"RISK AVERSION AND PRUDENCE:
THE CASE OF MEAN-VARIANCE PREFERENCES"

by

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Risk Aversion and Prudence: The Case of Mean-Variance Preferences

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Abstract

This paper studies the concepts of risk aversion and prudence in the mean-variance model and relates them to the corresponding concepts in the expected-utility model. The concepts of risk premia and precautionary premia have their counterparts in this framework and can be represented graphically in an intuitive fashion. We show that there is an isomorphic relationship between the concept of prudence and the concept of risk aversion in the mean-variance framework. We define the conditions under which a utility function is more risk averse (more prudent) than another one and the conditions under which a utility function exhibits increasing, decreasing and constant risk aversion (prudence). In the particular case of mean-variance preferences derived from expected utility and normal distributions, we show that utility is concave as a function of variance and mean if and only if it exhibits decreasing prudence.

1 Introduction

The concept of risk aversion is fundamental in the expected-utility analysis of choices under risk. Since the pioneering works of Pratt (1964) and Arrow (1965), the Arrow-Pratt coefficient has been the standard measure of risk aversion. Building on the theory of risk aversion, Kimball (1990a) introduced the concept of "prudence," which measures the strength of the precautionary saving motive. The precautionary saving motive is the motive to save when confronted with an unavoidable uncertainty about future income. Kimball showed that the theory of prudence is isomorphic to the Arrow-Pratt theory of risk aversion as applied to minus the derivative of the utility function. The theory of prudence extends the works of Leland (1968), Sandmo (1970), and Drèze and Modigliani (1972) on precautionary saving under uncertainty. Examples of other applications of prudence are the extension to risky securities and multiple periods in Kimball (1990b), the studies of the effects of income taxes in Elmendorf and Kimball (1988) and Kimball and Mankiw (1989), and Weil's (1992) explanation of the equity premium puzzle of Mehra and Prescott (1985).

In this paper, we develop the theory of risk aversion and prudence in the mean-variance framework, where the investor has a utility function which is a function of standard deviation and mean of his future wealth or consumption. In this framework, the theory of prudence is isomorphic to the theory of risk aversion as applied to minus the derivative with respect to the mean of the utility function of standard deviation and mean.

Two particular results that we derive are the following. Suppose utility functions that are functions of variance and mean are derived from expected utility based on von Neumann-Morgenstern utility functions and a two-parameter familiy of distributions. We show that one von Neumann-Morgenstern utility function is more risk averse than another if and only if the same relation holds between the corresponding utility functions of variance and mean. This means that in order to verify the relation of "more risk averse," it suffices to check the decision makers' attitudes toward a limited two-parameter family of gambles. Secondly, we find an economic interpretation of the concavity of a utility function which is a function of variance and mean, in the case where distributions are normal: it turns out that concavity of this function is equivalent to decreasing prudence.

The assumption of mean-variance preferences has been important in elementary financial economics since the works of Markowitz (1952), Sharpe (1964), Lintner (1965) and Tobin (1957). While restrictive, this assumption has been popular because of the simple one-dimensional representation of risk by standard deviation. It allows us to study risk aversion and prudence in an intuitive manner. It allows simple graphical representations of the compensating and equivalent risk premia (and precautionary premia) and of the indifference curves of a utility function that exhibits increasing, decreasing or constant risk aversion.

We do not treat mean-variance behavior as a special case of expected utility maximization but as an alternative. Mean-variance preferences derived from expected utility are a special case of both. Epstein (1985) shows that within a non-expected utility framework, mean-variance utility is a consequence of certain behavioral assumptions reminiscent of decreasing absolute risk aversion.

We use the marginal rate of substitution between mean and standard deviation of return as the natural measure of risk aversion. This allows us to measure risk aversion at any level of initial risk, while the Arrow-Pratt coefficient measures risk aversion only for initial situations without risk. Thus, the mean-variance framework enables us easily to take into account the existence of "background" risk. Most studies in the expected utility framework have focused on the impact of the introduction of an additional source of risk while starting from a riskless situation. Only recently has "background" risk been considered in studies such as Pratt and Zeckhauser (1987), Elmendorf and Kimball (1988), and Kimball (1993).

In the case where mean-variance preferences are derived from expected utility, we show that the usual definitions of the relation of "more risk averse" and of decreasing, increasing and constant risk aversion based on the von Neumann-Morgenstern utility function are consistent with our definitions based on the utility function for standard deviation and mean. We also show this consistency for the relation of "more prudent" and for decreasing, increasing and constant prudence.

In the mean-variance model, an individual's (compensating) precautionary premium is the addition to wealth he requires in order to be willing to keep his consumption constant while the standard deviation of his future income goes up. One individual is more prudent than another if he always requires a higher precautionary premium when they face similar situations. An individual has decreasing prudence if his precautionary premium is a decreasing function of his wealth.

