premium proposed by Kimball (1990), which characterize the behavior of the *marginal utility* function.

The paper first derives the properties of the precautionary premium in the presence of idiosyncratic risk and different levels of marketable income. It then analyzes the optimal behavior of the agent and shows that the higher the non-insurable risk, the more the agent purchases claims in states with low aggregate income and sells claims in states with high aggregate income - a type of

insurance. In this context, the equilibrium is characterized by the sharing rules of the various agents in the economy. Then, the sensitivity of the equilibrium to an increase in the non-insurable risk across all investors is analyzed.

These results are made more specific by considering the special case of the Hyperbolic Absolute Risk Aversion (HARA) family of utility functions. In this case, the precautionary premium can be defined more precisely and the sharing rules can be derived explicitly.

The results in the paper have a number of applications such as in the analysis of the demand for portfolio insurance, the hedging decisions of corporations, and the impact of an aggregate "shock" in idiosyncratic risk on the allocation of risk bearing in an economy.

1 INTRODUCTION

The purpose of this paper is to analyze the effects of non-insurable idiosyncratic risk on the optimal sharing rules of agents in an economy. In this economy, agents are allowed to buy and sell claims contingent upon aggregate marketable income in the presence of an idiosyncratic risk which cannot be hedged. We investigate the optimal behavior of an individual agent in such a situation and, in particular, the agent's sharing rule defined in terms of claims on the aggregate marketable income. We also study the effect of idiosyncratic risk on the relative pricing of claims in the economy.

It has increasingly been recognized in the literature that an agent's choice between a risky and a riskless asset is complicated by the existence of other unavoidable risks. As Kihlstrom et al (1981) point out, it is rare for decisions on the purchase of risky assets to be taken in the absence of wage income risk, for example. This has led Ross (1981), Kihlstrom et al (1981), Nachman (1982), and Pratt and Zeckhauser (1987) to consider the robustness of the Pratt (1964) - Arrow (1965) theory of risk aversion in the presence of such an additional income risk. Essentially, Kihlstrom et al (1981) and Nachman (1982) investigate the following issue: An investor who is more risk-averse invests less in the risky asset than one who is less risk-averse. Under what conditions do investors also exhibit the same behavior in the face of idiosyncratic risk? They conclude that the condition amounts to the restriction that the agent's utility function must exhibit non-increasing risk aversion for all levels of wealth.

In this paper, we are also concerned with the agent's *response* to the existence of an additional non-insurable income risk. The agent chooses a state-dependent share $x=g(X)$ of the aggregate marketable income, X . We assume that the capital market is perfect and complete with respect to the *marketable* aggregate income. The independent income risk ε faced by the agent neither can be hedged nor diversified away and, therefore, is called an idiosyncratic risk, ε . However, the agent can modify his optimal purchases of claims on the aggregate marketable income, in the presence of the idiosyncratic risk. The focus here is on the effect of ε on the sharing rule $g(X)$. For example, the agent may choose $g(X)$ so as to receive a greater proportion of the aggregate income in the lower states than in the higher states. A low (high) state is defined as a state in which the aggregate marketable income, X , is low (high). In contrast, the papers by Pratt and Zeckhauser, Kihlstrom et al, Ross, and Nachman consider only the choices between a risky and a risk-free-asset. In that literature, the risky asset is not necessarily a marketable claim.

In a paper related to ours, Kimball (1990) discusses the demand for precautionary savings employing the concept of a "precautionary risk premium." His analysis is also restricted to risk-free claims as a vehicle for saving. Kimball emphasizes the degree of absolute prudence defined as $-v'''(y)/v''(y) = \eta(y)$ associated with the agent's utility function $v(y)$, where y is the consumption. He shows that the shift in the consumption function induced by an idiosyncratic future income risk depends upon absolute prudence. He also shows that absolute risk aversion decreases if and only if absolute prudence exceeds absolute risk aversion.

We apply Kimball's concept of the precautionary risk premium in our analysis to characterize the optimal behavior of an agent facing idiosyncratic risk. We analyze the comparative statics of the optimal behavior of the agent who faces idiosyncratic risk. We use these results to define the sharing rule of the agent, given the pricing of claims on aggregate marketable income.

The economic setting of the paper is as follows. The agent has an opportunity to buy claims on a single marketable aggregate income, X . The agent's consumption y at the end of a single period is the chosen amount of the marketable income, $x=g(X)$, plus an independent non-marketable income ε , with a zero mean.¹ Hence, $y=g(X)+\varepsilon$, where the scalar, σ , is used as a metric for the size of the non-marketable risk in the analysis that follows. The agent's problem is to choose the functional form of $g(X)$ so as to maximize a utility function $E[v(y)]$.

The agent's demand for claims on aggregate marketable income in the face of non-marketable, idiosyncratic risk depends upon the absolute risk aversion and the absolute prudence of the utility function. $a(y) = -v''(y)/v'(y)$ is the Arrow-Pratt coefficient of absolute risk aversion. The degree of risk aversion is reflected in the risk premium which is defined by π in $E[v(y)] = v[E(y)-\pi]$. Analogously, the precautionary risk premium, or simply, the precautionary premium, is defined as Ψ in the relation, $E[v'(y)] = v'[E(y)-\Psi]$. It is the coefficient of absolute

¹ The results of the paper do not change if the mean of ε conditional on X depends on X . In a complete market, the agent can always sell these means.

prudence $\eta(y) = -v'''(y)/v''(y)$, which determines the precautionary risk premium Ψ and, hence, the agent's response in terms of optimal decisions to the existence of non-marketable risks. This follows from the observation that optimal decisions depend on *marginal utility* and not on the utility itself.

Although the concepts of risk aversion and prudence are important in characterizing the behavior of agents in the presence of idiosyncratic risk, we should be careful not to make strong assumptions at the outset. This is because certain stylized facts could be explained by rather weak restrictions, while others may require stronger assumptions. Specifically, as we shall see, assumptions on the degree of risk aversion and prudence may be overly strong in order to characterize certain types of behavior in the face of idiosyncratic risk. Therefore, in section 2 of the paper, we begin our analysis by starting with the weakest set of assumptions: only that $v'(y) > 0$. We then progressively add further restrictions, on the signs of the higher order derivatives until the fourth derivative. In the most restrictive scenario we consider, we assume that $v'(y) > 0$, $v''(y) < 0$, $v'''(y) > 0$ and $v''''(y) < 0$.² In section 2A, we define the

² These restrictions are implied by several stylized facts. Non-satiation implies that $v'(y) > 0$. Positive risk aversion [$a(y) > 0$] implies that $v''(y) < 0$. The additional assumption of either decreasing absolute risk aversion [$a'(y) < 0$] or positive prudence [$\eta(y) > 0$] implies that $v'''(y) > 0$. Analogously, requiring additionally that absolute prudence be decreasing [$\eta'(y) < 0$] implies that $v''''(y) < 0$. Kimball (1989) gives many examples which suggest that agents have both positive decreasing absolute risk aversion and positive, decreasing absolute prudence.

precautionary premium in our context characterize some of its properties. In section 2B, we discuss the properties of risk aversion in the presence of idiosyncratic risk. Specifically, we explore how the derived utility function is affected by the presence of idiosyncratic risk. In the next section, section 2C, we derive the effect of an increase in the idiosyncratic risk on the optimal sharing rule. In particular, we show that the agent buys more claims in states with low aggregate marketable income and less claims in states with high aggregate marketable income. Next, in section 2D, we consider the effects of changes in the idiosyncratic risk on the relative pricing of claims in different states. We show that the prices of claims in low states go up in relation to those of claims in high states in response to an aggregate shock to idiosyncratic risk if the aggregate derived risk tolerance goes down in every state.

In order to make more precise statements about the behavior of the precautionary premium and the form of the sharing rule, we need to make more specific assumptions about the preferences of agents. In section 3, we assume that all agents have utility functions belonging to the Hyperbolic Absolute Risk Aversion class (HARA) and make more specific our results of the previous section. We show that for any agent, the sharing rule is a linear function of the agent's precautionary premium and the aggregate precautionary premium. However, the sharing rule of the agent is non-linear in aggregate marketable income. For an agent without idiosyncratic risk, the sharing rule is strictly concave. A general rise in idiosyncratic risk renders claims in low states more expensive relative to those in high states if welfare is reduced more in the low states. In each state, welfare

is measured by the exogenous aggregate marketable income minus the endogenous aggregate precautionary premium. In section 4, we conclude with an interpretation of the results.

2 PRECAUTIONARY PREMIUM AND THE SHARING RULE: THE GENERAL CASE

In this section, we first discuss some properties of the precautionary premium which are helpful in the analysis. We then start with the weakest set of restrictions and then make progressively stronger assumptions to characterize the behavior of the agent in the presence of idiosyncratic risk.