We identify a series of equivalent conditions for one individual to be more prudent than another. We then define increasing (decreasing, constant) prudence, and again we identify a series of equivalent conditions. Some of these conditions (in Proposition 11) compare the measure of prudence with the measure of risk aversion or compare the precautionary premia with the risk premia.

Decreasing prudence is considered more plausible than increasing prudence, like decreasing absolute risk aversion is more plausible than increasing absolute risk aversion.

Finally, we use the concept of prudence to address an old problem in mean-variance analysis: what is the economic interpretation of concavity of the utility function of variance and mean? Concavity of this function is a stronger requirement than concavity of the utility function of standard deviation and mean. For example, it implies that the portfolio selection problem always has an optimal solution. In the CAPM without a riskless asset, it also implies that there necessarily exists a satiation portfolio for the investor. Allingham (1991), in his study of existence of equilibrium in CAPM, assumed concavity of the utility function of variance and mean, whereas Nielsen (1990b, 1992) assumed concavity (or quasi-concavity) only of the utility function of standard deviation and mean. Chipman (1973) found a sufficient condition for the utility function of variance and mean to be concave in the expected-utility case with normal distributions, but he did not give an economic interpretation.

What we show is that in the case of normal distributions, the utility function of variance and mean is concave if and only if the preferences exhibit decreasing prudence.

The plan of the paper is the following. In Section 2, we set up the mean-variance framework.

In Section 3, we study the concept of risk aversion in this framework. We define and represent graphically the compensating as well as the equivalent risk premium. We define what it means for one utility function for standard deviation and mean to be more risk averse than another and identify a series of equivalent conditions. We then define increasing (decreasing, constant) risk aversion, and again we identify a series of equivalent conditions. Finally, we identify a sufficient condition for a utility function to exhibit increasing (decreasing, constant) risk aversion, which compares the compensating and the equivalent risk premia.

Section 4 links the mean-variance framework to expected utility.

The findings about risk aversion in Sections 3 and 4 are used and reinterpreted in terms of prudence in the following sections. The results about risk aversion are applied to minus the derivative of the utility function with respect to the mean.

In Section 5, we define and study the compensating and the the equivalent precautionary premia, the notion of one utility function for standard deviation and mean being more prudent than another, and the notion of increasing (decreasing, constant) prudence.

In Section 6, we link the mean-variance framework to the expected utility setting. We consider the special case of normal distribution and show the equivalence of decreasing prudence and concavity of the utility function for variance and mean.

2 Prerequisites

We consider a consumer or investor who bases his choices on a utility function $W(v, \mu)$, which is a function of the variance and mean of total portfolio return or total future wealth. It is defined for $v \geq 0$ and for all values of μ. The corresponding utility function for standard deviation and mean is

$$U(\sigma, \mu) = W(\sigma^2, \mu).$$

It is defined for $\sigma \geq 0$ and for all values of μ.

Assumption 1 *W is continuously differentiable (also at $v = 0$) with $W_v < 0$ and $W_\mu > 0$, and U is strictly quasi-concave.*

Continuous differentiability of W is not strictly necessary, but it is convenient, and little interesting generality is lost by maintaining Assumption 1. In fact, the main reason for introducing the function W instead of just working with U is the convenience of the differentiability assumption on W. It implies continuous differentiability of U, but continuous differentiability of U does not imply differentiability of W with respect to v, when $v = 0$.

Mean-variance behavior is consistent with expected utility maximization with general utility functions if the total returns follow the distributions described by Chamberlain (1983) and Owen and Rabinovitch (1983), which include the normal distributions. In the present paper, mean-variance behavior is treated not as a special case of expected utility maximization but as an alternative, normal distributions being a special case of both.

Concavity of U is a stronger assumption than convexity of the induced preferences in (σ, μ)-space. For example, concavity of U implies that all the indifference curves in (σ, μ)-space have the same asymptotic slope at large values of σ. If U results from expected utility based on normal distributions and a risk averse von Neumann–Morgenstern utility function, then U is concave.

Let $S(\sigma, \mu)$ denote the slope of the investor's indifference curve in (σ, μ)-space at (σ, μ). To define it formally, note that for each value b of U, there is a unique function I_b, defined on an interval of the form $[0, s)$, $0 < s \leq \infty$, such that the graph of I_b is the indifference curve $U(\sigma, \mu) = b$:

$$\{(\sigma, \mu) : \sigma \geq 0 \text{ and } U(\sigma, \mu) = b\} = \{(\sigma, I_b(\sigma)) : 0 \leq \sigma < s\}.$$

This follows from Assumption 1. Figure 1 illustrates a situation where s is finite: the indifference curve goes to infinity as σ approaches s from the left.

Since Assumption 1 ensures that U is continuously differentiable with $U_\mu > 0$,

$$S(\sigma, \mu) = -U_\sigma(\sigma, \mu)/U_\mu(\sigma, \mu).$$

Note that every value b of U equals $U(0, \mu)$ for some μ (this means that every indifference curve touches the vertical axis). This can be formally proved as follows (see Figure 2). Suppose $b = U(\sigma_0, \mu_0)$. Then I_b is defined at least on $[0, \sigma_0]$. Set $S = S(\sigma_0, \mu_0)$. Since preferred sets are convex, the indifference curve lies above its tangent: $I_b(\sigma) \geq \mu_0 + (\sigma - \sigma_0)S$ for $0 \leq \sigma \leq \sigma_0$. So,

$$U(0, \mu_0) \geq b \geq U(0, \mu_0 - S\sigma_0).$$

By continuity, there is some μ between $\mu_0 - S\sigma_0$ and μ_0 such that $b = U(0, \mu)$.

In order to ensure that $s = +\infty$, so that I_b is defined on the entire non-negative real line, for all values b of U, we need to make the following assumption about the utility function. The assumption rules out the situation illustrated in Figure 1. It will also ensure that the compensating risk premium, to be defined in the following section, always exists.

Assumption 2 *For every σ_0 and σ_1 with $\sigma_1 \geq \sigma_0 \geq 0$, and every μ_0, there exists μ_1 such that $U(\sigma_1, \mu_1) \geq U(\sigma_0, \mu_0)$.*

It is easily seen that given Assumption 1, Assumption 2 is equivalent to the requirement that for each $\sigma \geq 0$,

$$U(\sigma, \mu) \to \sup U \text{ as } \mu \to \infty$$

Proposition 1 *If $S(\sigma, \mu)$ is a decreasing function of μ, then Assumption 2 is satisfied.*

PROOF: See Figure 3. Let $S = S(\sigma_1, \mu_0)$ be the slope of the indifference curve through (σ_1, μ_0). Set $\mu_1 = \mu_0 + S\sigma_1$. Since $S(\sigma_1, \mu_1) \leq S$, the line segment