A SOME PROPERTIES OF THE PRECAUTIONARY RISK PREMIUM

Suppose now that $y = x + \sigma\varepsilon$ where ε is a zero mean risk with standard deviation of unity; ε is independent of X and hence of $x = g(X)$ and σ is a positive scalar denoting the standard deviation of $\sigma\varepsilon$. Hence, an increase in σ denotes the addition of a mean-preserving spread in the sense of Rothschild and Stiglitz (1970) to the income y , conditional on the marketable income x . The utility function, $v(y)$, is defined over the range of real numbers $[Y^-, Y^+]$. We first define the risk premium Π conditional on x by the relation $E_x(v(y)) = v(x - \Pi)$, where $\Pi = \Pi(x, \sigma)$ and $v(x - \Pi)$ is the derived utility function as defined in Nachman (1982). We then define, as in Kimball (1990), the precautionary risk premium Ψ

conditional on x by the relation $E_x(v'(y)) = v'(x - \Psi)$ where $\Psi = \Psi(x, \sigma)$.^{3,4}

First, Ψ is positive and a strictly decreasing function of x , if the absolute prudence is positive and decreasing, i.e.

$$\Psi > 0 \quad \leftrightarrow \quad \eta(y) > 0 \quad ;$$

$$\frac{\partial \Psi}{\partial x} < 0 \quad \leftrightarrow \quad \eta'(y) < 0 \quad .$$

This is discussed by Kimball (1990) and follows directly by applying the argument of Pratt (1964).

We next look at the effect of an increase in idiosyncratic income risk ($\sigma\varepsilon$) on the precautionary risk premium in order to determine the effect on the optimal

³ In Kimball's context, x is period 1 income and $\sigma\varepsilon$ represents a risk to income in period 2. For the problem that we address here, $\sigma\varepsilon$ represents a contemporaneous income risk which is idiosyncratic to the agent. Kimball shows that the properties of the precautionary premium Ψ are analogous to those of the Pratt - Arrow risk premium.

⁴ The use of the precautionary premium rather than just the risk premium simplifies the notation and makes the analysis more intuitive. To see this differentiate $E_x(v(y)) = v(x - \Pi)$ with respect to x which yields $E_x(v'(y)) = v'(x - \Pi)(1 - d\Pi/dx) = v'(x - \Psi)$.

allocation of claims. We have $\partial\Psi/\partial\sigma > 0$ ⁵. In fact, for a small income risk,

$$\Psi - \frac{1}{2}\eta(x)\sigma^2$$

so that

$$\frac{\partial\Psi}{\partial\sigma^2} - \frac{1}{2}\eta(x) > 0 .$$

The precautionary premium $\Psi = \Psi(x,\sigma)$ is the crucial determinant of the agent's demand for risky assets in the face of idiosyncratic income risk. It is relevant, therefore, to consider further characteristics of Ψ . If the coefficient of absolute prudence is declining in y , it follows from the last equation for a small income risk that

$$\frac{\partial^2\Psi}{\partial\sigma^2\partial x} - \frac{1}{2}\eta'(x) < 0.$$

For large risks, these equations do not hold. But, it is possible to derive some statements on the derived marginal utility, $v'(x - \Psi)$, which are relevant for the

⁵ Rothschild and Stiglitz (1970) have shown for utility functions with positive absolute risk aversion that the addition of a mean preserving spread raises the risk premium. Using the Kimball analogy, it follows that an increase in σ raises the idiosyncratic income risk, and hence the precautionary risk premium, if $\eta(y) > 0$.

development of the sharing rules of the agent. It is convenient to analyze the sensitivities of growth rates of $v'(x - \Psi)$, i.e. the sensitivities of $\ln v'(x - \Psi)$, with respect to income x and idiosyncratic risk σ .

B PROPERTIES OF RISK AVERSION IN THE FACE OF IDIOSYNCRATIC RISK

We start with the weakest assumption and derive our first result, lemma 1. We first examine the effect of idiosyncratic risk on the risk attitudes of the agent. To be specific, does the risk aversion of the agent remain positive even in the presence of idiosyncratic risk?

Assumption 1:

The utility function has a positive first derivative, $v'(y) > 0$, and the sign of the second derivative, $v''(y)$, is the same everywhere.

Lemma 1:

The derived marginal utility function in the presence of idiosyncratic risk preserves the sign of the risk aversion of the agents' utility function, i.e.

$$\text{sign } \hat{a}(x) = \text{sign } a(x)$$

where:

$$\hat{a}(x) = -\frac{\partial \ln v'(x - \Psi)}{\partial x} = -\frac{v''(x - \Psi)}{v'(x - \Psi)} \left[1 - \frac{d\Psi}{dx} \right]$$

Proof:

From the definition of $\hat{a}(x)$,

$$\hat{a}(x) = a(x - \Psi) \left[1 - \frac{\partial \Psi}{\partial x} \right].$$

Hence, we need to show that $1 - \frac{\partial \Psi}{\partial x} > 0$ everywhere .

From the definition of the precautionary risk premium,

$$E_x[v'(x + \sigma \varepsilon)] = v'(x - \Psi)$$

it follows that

$$\begin{aligned} \frac{dE_x[v'(x + \sigma \varepsilon)]}{dx} &= E_x[v''(x + \sigma \varepsilon)] \\ &= v''(x - \Psi) \left(1 - \frac{\partial \Psi}{\partial x} \right) \end{aligned}$$

Since $v''(y)$ has the same sign everywhere, $E[v''(y)]$ has the same sign as $v''(y)$. Hence, $1 - \frac{\partial \Psi}{\partial x} > 0$ and it follows that $\text{sign } \hat{a}(x) = \text{sign } a(x)$.⁶ ■

Lemma 1 says that the agents' risk attitude towards marketable risk is unaffected by the presence of idiosyncratic risk. Hence, all the results in the literature that characterize the behavior of the agent based on the sign of the risk aversion coefficient apply even in the presence of idiosyncratic risk. In particular, if the sign is positive, the agent exhibits risk aversion also in the presence of

⁶ This proof is in the spirit of Kihlstrom et. al. (1981) and Nachman (1982).

idiosyncratic risk.

We can now characterize the derived marginal utility function as a function of the size of the idiosyncratic risk. In order to do this, we need to make an additional assumption about the agents' utility function:

Assumption 2:

The utility function has a negative second derivative $v''(y) < 0$.

This leads directly to lemma 2.

Lemma 2:

The derived marginal utility increases with the size of the idiosyncratic risk if and only if the third derivative of the utility function, $v'''(y) > 0$.

Proof:

$$\frac{\partial \ln v'(x-\Psi)}{\partial \sigma} = a(x-\Psi) \frac{\partial \Psi}{\partial \sigma} .$$

As $a(x-\Psi) > 0$, $\frac{\partial \ln v'(x-\Psi)}{\partial \sigma} > 0$ iff $\frac{\partial \Psi}{\partial \sigma} > 0$, which is equivalent to

$v'''(y) > 0$. ■

Lemma 2 says that the rate of change of the derived marginal utility increases with idiosyncratic risk, since an increase in idiosyncratic risk raises the precautionary premium and, thus, has the same effect as a decline in income. The logic of the proof is analogous to the arguments used by Pratt (1964) and Arrow (1965) to

show that the effect of an increase in the size of the risk on the risk premium is determined by the sign of the second derivative of the utility function, i.e. risk aversion. The only difference is that here we are talking about the effect of the size of the risk on the precautionary risk premium being equivalent to the sign of the third derivative of the utility function, i.e. prudence. If prudence is positive, the marginal utility increases with the size of the idiosyncratic risk.

In order to determine the effects of an increase in idiosyncratic risk on the sharing rule, we also need to know the second derivatives of $\ln v'(x - \Psi)$ with respect to income x and idiosyncratic risk σ .

Lemma 3:

The derived utility function exhibits decreasing absolute risk aversion if and only if the utility function also does.

$$\frac{\partial \hat{a}(x)}{\partial x} = - \frac{\partial^2 \ln v'(x - \Psi)}{\partial x^2} < 0 \quad \text{iff} \quad \frac{da}{dx} < 0, \quad \forall x.$$

Proof:

See Appendix A.

Lemma 3 says that there is an equivalence between the risk aversion of the original utility function and that of the derived utility function. Hence, every result in the literature that is based on decreasing risk aversion also holds in the

presence of idiosyncratic risk.

We now impose the additional restriction that prudence is positive which is necessary condition for decreasing absolute risk aversion.⁷

Assumption 3:

The utility function has a positive third derivative $v'''(y) > 0$.

We also define an additional characteristic of the utility function $\theta = \frac{v'''(y)}{v'(y)}$ as a *combined* prudence/risk aversion measure. This measure defines the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion.

Lemma 4:

The derived risk aversion is positive. It increases with an increase in idiosyncratic risk if and only if the product of the coefficient of risk aversion and the coefficient of absolute prudence is decreasing in total income, for all levels of income, i.e.,

$$\frac{\partial \hat{\alpha}(x)}{\partial \sigma} > 0 \quad \text{iff} \quad \frac{d\theta}{dy} - \frac{d\left[\frac{v'''(y)}{v'(y)}\right]}{dy} < 0, \quad \forall y$$

⁷ Positive prudence is not sufficient for decreasing absolute risk aversion, since, for decreasing absolute risk aversion, we would require additionally that $[v'(y)][v'''(y)] > [v''(y)]^2$, which places a restriction on the relative magnitude of $v'''(y)$.

Proof:

$$\text{Necessity: } \frac{\partial \hat{a}(x)}{\partial \sigma} > 0 \quad - \quad \frac{d \left[\frac{v'''(y)}{v'(y)} \right]}{dx} < 0$$

Consider a small risk. Then $\Psi(x) = \frac{1}{2} \eta(x) \sigma^2$.

$$\begin{aligned} \frac{\partial \ln v'(x-\Psi)}{\partial \sigma^2} &= \frac{-v''(x-\Psi)}{v'(x-\Psi)} \cdot \frac{1}{2} \eta(x) \\ &= \frac{-v''(x)}{v'(x)} \cdot \frac{v''(x)}{-v''(x)} \cdot \frac{1}{2} \\ &= \frac{1}{2} \frac{v'''(x)}{v'(x)}. \end{aligned}$$

Hence, for a small risk,

$$\frac{\partial \hat{a}(x)}{\partial \sigma^2} = \frac{\partial^2 \ln v'(x-\Psi)}{\partial \sigma^2 \partial x} = \frac{1}{2} \frac{d \left[\frac{v'''(x)}{v'(x)} \right]}{dx}$$

It follows, therefore, that $\frac{\partial \hat{a}(x)}{\partial \sigma} = \frac{2\sigma}{\partial \sigma^2} \frac{\partial \hat{a}(x)}{\partial \sigma^2} > 0$ only if $\frac{d \left[\frac{v'''(x)}{v'(x)} \right]}{dx} < 0$

$$\text{Sufficiency: } \frac{d \left[\frac{v'''(y)}{v'(y)} \right]}{dx} < 0 \quad - \quad \frac{\partial \hat{a}(x)}{\partial \sigma} > 0$$

Integrating $\frac{\partial \hat{a}(x)}{\partial \sigma^2}$ with respect to σ^2 yields

$$\hat{a}(x) - a(x) - \frac{1}{2} \sigma^2 \frac{-d\left(\frac{v'''(x)}{v'(x)}\right)}{d(x)}$$

for a small risk. Hence we may interpret $\hat{a}(x) - a(x)$ as the risk premium of a

small risk for a utility function with absolute risk aversion $\frac{-d\left(\frac{v'''(x)}{v'(x)}\right)}{d(x)}$. If this is positive, then an increase in σ^2 raises the risk premium for a small risk.

If this is positive everywhere, then it follows from Rothschild and Stiglitz that every mean preserving spread, therefore every increase in σ , raises $\hat{a}(x)$ ■

Lemma 4 says that derived risk aversion increases with idiosyncratic risk if and only if $v'''(x)/v'(x)$ is decreasing. Hence, *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for the result to obtain. However, the combination of these conditions is sufficient for the lemma to hold, since the requirement is that the product of the two must be decreasing. Hence, this lemma tells us that an investor facing the choice between a riskless and a risky asset buys less of the risky asset if his idiosyncratic risk increases provided that v'''/v' is decreasing.

It should be noted that Kimball's assumptions of decreasing prudence and decreasing absolute risk aversion, although sufficient, are overly strong to characterize the effect of increasing risk on the derived risk aversion. Furthermore, this lemma immediately gives us the main results of Pratt and Zeckhauser, without using their assumptions of alternating signs of the derivatives of the utility function to show "proper" risk aversion.⁸

Lemma 4 states in conjunction with lemma 3 that the negative proportionate change in derived marginal utility due to an increase in marketable income grows with idiosyncratic risk, but declines as income rises. Also, from lemmas 2 and 4, it follows that an investor who can buy a risk-free and a risky asset, invests less in the risky asset when his idiosyncratic risk increases. The intuition behind Lemma 4 is that an increase in idiosyncratic risk is equivalent to a decrease in marketable income and hence derived risk aversion behaves in a similar manner under both changes. This raises the question of substitution between these two effects. In other words, what can be said about the marginal rate of substitution between an increase in risk and a decrease in income? This takes us to lemma 5.

⁸ Proper risk aversion is a property of utility functions whereby "an undesirable lottery can never be made desirable by the presence of an undesirable independent lottery" (p. 143). A subset of these functions has positive odd derivatives and negative even derivatives over some interval of wealth.

Lemma 5:

The marginal "rate of substitution" between idiosyncratic risk and marketable income decreases with marketable income if and only if absolute prudence is decreasing in total income, for all levels of income, i.e.,

$$\frac{\partial \left[- \frac{\partial \ln v'(x - \Psi) / \partial \sigma}{\partial \ln v'(x - \Psi) / \partial x} \right]}{\partial x} < 0, \quad \text{iff} \quad \frac{d\eta}{dy} < 0.$$

Proof:

See Appendix B.

Lemma 5 considers the marginal rate of substitution between changes in idiosyncratic risk and income x which leave derived marginal utility unaffected. The lemma says that the marginal rate of substitution (which has been defined to be positive) is lower, the higher is the income x . In other words, at higher levels of marketable income, the agent is willing to accept a smaller increase in income to compensate for a small increase in idiosyncratic risk to maintain the same level of utility.

C IDIOSYNCRATIC RISK AND THE OPTIMAL SHARING RULE

After establishing the basic results on the behavior of an agent who faces idiosyncratic risk, we can now characterize his optimal portfolio decisions with respect to his *marketable* income. We first consider a setting in which the agent i faces idiosyncratic risk, but can invest in a riskless and a risky asset in the capital market. In other words, the agent can adjust his portfolio decisions to take into account the presence of non-marketable risk.

We assume that the utility function of the agent satisfies assumptions 1, 2 and 3 of the previous section, $v' > 0$, $v'' < 0$, $v''' > 0$ and $v'''' > 0$. The agent solves the following maximization problem:

$$\max_{\alpha} E[v(x_i + \sigma_i \varepsilon)] \quad (1)$$

subject to

$$x_i - w_i r + \alpha(\tilde{r}_m - r)$$

where \tilde{r}_m and r are respectively, the rate of return on the risky asset and the rate of return on the riskless asset, α is the dollar amount invested in the risky asset, and w_i the agent's initial endowment. We can now state and prove the following proposition.

Proposition 1:

Assume that an agent who faces idiosyncratic risk can only buy a riskless and a risky asset in the capital market. Suppose the size of the idiosyncratic risk increases. Then, he invests less in the risky asset if and only if the product of the coefficient of absolute risk aversion and the coefficient of absolute prudence is decreasing in total income, for all levels of income.

Proof:

By lemma 4, the derived risk aversion of the agent in the presence of idiosyncratic risk is positive and increases with the size of the idiosyncratic risk. Hence, by Theorem 7 of Pratt (1964), the dollar investment in the risky asset declines in the presence of a higher idiosyncratic risk. ■

We next expand the portfolio opportunity set of the agent by permitting him to invest in state-contingent claims on marketable aggregate income. We assume that the capital market is perfect and complete with respect to marketable aggregate income X . An agent i in the economy solves the following maximization problem

$$\max_{x_i - g_i(X)} E[v(x_i + \sigma_i \varepsilon)] \quad (2)$$

subject to

$$w_i - E[\Phi(X) x_i]$$

where w_i is the agent's initial endowment and $\Phi(X)$ is the market pricing function

for state contingent claims on X .⁹ The first order condition for a maximum is

$$E_x[v'(x_i + \sigma_i \varepsilon)] - \lambda_i \Phi(X) \quad (3)$$

where $E_x(\cdot)$ is the conditional expectation given X , and λ_i is a positive state independent Lagrange multiplier reflecting the tightness of the budget constraint. Notice that the derived marginal utility of the agent is proportional to the price $\Phi(X)$. If all agents are risk averse, it follows immediately that $\Phi'(X) < 0$. To see this, if we differentiate (2) with respect to X we have (dropping the agent index i)

$$\frac{dx}{dX} - \frac{dg(X)}{dX} = \frac{\lambda \Phi'(X)}{E_x[v''(x + \sigma \varepsilon)]}$$

Thus, $dg(X)/dX$ has the same sign for each agent. Since it must be positive, it follows immediately that $\Phi'(X) < 0$. This establishes our first result regarding the optimal sharing rule for the individual agent in this economy. Agents buy increasing claims on the aggregate income X . The sharing rule has a positive slope as it does in the absence of idiosyncratic risk (see Rubinstein (1974)).

Equation (2) can be written, using the precautionary premium, as

$$v'[x_i - \Psi_i(x_i, \sigma_i)] - \lambda_i \Phi(X) \quad (4)$$

In (3), $v'(\cdot)$ is the derived marginal utility of the agent.

Totally differentiating the left hand side of equation (3) with respect to σ yields

⁹ Note that the market pricing function gives the market price of a state contingent claim divided by the probability of the state occurring.

$$\frac{dv'(\cdot)}{d\sigma} = \frac{\partial v'(\cdot)}{\partial \sigma} + \frac{dg(X)}{d\sigma} \frac{\partial v'(\cdot)}{\partial g(X)}$$

It follows from equation (3) that $dv'(\cdot) / d\sigma = v'(\cdot) d\ln\lambda / d\sigma$. Hence the effect of the idiosyncratic risk on the sharing rule is given by

$$\frac{dg(X)}{d\sigma} = \frac{\frac{d\ln\lambda}{d\sigma} v'(\cdot)}{\frac{\partial v'(\cdot)}{\partial g(X)}} - \frac{\frac{\partial v'(\cdot)}{\partial \sigma}}{\frac{\partial v'(\cdot)}{\partial g(X)}} \quad (5)$$

The effect on the sharing rule in equation (4) is the sum of two factors. The first is analogous to an "income effect". The second term is the pure "substitution effect" of the change in σ . If we were to compare the sharing rules of two agents, cross-sectionally within an equilibrium, with differing σ_i but with the same λ_i , then consideration of only the "substitution effect" would be relevant.