$$\{(\sigma, \mu_0 + S\sigma) : \sigma \leq \sigma_1\}$$

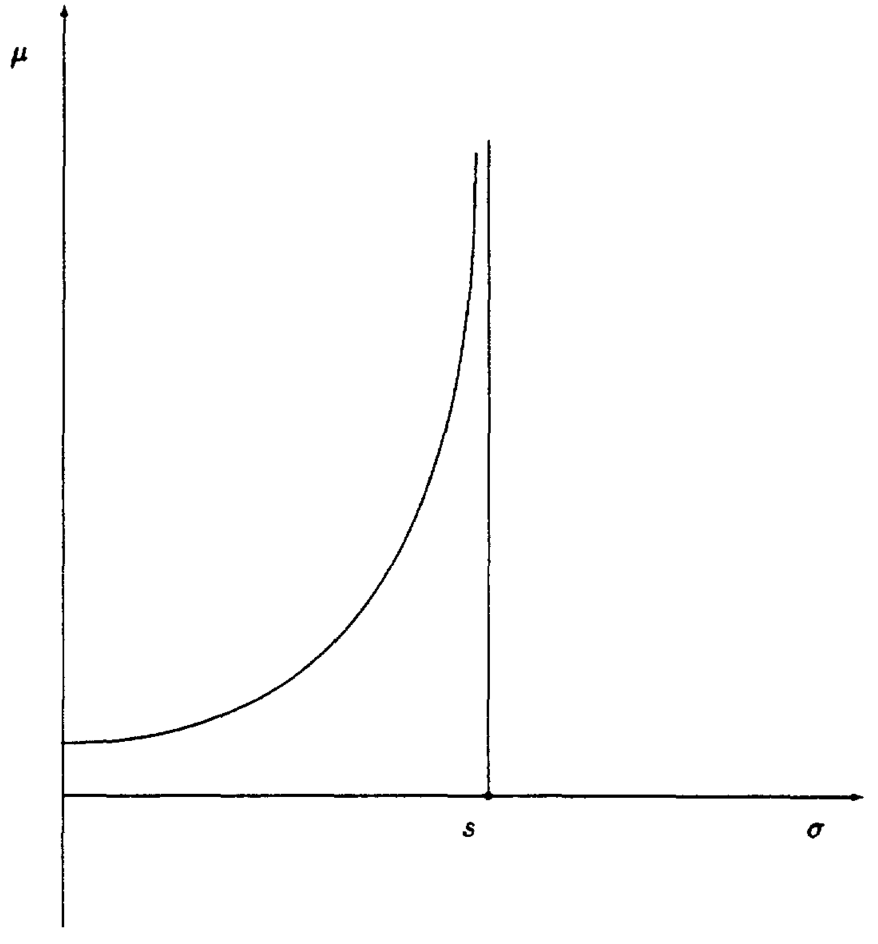

Figure 1: I_b defined only for $0 \leq \sigma < s$

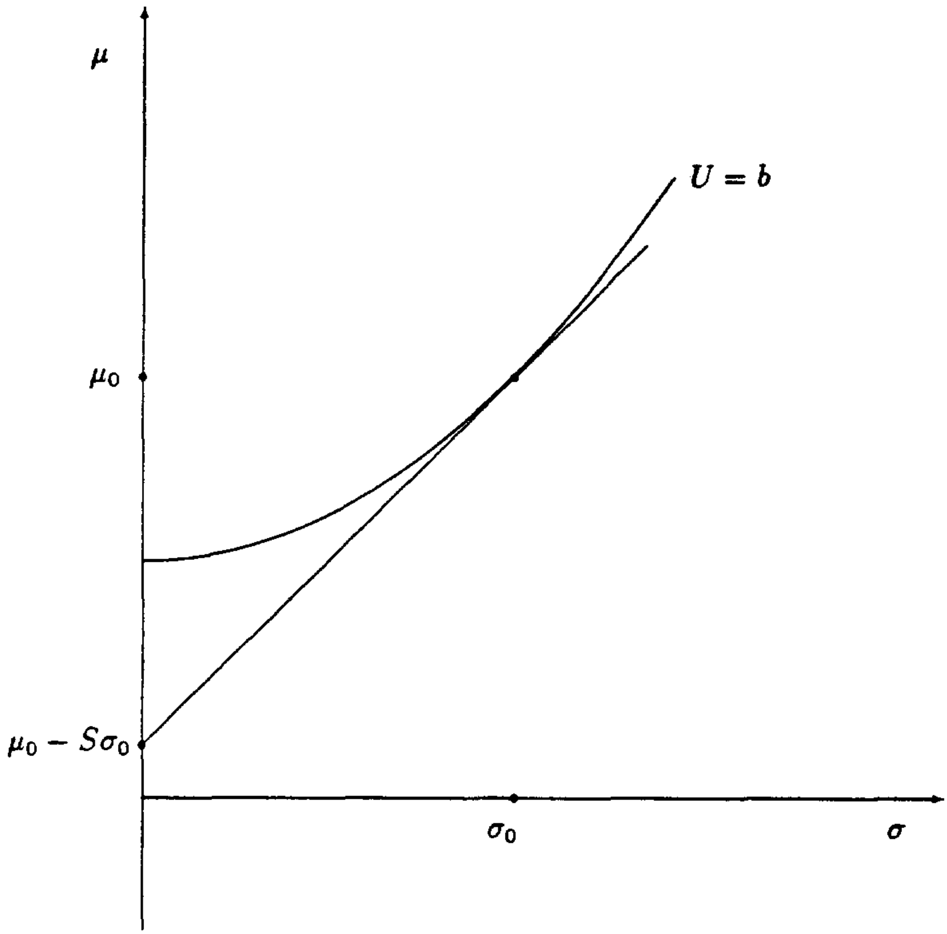

Figure 2: Existence of μ with $U(0, \mu) = U(\sigma_0, \mu_0)$

lies below the segment

$$\{(\sigma, \mu) : U(\sigma, \mu) = U(\sigma_1, \mu_1) \text{ and } \sigma \leq \sigma_1\}$$