An alternative question to ask is the following: How does an agent react if idiosyncratic risk increases? In this case, both the "income effect" and the "substitution effect" are relevant since we are talking about a comparative statics shift. In order to distinguish these two questions, we establish proposition 2 and 3.

Proposition 2:

Assume that an agent who faces idiosyncratic risk can buy state-contingent claims on marketable aggregate income. Suppose the size of the idiosyncratic risk increases, but the tightness of the budget constraint is the same. Then, there is only a substitution effect such that $dx/d\sigma$ is higher, the smaller is the agent's marketable income, if and only if his absolute prudence is decreasing in total income, for all levels of income.

Proof:

If λ is constant, only the "substitution effect" is relevant. From equation (5), in this case

$$\frac{dg(X)}{d\sigma} = - \frac{\frac{\partial v'(\bullet)}{\partial \sigma}}{\frac{\partial v'(\bullet)}{\partial g(X)}} \quad (6)$$

and

$$\frac{d^2g(X)}{d\sigma dX} = \frac{d}{dg(X)} \left[- \frac{\frac{\partial v'(\bullet)}{\partial \sigma}}{\frac{\partial v'(\bullet)}{\partial g(X)}} \right] \frac{dg(X)}{dX} \quad (7)$$

From lemma 5, the first term on the right hand side of equation (7) is negative. The second term is positive and hence the derivative in (7) is negative. The statement in proposition 2 then follows.■

We now examine the other factor in the change in the optimal portfolio decision of the agent, the income effect.

Proposition 3:

Suppose the tightness of the budget constraint measured by λ increases. Then, there is an income effect such that $dx/d\lambda$ is smaller, the higher is the marketable income, if and only if the agent exhibits decreasing absolute risk aversion for all levels of marketable income.

Proof:

Take the logarithm of the optimality condition (4) and differential it with respect to $\ln\lambda$. This yields

$$- \hat{a} [x - \Psi(x, \sigma)] \frac{dx}{d \ln \lambda} = 1$$

Hence, $dx/d \ln\lambda$ is decreasing in x and hence, in X , if and only if the derived risk aversion is decreasing also. Next, we translate this into the risk aversion of the basic utility function using lemma 3. Hence, proposition 3 follows.■

We can combine the results of propositions 2 and 3 by considering the *total* effect of an increase in the size of the idiosyncratic risk for an agent.

Proposition 4:

Assume that $v(y)$ has the properties of positive and declining absolute risk aversion and prudence, i.e. $a(y) > 0$, $a'(y) < 0$, $\eta(y) > 0$, $\eta'(y) < 0$. Consider an increase in σ for a *given* agent. Then $\frac{d^2g(X)}{d\sigma dX} < 0$. Also $\exists X^*$ such that

$$\frac{dg(X)}{d\sigma} \geq < 0 \text{ for } X \leq > X^* .$$

Proof:

Here we consider both, the "income" and the "substitution effect" in equation (5).

However, since the "substitution effect" is positive (which follows from

$$\frac{\partial v'}{\partial g(X)} < 0 \text{)}, \text{ the "income effect" must be negative since otherwise } \frac{dg(X)}{d\sigma}$$

would be positive in all states, violating the budget constraint. Hence

$\frac{d \ln \lambda}{d\sigma} > 0$. To establish the proposition, note that in this case

$$\frac{d^2g(X)}{d\sigma dX} = \frac{d}{dg(X)} \left[\frac{\frac{d \ln \lambda}{d\sigma} v'(\bullet)}{\frac{\partial v'(\bullet)}{\partial g(X)}} - \frac{\frac{\partial v'(\bullet)}{\partial \sigma}}{\frac{\partial v'(\bullet)}{\partial g(X)}} \right] \frac{dg(X)}{dX} \quad (8)$$

From proposition 2, we know that the derivative of the second term in the bracket is negative. From proposition 3, the derivative of the first term in the bracket is also negative under the assumed conditions and proposition 4 follows directly.

Finally, since the purchase and sales of state contingent claims must be self financing, it follows that there must exist a critical level X^* such that $dg(X)/d\sigma$ is positive for $X < X^*$ and negative for $X > X^*$. ■

It should be emphasized that the conditions assumed are overly strong to obtain the result in proposition 4. Neither decreasing absolute risk aversion nor decreasing absolute prudence is *necessary* for the result to hold. Of course, they are sufficient by the lemmas of the previous section.

The significance of proposition 4 is that it tells us, *within an equilibrium*, how the sharing rules of agents differ in response to differences in idiosyncratic risk. Given the utility function and tightness of the budget constraint, proposition 2 says that agents with high idiosyncratic risk will tend to buy relatively more contingent claims in low states and relatively less claims in high states. If we compare a situation without idiosyncratic risk with one with such a risk, then we could say that the agents with high σ_i tend to buy "insurance" (i.e. claims in low states) from those with low σ_i . In contrast, proposition 4 tells us the effect of an increase in σ_i on an agent taking into account the effects of σ_i on the budget constraint and on λ_i . Again the effect is unambiguous. The agent increases the purchase of claims in the low states of X . The implications of this behavior by agents in general will be looked at in the following section.

D PRICING EFFECTS OF CHANGES IN IDIOSYNCRATIC RISK

In the preceding section, sharing rules of agents have been investigated with prices of contingent claims being given. From proposition 4, it follows that an

agent demands more claims in the states with low aggregate marketable income and less claims in the states with high aggregate marketable income if his idiosyncratic risk increases. Suppose now that there is an aggregate shock such that idiosyncratic risk increases for every agent in the economy. Then, every agent wishes to buy more claims in the "low" states at the *old* prices. But this is impossible since the agents' additional demand for marketable claims must sum to zero for every state. Hence, the prices of all claims must change to reflect the change in demand. Proposition 4 suggests that claims in "low" states become relatively more expensive compared to claims in "high" states. A somewhat weak qualification is necessary, however, for this implication to go through. This qualification is a restriction on the agents' derived risk tolerance. Agent i 's derived risk tolerance $\hat{e}_i(x_i)$ is the inverse of his derived risk aversion $\hat{a}_i(x_i)$, i.e.

$$\hat{e}_i(x_i) = 1/\hat{a}_i(x_i) . \text{ As } x_i = g_i(X) , \hat{e}_i(x_i) = \hat{e}_i(X) . \text{ We now state the}$$

equilibrium implication as a proposition.

Proposition 5:

A general rise in idiosyncratic risk across many agents in the economy leads to an increase in the price of claims contingent upon marketable aggregate income in state s , X_s , relative to the price of claims contingent upon a higher marketable aggregate income in state t , X_t , if and only if the aggregate derived risk tolerance decreases in every state. More precisely, for a small general rise in idiosyncratic risk,

$$\frac{d(\Phi_s/\Phi_t)}{d\sigma} > 0 \quad \forall [(s,t):X_s < X_t] \quad \leftrightarrow \quad \frac{d\sum_i \hat{e}_i(X)}{d\sigma} < 0 \quad \forall X .$$

Proof:

The first order condition for an optimal sharing rule of investor i is

$$v_i'(g_i(X) - \Psi_i(X, \sigma_i)) - \lambda_i \Phi(X)$$

Take logarithms of this equation, then differentiating with respect to X , we have

$$\frac{dg_i(X)}{d(X)} - \hat{e}_i(X) \frac{d \ln \Phi(X)}{dX} ; \quad \forall X ,$$

where $\hat{a}_i(X) = 1/\hat{e}_i(X)$ is defined in lemma 1.

Then, aggregation over all investors yields

$$1 = - \frac{d \ln \Phi(X)}{dX} \sum_i \hat{e}_i(X) .$$

An increase in idiosyncratic risk affects both factors on the right hand side of the equation in an offsetting manner: an increase in idiosyncratic risk across many

agents decreases $\frac{d \ln \Phi(X)}{dX}$ if and only if it decreases $\sum_i \hat{e}_i(X)$. In other

words, $d^2 \ln \Phi(X) / dX d\sigma < 0 \quad \forall X$ means that the growth rate of $\Phi(X_t)$ is less than that of $\Phi(X_s)$ for every pair (s,t) with $X_s < X_t$. $d^2 \ln \Phi(X) / dX d\sigma < 0 \quad \forall X$

is true if and only if $d \sum_i \hat{e}_i(X) / d\sigma < 0 \quad \forall X$. Hence, the proposition follows. ■

The intuition behind proposition 5 is straightforward. Since the excess demand in state s is higher than that in state t , the price relative Φ_s / Φ_t must change in order to make these excess demands disappear. Clearly, we would expect the price relative to increase. But price changes can have various feedback effects, for example, on the agents' initial endowments. Moreover, the price change of one state interacts with the price changes in other states. Therefore these feedback effects need to be constrained in order to get a unique answer

[Deaton/Muellbauer (1981)].

The necessary and sufficient condition established in the proposition is that *aggregate* derived risk tolerance is a decreasing function of idiosyncratic risk. This condition is quite plausible since, without trade and without endowment changes, an increase in an agent's idiosyncratic risk raises the precautionary premium and, thus, the derived risk aversion. Hence the derived risk tolerance of the agent is reduced. If all agents are "reasonably" similar in terms of endowments, risk aversion and idiosyncratic risk, then trade and endowment effects cannot overturn this. But if agents are very different in terms of endowments or idiosyncratic risk, it is possible that after trade, aggregate derived risk tolerance might increase in some states. This possibility is ruled out by the condition that aggregate derived risk tolerance decreases in every state.

3 FURTHER RESULTS IN THE HARA-CASE

In this section, we assume that agents have utility functions belonging to the class of functions with hyperbolic absolute risk aversion (HARA-class). This allows us to derive more specific results and to provide some illustration for the propositions of the previous section. As in the general case, we first derive some additional properties of the precautionary premium.

A PROPERTIES OF THE PRECAUTIONARY PREMIUM FOR HARA UTILITY

The HARA-class of utility functions is defined by

$$v(y) - \frac{1-\gamma}{\gamma} \left(A + \frac{y}{1-\gamma} \right)^\gamma > 0$$

with A being a constant and $A + y/(1-\gamma) > 0$.¹⁰ Then

$$v'(y) - \left(A + \frac{y}{1-\gamma} \right)^{\gamma-1} > 0 \quad ,$$

$$a(y) - \left(A + \frac{y}{1-\gamma} \right)^{-1} \quad ,$$

$$\eta(y) - \frac{\gamma-2}{\gamma-1} \left(A + \frac{y}{1-\gamma} \right)^{-1} - \frac{\gamma-2}{\gamma-1} a(y).$$

If $1 > \gamma > -\infty$, then $\eta(y) > 0$ and $\eta'(y) < 0$. Then, as shown by Pratt and Zeckhauser, the utility function exhibits proper risk aversion. $\gamma > 1$ implies utility functions with negative marginal utility for high values of y , and thus, is irrelevant for our analysis.

For the HARA-class, stronger results hold than for the class of all utility functions with $\eta(y) > 0$ and $\eta'(y) < 0$. For the HARA-class with $1 > \gamma > -\infty$, the precautionary premium $\Psi(x, \sigma)$ has the following properties besides $\frac{\partial \Psi}{\partial x} < 0$ (for the proof see appendix B):

$$\frac{\partial^2 \Psi}{\partial x^2} > 0 \quad ,$$

¹⁰ This constraint can be satisfied only if ε is bounded from below.

$$\frac{\partial^2 \Psi}{\partial \sigma \partial x} < 0 \quad ,$$

$$\frac{\partial^3 \Psi}{\partial \sigma \partial x^2} > 0 \quad .$$

The first property says that the precautionary premium is a decreasing, convex function in x . The second and the third properties characterize the effects of an increase in the idiosyncratic risk on the precautionary premium. The increase in the premium is the smaller, the higher the marketable income x . Moreover, the convexity of the premium in x increases with the idiosyncratic risk. This is significant when we consider optimal sharing rules for the HARA-class in section 4. Figure 1 illustrates these properties.¹¹

¹¹ For exponential utility $\gamma = -\infty$, absolute risk aversion and absolute prudence are constant. Hence, the precautionary premium is independent of x .

For $\gamma=2$ (quadratic utility), absolute prudence is zero and, therefore, the precautionary premium is zero. Generally speaking, idiosyncratic risk which is independent of the marketable risk, X , is completely irrelevant for the sharing rule in a mean-variance world. This follows from the derivation of efficient sharing rules. The objective is to minimize the variance. In this objective function idiosyncratic risk is an additive constant and, thus, cannot affect the efficient sharing rule. Therefore, mean-variance theory cannot capture the effects of idiosyncratic risk as discussed in this paper.

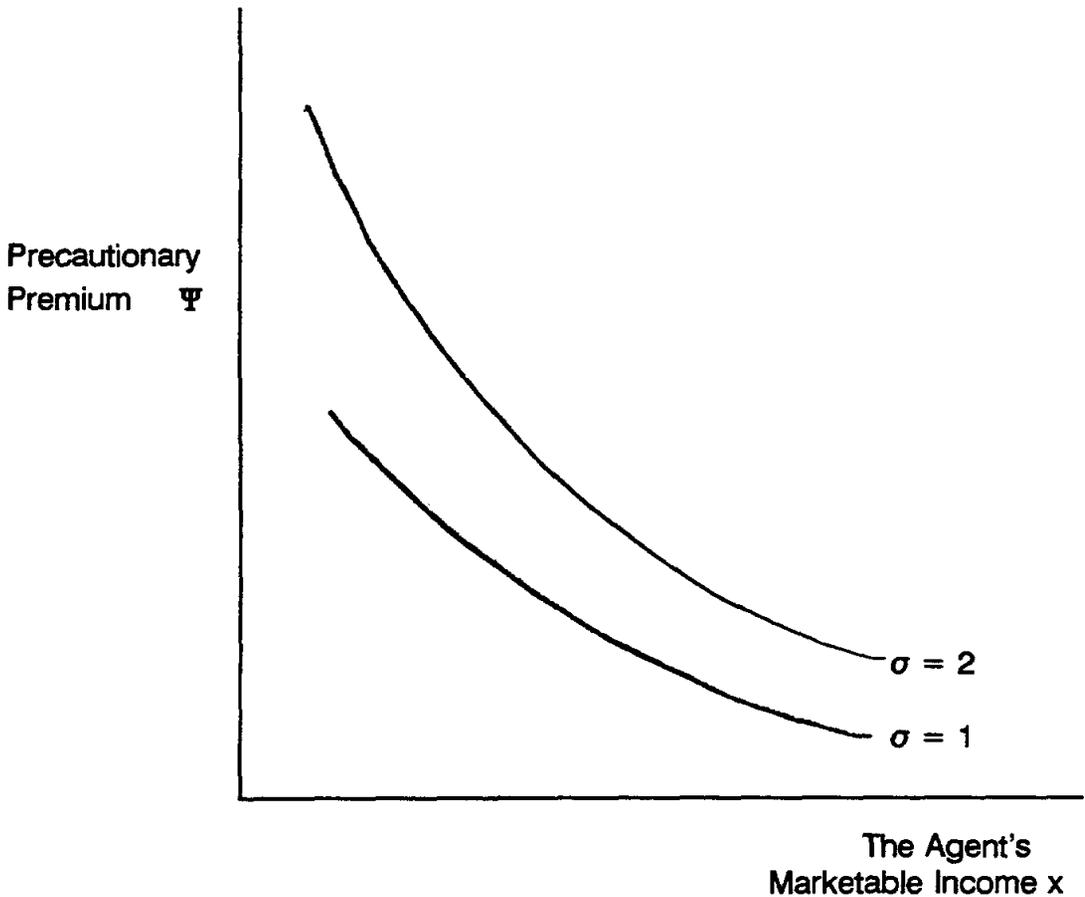


Figure 1: The precautionary premium is a decreasing, convex function of the agent's marketable income x if the utility function belongs to the HARA-class with $-\infty < \gamma < 1$.

A further result is important in the subsequent analysis. It follows from the monotonicity of the HARA-functions. If idiosyncratic risk grows by the factor q and x changes from x_0 to x_1 such that $q[A(1-\gamma)+x_0] = A(1-\gamma)+x_1$, then the precautionary premium grows by the factor q , $\Psi(x_1, q\sigma) = q\Psi(x_0, \sigma)$. This is also proved in Appendix C.

B IDIOSYNCRATIC RISK AND THE OPTIMAL SHARING RULE

In this section we investigate the optimal sharing rules of agents for the HARA-case. Consider, the first order condition (3) for the optimal sharing rule and insert the HARA-marginal utility,

$$\left(A + \frac{g(X) - \Psi(X, \sigma)}{1 - \gamma} \right)^{\gamma-1} = \lambda \Phi(X) ; \forall X .$$

Taking roots yields

$$x - \Psi(X, \sigma) = g(X) - \Psi(g(X), \sigma) = \left(\lambda^{\frac{1}{\gamma-1}} (\Phi(X))^{\frac{1}{\gamma-1}} - A \right) (1 - \gamma) ; \forall X . \quad (9)$$

As in proposition 1, we now investigate two questions. First, consider two agents with the same γ and the same tightness of the budget constraint (i.e. λ is the same for both), but the constant A need not be the same. Suppose that idiosyncratic risk is higher for the second agent ($\sigma_2 > \sigma_1$). Then equation (8) implies that the convexity or concavity of $[x - \Psi(x, \sigma)]$ measured by the second derivative with respect to X , is the same for both agents since $\lambda \Phi(X)$ is the same for both. In section A, we saw already that the precautionary premium $\Psi(x, \sigma)$ is convex and the convexity increases with σ . Hence the convexity of $\Psi_2(x, \sigma_2)$ exceeds that of $\Psi_1(x, \sigma_1)$ which implies that the convexity of $g_2(X)$ exceeds that of $g_1(X)$ since the convexity of $g(X) - \Psi(X, \sigma)$ is the same for both.

Second, we consider one agent only and ask the question how his sharing rule changes when his idiosyncratic risk increases. The difference between the

preceding question and this question is that now λ changes, too. It has been

shown before that $d\lambda/d\sigma > 0$ and hence $d\lambda^{\frac{1}{\gamma-1}}/d\sigma < 0$. Hence from

equation (8) it follows that $d^2g(X)/dX^2$ increases due to the increase in λ if

$[\Phi(X)]^{\frac{1}{\gamma-1}}$ is concave, i.e. $\Phi(X)$ is convex. This establishes

Proposition 6:

a) Consider two agents with HARA-utility functions such that λ is the same for both ($1 > \gamma > -\infty$) and the tightness of the budget constraint, λ , is the same for both.