of the indifference curve through (σ_1, μ_1). So,

$$U(\sigma_1, \mu_1) \geq U(0, \mu_0) \geq U(\sigma_0, \mu_0).$$

$\square$

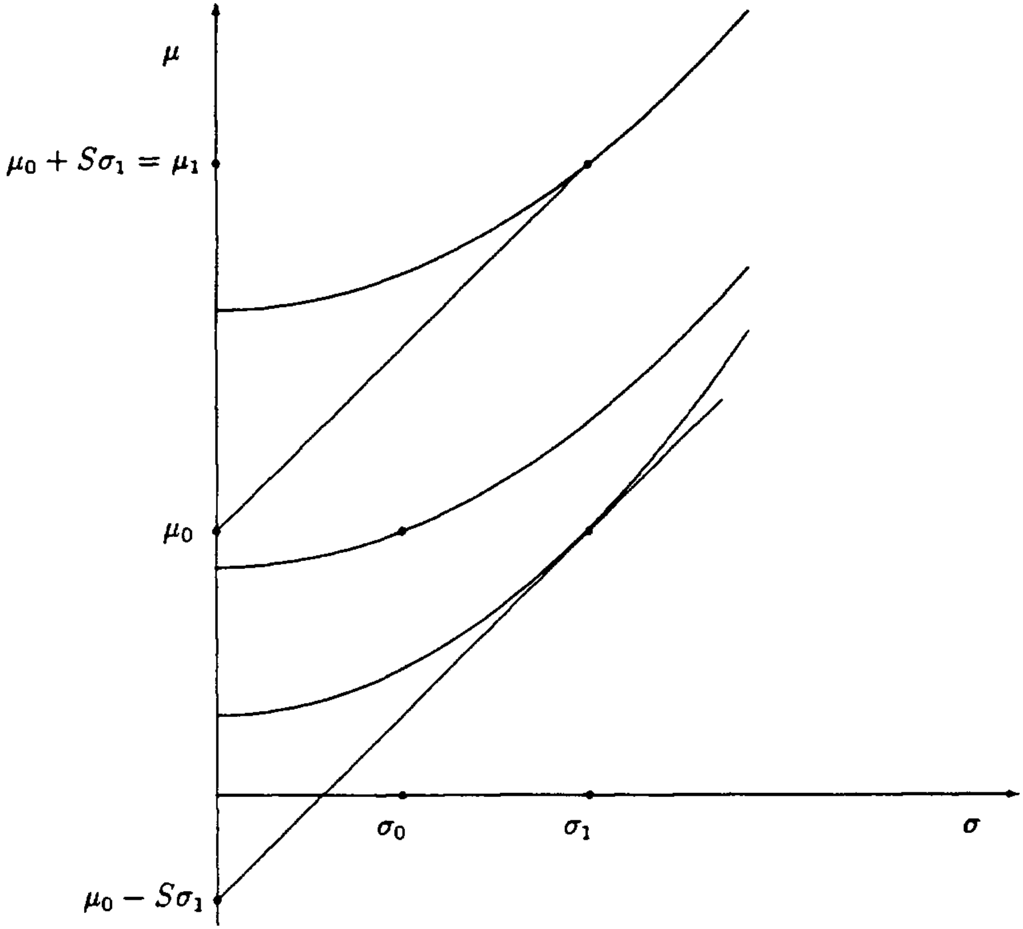

Figure 3: Assumption 2 follows if $S(\sigma, \mu)$ is decreasing in μ

3 Risk Aversion

This section defines the concept of risk aversion in the mean-variance framework, and the following section relates it to the usual concept of risk aversion in a two-parameter expected-utility model. The ideas and results developed here will be reinterpreted in terms of prudence in Section 5.

Let U and $\bar{U}$ be two utility functions for standard deviation and mean. Say that U is *more risk averse* than $\bar{U}$ if for all σ_0 and σ_1 with $0 \leq \sigma_0 \leq \sigma_1$, and all μ_0 and μ_1, $U(\sigma_0, \mu_0) \leq U(\sigma_1, \mu_1)$ implies $\bar{U}(\sigma_0, \mu_0) \leq \bar{U}(\sigma_1, \mu_1)$.

This means that if a random prospect is preferred to another random prospect with lower risk by an individual with utility function U, it will also be preferred by an individual with utility function $\bar{U}$. Graphically, the indifference curve for U through any point (σ_0, μ_0) is steeper than the indifference curve for $\bar{U}$ through the same point.

The *equivalent risk premium* for U, $\pi = \pi(\sigma_0, \mu_0; \sigma)$, is defined for any $\sigma \geq \sigma_0 \geq 0$ and any μ_0 by

$$U(\sigma, \mu_0) = U(\sigma_0, \mu_0 - \pi(\sigma_0, \mu_0; \sigma))$$

It can be interpreted as the maximum amount of insurance the individual is willing to pay in order to decrease risk from σ to σ_0. It is analogous to the equivalent risk premium as defined by Pratt (1964) and Arrow (1965), and thus $-\pi$ can be interpreted as the *ask price* for risk $\sigma - \sigma_0$. Because U is strictly increasing in μ, π is unique if it exists.

The *compensating risk premium* for U, $\pi^* = \pi^*(\sigma_0, \mu_0; \sigma)$, is defined for any $\sigma \geq \sigma_0 \geq 0$ and any μ_0 by

$$U(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma)) = U(\sigma_0, \mu_0)$$

It is the addition to expected wealth required in order for the individual to be compensated for taking the increase in risk from σ_0 to σ. It is analogous to the compensating risk premium defined by Pratt (1964) and thus $-\pi^*$ can be interpreted as the *bid price* for risk $\sigma - \sigma_0$. Because U is strictly increasing in μ, π^* is unique if it exists.

The compensating risk premium and the equivalent risk premium are illustrated in Figure 4. Note that the compensating risk premium is equal to the equivalent risk premium at a higher mean. In fact, π^* can be written as

$$\pi^*(\sigma_0, \mu_0; \sigma) = \pi(\sigma_0, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma); \sigma)$$

Similarly, the equivalent risk premium is equal to the compensating risk premium at a lower mean. In fact, π can be written as

$$\pi(\sigma_0, \mu_0; \sigma) = \pi^*(\sigma_0, \mu_0 - \pi(\sigma_0, \mu_0; \sigma); \sigma)$$

In the situation in Figure 1, where an indifference curve goes to infinity as σ increases toward some fixed finite value s, the compensating risk premium $\pi^*(\sigma_0, \mu_0; \sigma)$ does not exist if (σ_0, μ_0) is on the indifference curve and if $\sigma \geq s$. However, Assumption 2 ensures that the compensating risk premium always exists. Indeed, π^* can be expressed in terms of the function whose graph is the indifference curve through (σ_0, μ_0): If $U(\sigma_0, \mu_0) = b$, then

$$\pi^*(\sigma_0, \mu_0; \sigma) = I_b(\sigma) - \mu_0$$

Observe that in one sense, our definitions of the risk premia are more general than those normally used in the expected-utility framework [1]: they refer to the comparison of two risky situations, whereas the usual definitions refer to the comparison of one risky and one riskless situation. In this sense, our results below about the risk premia will also be more general than the usual ones in the expected-utility framework.