Then the convexity of the sharing rule, i.e. the second derivative of the sharing rule with respect to the aggregate marketable income, X , is higher for the agent with the higher idiosyncratic risk, measured by σ .

b) Consider an agent with HARA-utility. Then the convexity of his sharing rule increases with his idiosyncratic risk if the pricing function $\Phi(X)$ is convex. ■

This is an interesting result since it says that a higher idiosyncratic risk not only implies portfolio adjustment as stated in proposition 1, but also greater convexity of the sharing rule.¹²

¹² This relates our findings to Leland's (1980) who defines portfolio insurance as convexity of the sharing rule. Note, however, that we do not prove convexity of the sharing rule. Proposition 3 shows that higher idiosyncratic risk raises the convexity although the sharing rule need not be convex. For a related discussion of portfolio insurance see also Brennan and Solanki (1981) and Benninga and Blume (1985).

Although proposition 6 relates the increase in convexity of the sharing rule to the increase in idiosyncratic risk, it remains open whether the sharing rule is convex, concave or of a mixed nature. More specific results can be obtained in a market equilibrium where every agent has a HARA-utility function such that is the same for every agent and all agents share the same expectations. It is well known from the work of Cass and Stiglitz (1970) and Rubinstein (1974) that, under these assumptions and in the absence of idiosyncratic risk, all agents have a linear sharing rule since the HARA-class implies separation, i.e. the sharing rules of different agents differ only through their level and their constant slope. In the following, we investigate the effects of idiosyncratic risk on the sharing rule. The main result is contained in proposition 4.

Proposition 7:

Assume that every agent has a utility function belonging to the HARA-class with $1 > \gamma > -\infty$ and is the same for every agent. Moreover, assume homogeneous expectations.

a) Then investor i 's sharing rule in equilibrium is

$$g_i(X) = [\alpha_i A - A_i](1 - \gamma) + \alpha_i X + [\Psi_i(X, \sigma_i) - \alpha_i \Psi(X)] \quad ; \quad \forall i; \quad (10)$$

with

$$\alpha_i = \lambda_i \frac{1}{r-1} / \sum_j \lambda_j \frac{1}{r-1}, \text{ so that } \sum_i \alpha_i = 1,$$

$$A = \sum_i A_i ; \quad \Psi(X) = \sum_i \Psi_i(X, \sigma_i) ,$$

b) the pricing function $\Phi(X)$ is a decreasing, convex function, i.e.

$$d\Phi(X) / dX < 0 , \quad d^2 \Phi(X) / dX^2 > 0 ,$$

c) the aggregate precautionary premium $\Psi(X)$ is a decreasing, convex function with $\Psi(X) \rightarrow 0$ for $X \rightarrow \infty$,

d) the agent's standardized "tightness" of the budget constraint, α_i is a function of A_i, σ_i, w_i such that

$$\alpha_i > 0, \quad \frac{\partial \alpha_i}{\partial \sigma_i} < 0, \quad \frac{\partial \alpha_i}{\partial A_i} > 0, \quad \frac{\partial \alpha_i}{\partial w_i} > 0 \quad \blacklozenge$$

Proof:

See Appendix D.

Proposition 7 says that every agent's sharing rule is composed of three elements, the risk-free asset, a constant fraction of the marketable aggregate income X plus a nonlinear term. This generalizes Rubinstein's results (1974). If no idiosyncratic risk exists in the economy, then the last term is zero so that a linear sharing rule follows. With idiosyncratic risk, the investor without idiosyncratic risk has a strictly concave sharing rule. Thus, if there are only two agents, one of whom has no idiosyncratic risk, then the agent with idiosyncratic risk must have a strictly convex

sharing rule.¹³

The sharing rule (10) also reveals that for high values of marketable aggregate

¹³ One might conjecture that under appropriate conditions there exists an agent with a linear sharing rule. This is very doubtful, however. The following example shows a situation in which such an agent cannot exist. There exist three agents. Agent 1 has no idiosyncratic risk. The other two agents have small idiosyncratic risks so that

$$\Psi_i(x_i, \sigma_i) = (1/2) \eta_i(x_i) \sigma_i^2; \quad i = 2, 3.$$

Now suppose that agent 2 has a linear sharing rule. Then

$$\Psi_2(x_2, \sigma_2) = \alpha_2 (\Psi_2(x_2, \sigma_2) + \Psi_3(x_3, \sigma_3))$$

follows from his sharing rule, or

$$\Psi_2(x_2, \sigma_2) (1 - \alpha_2) = \alpha_2 \Psi_3(x_3, \sigma_3).$$

For small risks it follows

$$\eta_2(x_2) \sigma_2^2 (1 - \alpha_2) = \alpha_2 \eta_3(x_3) \sigma_3^2.$$

In the HARA-case, this yields

$$\frac{\sigma_2^2 (1 - \alpha_2)}{A_2 (1 - \gamma) + x_2} = \frac{\alpha_2 \sigma_3^2}{A_3 (1 - \gamma) + x_3},$$

so that x_3 is linear in x_2 . Hence linearity of $x_2 = g_2(X)$ implies linearity of $x_3 = g_3(X)$. But then agent 1 must also have a linear sharing rule in equilibrium which contradicts proposition 7. Therefore, in this example a representative investor, i.e. an agent with a linear sharing rule, cannot exist.

income, all sharing rules become approximately linear, since the precautionary premia approach 0. This means that an agent who purchases many claims in the low states because of high idiosyncratic risk pays for these claims by accepting a relatively low slope (α_i) of his sharing rule in the high states as evidenced by $\partial\alpha_i/\partial\sigma_i < 0$.

Comparing two agents i and j who differ only in their constants A_i and A_j such that $A_i < A_j$ means that agent j is less risk averse and less prudent. Hence, agent j is affected less by idiosyncratic risk. The consequence is that agent j buys a larger share α of marketable aggregate income X as indicated by $\partial\alpha/\partial A > 0$. As a consequence, agent j invests less in purchasing other claims.

Finally, if the two agents differ only in their initial endowments such that $w_i < w_j$, then agent j buys more of the risk-free asset and of the marketable aggregate income as evidenced by $\partial\alpha/\partial w > 0$. In addition, agent j is less risk averse and less prudent so that the sharing rule adjustment for own idiosyncratic risk is reduced and in consequence this agent is more inclined to sell claims in the low states.

C PRICING EFFECTS OF CHANGES IN IDIOSYNCRATIC RISK

The effect of an aggregate shock that changes the idiosyncratic risk of many agents in the economy can be studied more precisely for the HARA-class of preferences. Since we now have a specific expression for the pricing function $\Phi(X)$, we can provide a more intuitive, sufficient condition for the changes in relative prices of state contingent claims.

Proposition 8:

Under the assumptions of proposition 4, an increase in idiosyncratic risk for many agents raises the price relative Φ_s/Φ_t for any states s and t with $X_s < X_t$ if it raises the aggregate precautionary premium at least as much in state s as in state t ♦

Proof:

In the aggregate, we have

$$\frac{A + \frac{X_s - \Psi(X_s)}{1 - \gamma}}{A + \frac{X_t - \Psi(X_t)}{1 - \gamma}} = \left[\frac{\Phi(X_t)}{\Phi(X_s)} \right]^{\frac{1}{1 - \gamma}}$$

Hence, if $\Psi(X_s)$ increases at least as much as $\Psi(X_t)$ with idiosyncratic risk, then the left hand side decreases, and, therefore, $\Phi(X_t)/\Phi(X_s)$ does ■

It is instructive to compare proposition 5 and proposition 8. Both are similar in spirit although proposition 5 gives a necessary and sufficient condition whereas proposition 8 gives a sufficient condition.

The condition in proposition 8 is that "Benthamite" welfare¹⁴, measured by $X - \Psi(X)$, is at least as much reduced by the increase in idiosyncratic risk in state s as in state t. Again, this condition appears to be innocuous as long as there are no large differences among investors in terms of endowments, risk attitudes as determined by A and idiosyncratic risks.

¹⁴ This is the modified Benthamite criterium of social welfare with compensation across agents based upon the precautionary premium.

4 CONCLUSIONS

The general approach in the paper has been to consider the optimal decision making by an agent who faces idiosyncratic risk but can buy/sell state-contingent claims on aggregate marketable income. This framework has several applications to problems in financial economics. The common feature of all these applications is that there are some risks that are non-marketable, but the agent is able to manage the overall risk through other tradeable claims.

Consider the case of the owner of a firm whose shares are not traded, but whose cash flow is dependent on several economy-wide variables such as interest rates, foreign exchange rates and commodity prices. In addition to these economy-wide risks, the cash flow of the firm is also affected by firm-specific factors which cannot be perfectly hedged. The question is how the entrepreneur should optimally hedge against the economy-wide risks, given the exposure to idiosyncratic risk.

An additional dimension can be introduced into this example by considering the behavior of the manager of a firm whose compensation is based on the cash flow of the firm as well as the cash flows of competing firms. The manager can alter the cash flow of the firm by buying claims on the marketable cash flow in the economy, but he is prohibited from trading in the firm's shares (no insider trading). The manager of the firm, therefore, chooses the firm's hedging policy to maximize his own expected utility given the risks of the firm's cash flows.

In all the above cases, idiosyncratic risk induces the agent to buy a type of insurance to optimize risk bearing: the insurance involves the purchase of claims

that pay off in "low" states of aggregate marketable income financed by the sale of claims that pay off in the "high" states. However, this type of insurance differs from conventional insurance in an important sense. Conventional insurance is defined as the purchase of a perfectly negatively correlated risk. Here, insurance means the purchase of claims so as to balance the effects of some independent risk on conditional expected marginal utility.

Idiosyncratic risk lowers the agent's risk taken in the market. When idiosyncratic risk increases for many agents, then prices of marketable claims must adjust. Under a mild condition, prices of claims in the "low" states of aggregate marketable income increase relative to the prices of claims in the "high" states. This price change motivates agents whose idiosyncratic risk has not increased, to take more risk in the market.

When all agents have utility functions of the HARA class with the same exponent, then agents without idiosyncratic risk have a concave sharing rule. But the other agents' sharing rules need not be linear, convex or concave. The convexity of an agent's sharing rule increases, however, with his idiosyncratic risk.

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APPENDIX A

Proof of Lemma 3

Lemma 3

$$\frac{\partial \hat{a}(x)}{\partial x} - \frac{\partial^2 \ln v'(x - \Psi)}{\partial x^2} < 0, \quad \text{iff } \frac{da}{dx} < 0, \quad (\text{A.1})$$

Proof:

$$\text{a) Sufficiency: } \frac{da}{dx} < 0 \quad \rightarrow \quad \frac{\partial \hat{a}(x)}{\partial x} < 0.$$

Inequality (A.1) is equivalent to

$$\frac{\partial \left(\frac{\partial v'(\cdot)/\partial x}{v'(\cdot)} \right)}{\partial x} > 0. \quad (\text{A.2})$$

where $v'(\cdot)$ is defined by

$$v'(\cdot) = v' [x - \Psi(x, \sigma)] = E_x[v'(x + \sigma \epsilon)] \quad (\text{A.3})$$

Hence

$$\frac{\partial v'(\cdot)/\partial x}{v'(\cdot)} = \frac{E[v''(x + \sigma \epsilon)]}{E[v'(x + \sigma \epsilon)]} \quad (\text{A.4})$$

where the expectation is conditional on x , but the subscript is dropped for notational simplicity.

In order to prove the sufficiency part of lemma 3, we need to show that the term in equation (A.4) strictly increases with x . We differentiate the fraction in equation (A.4) with respect to x .

This differential is positive if and only if

$$E[v'(\cdot)] E[v'''(\cdot)] > [E[v''(\cdot)]]^2 \quad (\text{A.5})$$

Decreasing absolute risk aversion [$da/dy < 0$] implies that

$$[v'''(\cdot)] [v'(\cdot)] > [v''(\cdot)]^2 \quad (\text{A.6})$$

Define

$$\hat{v}'(\cdot) = \frac{[v''(\cdot)]^2}{v'''(\cdot)} \quad (\text{A.7})$$

Then, from (A.6) and (A.7),

$$\hat{v}'(\cdot) < v'(\cdot) \quad (\text{A.8})$$

Hence, a sufficient condition for (A.5) to hold is

$$E[\hat{v}'(\cdot)] E[v'''(\cdot)] \geq [E[v''(\cdot)]]^2 - [E[\hat{v}'(\cdot) v'''(\cdot)]]^2 \quad (\text{A.9})$$

Cauchy's inequality implies that (A.9) is true since $\hat{v}'(\cdot)$ and $v'''(\cdot)$ are positive.

b) *Necessity:*
$$\frac{\partial \hat{a}(x)}{\partial x} < 0 \quad - \quad \frac{da}{dx} < 0 .$$

Consider a very small risk. Then $\hat{a}(x) \rightarrow a(x)$ for all levels of x .

Hence, $\frac{\partial \hat{a}(x)}{\partial x} < 0 \rightarrow \frac{da}{dx} \leq 0$, . However, $\frac{da}{dx} = 0$ in the neighborhood

of x leads to a contradiction, since it implies exponential utility, for which

$\hat{a}(x) - a(x) = \text{constant}$, so that $\frac{\partial \hat{a}(x)}{\partial x} < 0$ is impossible. Hence, $\frac{da}{dx} < 0$

is required. ■

APPENDIX B

Proof of Lemma 5

Lemma 5

$$\frac{\partial \left[- \frac{\partial \ln v'(x - \Psi) / \partial \sigma}{\partial \ln v'(x - \Psi) / \partial x} \right]}{\partial x} < 0, \quad \text{iff} \quad \frac{d\eta}{dy} < 0. \quad (\text{B.1})$$

Proof:

Inequality (B.1) is equivalent to

$$\frac{\partial \left[\frac{\partial v'(\cdot) / \partial \sigma}{\partial v'(\cdot) / \partial x} \right]}{\partial x} > 0 \quad (\text{B.2})$$

Using the definition of the precautionary premium as in (A.3), the term within square brackets in inequality (B.2) is equal to

$$\frac{E[v''(x + \sigma \varepsilon) \varepsilon]}{E[v''(x + \sigma \varepsilon)]}$$

Differentiate the above term with respect to x . The differential is positive if and only if

$$E[v''(x + \sigma \varepsilon)] E[v'''(x + \sigma \varepsilon) \varepsilon] > E[v''(x + \sigma \varepsilon) \varepsilon] E[v'''(x + \sigma \varepsilon)] \quad (\text{B.3})$$

which is the same as

$$\frac{E[v'''(x+\sigma\varepsilon)\varepsilon]}{E[v'''(x+\sigma\varepsilon)]} < \frac{E[v''(x+\sigma\varepsilon)\varepsilon]}{E[v''(x+\sigma\varepsilon)]} \quad (\text{B.4})$$

since $E[v''(x+\sigma\varepsilon)] < 0$ and $E[v'''(x+\sigma\varepsilon)] > 0$, by assumptions 2 and 3 respectively.

Decreasing absolute prudence is a necessary and sufficient condition for equation (B.4) to hold. In order to see this, consider the analogous problem where the following condition is valid if and only if the agent exhibits decreasing absolute risk aversion:

$$\frac{E[v''(x+\sigma\varepsilon)\varepsilon]}{E[v''(x+\sigma\varepsilon)]} < \frac{E[v'(x+\sigma\varepsilon)\varepsilon]}{E[v'(x+\sigma\varepsilon)]} \quad (\text{B.5})$$

Consider an investor facing the choice between a riskless and a risky asset, where the excess return on the risky asset is equal to $\mu + \varepsilon$, and μ is the expected excess return of the risky asset over the riskless rate. Then, if σ denotes the optimal dollar investment in the risky asset, the optimality condition is that the right hand side of inequality (B.5) equals $-\mu$, with x being the expected total income. If the initial wealth increases, there is a higher dollar investment in the risky asset if and only if the agent shows decreasing absolute risk aversion [Theorem 7 of Pratt (1964)]. Hence, the right hand side of inequality (B.5) evaluated at the new level of wealth is higher if and only if the agent exhibits decreasing absolute risk aversion. By analogy, inequality (B.4) holds if and only if absolute prudence is decreasing. ■

Appendix C

Properties of the Precautionary Premium for the HARA Class of

Preferences with $\gamma < 1$

1) Consider the case of $\gamma = -\infty$, where the agents have exponential utility. Then, the absolute risk aversion and the absolute prudence are constants, so that the risk premium and the precautionary premium are independent of marketable income x .

2) Assume that $1 > \gamma > -\infty$. We now prove

$$\frac{\partial^2 \Psi(x, \sigma)}{\partial x \partial \sigma} < 0 \quad . \quad (C.1)$$

Differentiate the definitional equation

$$v'[x - \Psi(x, \sigma)] = E[v'(x + \sigma \varepsilon)] \quad (C.2)$$

with respect to σ and obtain

$$\begin{aligned} \frac{\partial \Psi(x, \sigma)}{\partial \sigma} &= \frac{E[v''(x + \sigma \varepsilon) \varepsilon]}{-v''[x - \Psi(x, \sigma)]} \\ &= \frac{E[v''(x + \sigma \varepsilon) \varepsilon]}{E[-v''(x + \sigma \varepsilon)]} \cdot \frac{E[-v''(x + \sigma \varepsilon)]}{-v''[x - \Psi(x, \sigma)]} \end{aligned} \quad (C.3)$$

The second term on the right hand side of equation (C.3) is positive, given the

assumption of risk aversion. Since the left hand side is positive, both fractions on the right hand side of (C.3) are positive.

As shown in the proof of lemma 5, the first fraction decreases as x increases. Hence, a sufficient condition for inequality (C.1) to hold is that the second fraction decreases as x increases.