In Proposition 2 below, we shall identify a series of equivalent conditions for one utility function of standard deviation and mean to be more risk averse than another. One of the conditions describes the individuals' behavior in the following simple portfolio choice problem.

There are two assets, one of which is riskless while the other is risky. The riskless asset has gross return R_f per share. The gross return per share of the risky asset has expectation $\bar{R}$, and as a matter of choice of units we can assume that the standard deviation is one. The price per share of the riskless asset is normalized at one, and the price per share of the risky

[1] When we refer to the "expected-utility framework", we mean the general framework used by Pratt (1964) and Arrow (1965). In the next section, we shall introduce a two-parameter expected-utility model which will be a special case of both the mean-variance model and the (general) expected-utility model.

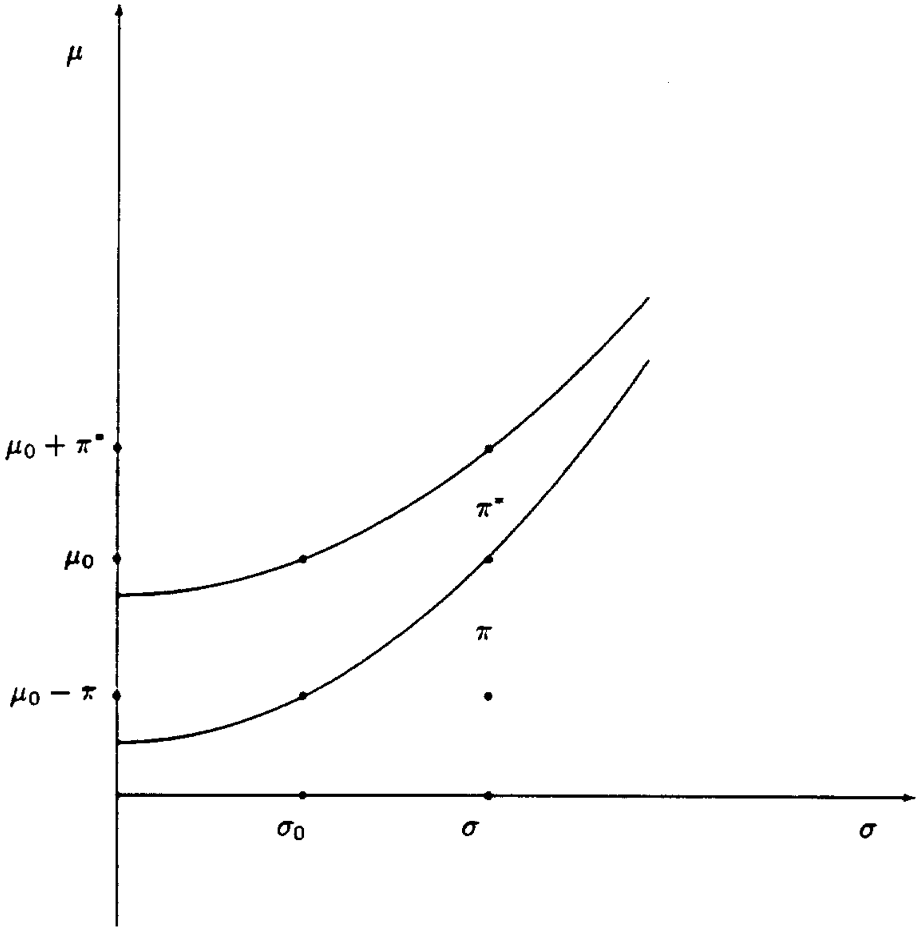

Figure 4: The compensating and equivalent risk premia

asset is p. An individual with utility function U chooses a portfolio so as to maximize his utility of standard deviation and mean of return given a budget constraint. There are no constraints on borrowing or short sales.

Let x_f and x be the number of shares demanded of the riskless asset and of the risky asset, respectively. Then the mean and standard deviation of the portfolio's return will be

$$\mu = x_f R_f + x\bar{R}$$

and

$$\sigma = x$$

If the investor's initial wealth is w, then the budget constraint is

$$w = x_f + px.$$

As a function of x, the mean return is

$$\mu = (w - px)R_f + x\bar{R}$$

The opportunity locus in (σ, μ)-space is

$$\begin{aligned} \mu &= (w - p\sigma)R_f + \sigma\bar{R} \\ &= wR_f + \sigma(\bar{R} - pR_f) \end{aligned}$$

It is represented in (σ, μ)-space by a half-line starting at $(0, wR_f)$ with slope $\bar{R} - pR_f$.