Define

$$v''[x - \varphi(x, \sigma)] - E[v''(x + \sigma \varepsilon)] \quad (\text{C.4})$$

where φ is the premium defined by the second derivative of the utility function. $[\pi$ is the premium defined by the utility function (risk premium) and Ψ is the premium defined by the first derivative (precautionary premium)]. Then, the second fraction in (C.3) can be rewritten as

$$\frac{E[-v''(x + \sigma \varepsilon)]}{-v''[x - \Psi(x, \sigma)]} - \frac{-v''[x - \varphi(x, \sigma)]}{-v''[x - \Psi(x, \sigma)]} \quad (\text{C.5})$$

For the HARA class of preferences, the right hand side of (C.5) can be written as

$$\frac{-v''[x - \varphi(x, \sigma)]}{-v''[x - \Psi(x, \sigma)]} - \left(\frac{A + \frac{x - \varphi(x, \sigma)}{1 - \gamma}}{A + \frac{x - \Psi(x, \sigma)}{1 - \gamma}} \right)^{\gamma - 2} \quad (\text{C.6})$$

Differentiate the right hand side of (B.6) with respect to x . The differential is negative (since $\gamma < 1$), if

$$\left[A + \frac{x - \varphi}{1 - \gamma} \right]^{-1} \left(1 - \frac{\partial \varphi}{\partial x} \right) > \left[A + \frac{x - \Psi}{1 - \gamma} \right]^{-1} \left(1 - \frac{\partial \Psi}{\partial x} \right) \quad (\text{C.7})$$

We substitute for $\frac{\partial \varphi}{\partial x}$ and $\frac{\partial \Psi}{\partial x}$ by differentiating (C.2) and (C.4) to obtain

$$\left[1 - \frac{\partial \Psi(x, \sigma)}{\partial x} \right] = \frac{E[v''(x + \sigma \varepsilon)]}{v''[x - \Psi(x, \sigma)]} \quad (\text{C.8a})$$

$$\left[1 - \frac{\partial \varphi(x, \sigma)}{\partial x} \right] = \frac{E[v'''(x + \sigma \varepsilon)]}{v'''[x - \varphi(x, \sigma)]} \quad (\text{C.8b})$$

We substitute (C.8a) and (C.8b) in (C.7) to yield

$$\frac{E \left[\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma} \right)^{\gamma - 3} \right]}{\left[A + \frac{x - \varphi}{1 - \gamma} \right]^{\gamma - 2}} > \frac{E \left[\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma} \right)^{\gamma - 2} \right]}{\left[A + \frac{x - \Psi}{1 - \gamma} \right]^{\gamma - 1}} \quad (\text{C.9})$$

Substitute for the denominators in the two sides of the inequality from (C.2) and (C.4) and obtain

$$E \left[\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma} \right)^{\gamma - 3} \right] E \left[\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma} \right)^{\gamma - 1} \right] > \left[E \left\{ \left(A + \frac{x + \sigma \varepsilon}{1 - \gamma} \right)^{\gamma - 2} \right\} \right]^2 \quad (\text{C.10})$$

Since

$$\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma}\right)^{\gamma - 3} \left(A + \frac{x + \sigma \varepsilon}{1 - \gamma}\right)^{\gamma - 1} - \left[\left(A + \frac{x + \sigma \varepsilon}{1 - \gamma}\right)^{\gamma - 2}\right]^2, \quad (\text{C.11})$$

it follows from Cauchy's inequality that (B.10) holds. Hence (B.1) follows.

3) Now we prove the convexity of Ψ relative to x . From equation (B.8a), it follows that

$$\frac{\partial^2 \Psi(x, \sigma)}{\partial x^2} > 0 \quad (\text{C.12})$$

if and only if the right-hand side in equation (B.8a) decreases as x increases. We have already shown this to be true in equations (B.5) through (B.11).

4) Lastly, we show that

$$\frac{\partial^3 \Psi}{\partial \sigma \partial x^2} > 0 \quad (\text{C.13})$$

First, note that convexity of Ψ approaches 0 as $\sigma \rightarrow 0$. Since Ψ is convex for any positive value of σ , it follows that convexity increases with σ for small changes from $\sigma=0$. We now use a monotonicity result to show that convexity increases with σ for any value of σ . We rewrite equation (C.2) for the HARA class and

multiply throughout by $\left[\frac{(1-\gamma)}{\sigma}\right]^{\gamma-1}$ to obtain

$$\left[\frac{[A(1-\gamma)+x]}{\sigma} - \frac{\Psi(x, \sigma)}{\sigma}\right]^{\gamma-1} - E\left[\left(\frac{[A(1-\gamma)+x]_{+\varepsilon}}{\sigma}\right)^{\gamma-1}\right] \quad (\text{C.14})$$

Multiply and divide equation (C.14) throughout by q , where $q > 0$, to yield

$$\left[\frac{q[A(1-\gamma)+x]}{q\sigma} - \frac{q\Psi(x,\sigma)}{q\sigma} \right]^{\gamma-1} = E \left[\left(\frac{q[A(1-\gamma)+x]}{q\sigma} + \varepsilon \right)^{\gamma-1} \right] \quad (\text{C.15})$$

Define x_1 such that

$$q[A(1-\gamma)+x_0] = A(1-\gamma)+x_1 .$$

Then, using subscript 0 for x in equation (C.15) yields

$$\left[\frac{[A(1-\gamma)+x_1]}{q\sigma} - \frac{q\Psi(x_0,\sigma)}{q\sigma} \right]^{\gamma-1} = E \left[\left(\frac{q[A(1-\gamma)+x_0]}{q\sigma} + \varepsilon \right)^{\gamma-1} \right] \quad (\text{C.16})$$

In words, if σ changes from σ to $q\sigma$ and x changes from x_0 to x_1 , then the new precautionary premium $\Psi(x_1, q\sigma) - q\Psi(x_0, \sigma)$.

In order to show that the convexity of Ψ grows with σ , suppose that σ is raised from a level arbitrarily close to 0. Then, the convexity of $\Psi(x_0, \sigma)$ increases. Hence, the convexity of $\Psi(x_1, q\sigma)$ increases by the factor q . As q can be arbitrarily large, the convexity of $\Psi(x_1, q\sigma)$ increases monotonically with $q\sigma$. ■

Appendix D

Proof of Proposition 7

a) Aggregation of the individual agents' sharing rules (9) yields

$$X - \sum_i \Psi_i(X, \sigma_i) = \left[\left(\sum_i \lambda_i^{\frac{1}{\gamma-1}} \right) \Phi(X)^{\frac{1}{\gamma-1}} - \sum_i A_i \right] (1-\gamma) \quad (\text{D.1})$$

or, substituting for $\Phi(X)$ and simplifying the sharing rule for agent j yields

$$x_j - \Psi_j(X, \sigma_j) = \frac{\lambda_j^{\frac{1}{\gamma-1}}}{\sum_i \lambda_i^{\frac{1}{\gamma-1}}} \left[\sum_i A_i (1-\gamma) + X - \sum_i \Psi_i(X, \sigma_i) \right] - A_j (1-\gamma) \quad (\text{D.2})$$

which, on simplification, and using $x_i = g_i(X)$, yields proposition 7 a.

b) Now we prove the convexity of the pricing function. Differentiating the sharing rule for agent j with respect to X yields

$$\frac{dg_j(X)}{dX} \left[1 - \frac{\partial \Psi_j}{\partial g_j(X)} \right] = (1-\gamma) \lambda_j^{\frac{1}{\gamma-1}} \frac{d[\Phi(X)]^{\frac{1}{\gamma-1}}}{dX} \quad (\text{D.3})$$

Aggregating the previous equation across all agents leads to

$$1 - \sum_i \frac{dg_i(X)}{dX} = (1-\gamma) \frac{d[\Phi(X)]^{\frac{1}{\gamma-1}}}{dX} \sum_i \lambda_i^{\frac{1}{\gamma-1}} \left[1 - \frac{\partial \Psi_i}{\partial g_i(X)} \right]^{-1} \quad (\text{D.4})$$

since the sum of the changes in the sharing rules across agents has to net to the change in the aggregate marketable income. Rearranging the above expression yields

$$\frac{1}{d\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]} = (1-\gamma) \sum_i \lambda_i^{\frac{1}{\gamma-1}} \left[1 - \frac{\partial \Psi_i}{\partial g_i(X)}\right]^{-1} \quad (\text{D.5})$$

Differentiating again with respect to X, we obtain

$$\begin{aligned} & \frac{-1}{\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]^2} \frac{d^2\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]}{dX^2} \\ & - (1-\gamma) \sum_i \lambda_i^{\frac{1}{\gamma-1}} \frac{\left[\frac{\partial^2 \Psi_i}{\partial g_i(X)^2}\right] \left[\frac{dg_i(X)}{dX}\right]}{\left[1 - \frac{\partial \Psi_i}{\partial g_i(X)}\right]^2} \end{aligned} \quad (\text{D.6})$$

Since $\frac{dg_i(X)}{dX} > 0$ Ψ_i is convex in $g_i(X)$ as shown in Appendix C, the right-hand side of the equation is positive. Hence, $\Phi(X)^{\frac{1}{\gamma-1}}$ is a concave function or $\Phi(X)$ is a convex function.

c) Rearranging equation (D.1) yields

$$A + \frac{X - \Psi(X, \sigma)}{1 - \gamma} - \left[\sum_i \lambda_i^{\frac{1}{\gamma-1}} \right] \Phi(X)^{\frac{1}{\gamma-1}} \quad (\text{D.7})$$

Hence strict convexity of $\Phi(X)$ [or concavity of $\Phi(X)^{\frac{1}{\gamma-1}}$] implies strict convexity of $\Psi(X)$.

d) In order to avoid obvious but tedious calculations, we provide only a verbal argument for the comparative statics of α_i . Note that the sign of the change in α_i is opposite to that of λ_i because of the negative exponent of λ_i in the definition of α_i . First, it has already been shown in the proof of Proposition 4 that $d\lambda/d\sigma > 0$. Second, an increase in A_i and w_i increases expected utility and, thus, reduces the increase in expected utility from an additional unit of endowment. ■