The investor chooses $x = \sigma$ so as to maximize

$$U((w - px)R_f + x\bar{R}, x) = U(wR_f + \sigma(\bar{R} - pR_f), \sigma)$$

The optimal combination of mean and standard deviation will be given by the tangency point of the budget constraint and an indifference curve, provided that such a tangency point exists. If it exists, then it is unique. The optimal combination of mean and standard deviation will correspond to the unique optimal portfolio.

As analyzed in Nielsen (1987), the existence of an optimum depends on the relation between the slope of the opportunity locus and the slopes of the indifference curves in (σ, μ)-space. Let α be the supremum of possible slopes of the indifference curves:

$$\alpha = \sup_{\{(\sigma,\mu):\sigma \geq 0\}} S(\sigma, \mu)$$

If $\alpha \leq \bar{R} - pR_f$, then there is no optimal portfolio. If U is concave, then all indifference curves have the same asymptotic slope α at large values of σ, and there exists an optimal portfolio if and only if $\alpha > \bar{R} - pR_f$.

The first-order condition is

$$-\frac{U_\sigma}{U_\mu} = S(\sigma, \mu) = \bar{R} - pR_f$$

Since S is always positive, the slope $\bar{R} - pR_f$ must be positive if an optimum exists.

Consider two individuals with utility functions U and $\bar{U}$. Let $\pi^*(\sigma_0, \mu_0; \sigma)$ and $\bar{\pi}^*(\sigma_0, \mu_0; \sigma)$ denote the compensating risk premia for U and $\bar{U}$, respectively. Let $S(\sigma, \mu)$ and $\bar{S}(\sigma, \mu)$ denote the slopes of their respective indifference curves, and let α and $\bar{\alpha}$ be the suprema of these slopes. The portfolio choice problem is specified by the parameters $(w, R_f, \bar{R}, p)$. If both individuals can find an optimal choice in this problem, then let x and $\bar{x}$ be their respective optimal holdings of the risky asset (measured as numbers of shares).

The following proposition restates, in the present mean-variance framework, results from Pratt (1964), Arrow (1965) and Kimball (1990).

Proposition 2 *The following statements are equivalent:*

1. *U is more risk averse than $\bar{U}$.*
2. *$S(\sigma, \mu) \geq \bar{S}(\sigma, \mu)$ for all (σ, μ) with $\sigma \geq 0$.*
3. *$\pi^*(\sigma_0, \mu_0; \sigma) \geq \bar{\pi}^*(\sigma_0, \mu_0; \sigma)$ for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$.*
4. *$\pi(\sigma_0, \mu_0; \sigma) \geq \bar{\pi}(\sigma_0, \mu_0; \sigma)$ for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$.*
5. *If $(w, R_f, \bar{R}, p)$ is such that the portfolio selection problem has an optimal solution for $\bar{U}$, then it also has an optimal solution for U, and $x \leq \bar{x}$.*

PROOF:

(1) equivalent to (2):

Follows immediately from the geometric interpretation of the definition of "more risk averse."

(1) implies (3):

For all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$,

$$U(\sigma_0, \mu_0) = U(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma))$$

and

$$\bar{U}(\sigma_0, \mu_0) = \bar{U}(\sigma, \mu_0 + \bar{\pi}^*(\sigma_0, \mu_0; \sigma)).$$

Since U is more risk averse than $\bar{U}$, this implies that

$$\bar{U}(\sigma, \mu_0 + \bar{\pi}^*(\sigma_0, \mu_0; \sigma)) = \bar{U}(\sigma_0, \mu_0) \leq \bar{U}(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma)).$$

Since $\bar{U}$ is strictly increasing in its second argument, it follows that

$$\pi^*(\sigma_0, \mu_0; \sigma) \geq \bar{\pi}^*(\sigma_0, \mu_0; \sigma).$$

(3) implies (1):

Let $\sigma \geq \sigma_0 \geq 0$. If $U(\sigma_0, \mu_0) \leq U(\sigma, \mu)$, then

$$\mu \geq \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma) \geq \mu_0 + \bar{\pi}^*(\sigma_0, \mu_0; \sigma),$$

and so

$$\bar{U}(\sigma, \mu) \geq \bar{U}(\sigma, \mu_0 + \bar{\pi}^*(\sigma_0, \mu_0; \sigma)) = \bar{U}(\sigma_0, \mu_0).$$

This shows that U is more risk averse than $\bar{U}$.

(1) implies (4):

For all $(\sigma_0, \mu; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$

$$U(\sigma, \mu) = U(\sigma_0, \mu - \pi(\sigma_0, \mu; \sigma))$$

and

$$\bar{U}(\sigma, \mu) = \bar{U}(\sigma_0, \mu - \bar{\pi}(\sigma_0, \mu; \sigma))$$

Since U is more risk averse that $\bar{U}$, this implies that

$$\bar{U}(\sigma_0, \mu - \pi(\sigma_0, \mu; \sigma)) \leq \bar{U}(\sigma, \mu) = \bar{U}(\sigma_0, \mu - \bar{\pi}(\sigma_0, \mu; \sigma))$$

Since $\bar{U}$ is strictly increasing in its second argument, it follows that

$$\pi(\sigma_0, \mu; \sigma) \geq \bar{\pi}(\sigma_0, \mu; \sigma)$$

(4) implies (1):

Let $\sigma \geq \sigma_0 \geq 0$. If $U(\sigma_0, \mu_0) \leq U(\sigma, \mu)$, then

$$\mu_0 \leq \mu - \pi(\sigma_0, \mu; \sigma) \leq \mu - \bar{\pi}(\sigma_0, \mu; \sigma)$$

and so

$$\bar{U}(\sigma, \mu) = \bar{U}(\sigma_0, \mu - \bar{\pi}(\sigma_0, \mu; \sigma)) \geq \bar{U}(\sigma_0, \mu_0)$$

(2) implies (5):

Let $(\bar{\sigma}, \bar{\mu})$ be the standard deviation and mean of the optimal portfolio of an individual with utility function $\bar{U}$. Since

$$\bar{R} - R_f p = \bar{S}(\bar{\sigma}, \bar{\mu}) \leq S(\bar{\sigma}, \bar{\mu}),$$

the set

$$\{(\sigma, \mu) : U(\sigma, \mu) \geq U(\bar{\sigma}, \bar{\mu}) \text{ and } \mu \leq wR_f + \sigma(\bar{R} - pR_f)\}$$

is contained in

$$\{(\sigma, \mu) : \sigma \leq \bar{\sigma} \text{ and } \bar{\mu} - (\bar{\sigma} - \sigma)S(\bar{\sigma}, \bar{\mu}) \leq \mu \leq wR_f + \sigma(\bar{R} - pR_f)\}$$

which is bounded. Hence, an optimum exists for U. Let (σ, μ) be its standard deviation and mean. Then $\bar{\sigma} \geq \sigma$, and hence, $\bar{x} \geq x$.

(5) implies (2):

Consider a point $(\bar{\sigma}, \bar{\mu})$ with $\bar{\sigma} \geq 0$. Construct $(w, R_f, \bar{R}, p)$ such that $(\bar{\sigma}, \bar{\mu})$ is the standard deviation and mean of the optimal portfolio chosen by an individual with utility function $\bar{U}$:

$$\bar{R} - pR_f = \bar{S}(\bar{\sigma}, \bar{\mu})$$

and

$$wR_f + \bar{\sigma}(\bar{R} - pR_f) = \bar{\mu}$$

For example, set $p = R_f = 1$ and use these two equations to solve for $\bar{R}$ and w. Let (σ, μ) be the standard deviation and mean of return of an optimal portfolio chosen by an individual with utility function U. Then

$$\bar{\sigma} = \bar{x} \geq x = \sigma$$

Since the indifference curve for $\bar{U}$ is tangent to the budget constraint to the right of (σ, μ),

$$\bar{S}(\sigma, \mu) \leq \bar{R} - pR_f = S(\sigma, \mu)$$

□

Because of condition (2) in Proposition 2, we can think of the slope $S(\sigma, \mu)$ as a measure of the degree of risk aversion of an individual who has utility function U. It corresponds to the Arrow-Pratt coefficient of absolute risk aversion, $a = -u''/u'$. However, the latter measures only aversion to taking on risk starting from a riskless initial situation, while the slope is also defined for risky initial situations.

Now say that U exhibits

- *decreasing risk aversion* if for all h, all μ_0, μ_1 with $\mu_0 \leq \mu_1$, all σ_0, σ_1 with $0 \leq \sigma_0 \leq \sigma_1$, $U(\sigma_0, \mu_0) \leq U(\sigma_1, \mu_0 + h)$ implies $U(\sigma_0, \mu_1) \leq U(\sigma_1, \mu_1 + h)$;

- *increasing risk aversion* if for all h, all μ_0, μ_1 with $\mu_0 \leq \mu_1$, all σ_0, σ_1 with $0 \leq \sigma_0 \leq \sigma_1$, $U(\sigma_0, \mu_1) \leq U(\sigma_1, \mu_1 + h)$ implies $U(\sigma_0, \mu_0) \leq U(\sigma_1, \mu_0 + h)$; and

- *constant risk aversion* if U exhibits both increasing and decreasing risk aversion.

Note that U exhibits increasing risk aversion if and only if for every $k > 0$, the function U_k given by $U_k(\sigma, \mu) = U(\sigma, \mu + k)$ is more risk averse than U; and U exhibits decreasing risk aversion if and only if for every $k > 0$, the function U_k is less risk averse than U.

In the portfolio selection problem, note that if $(w, R_f, \bar{R}, p)$ is such that the portfolio selection problem has an optimal solution for U, then it also has an optimal solution at any higher wealth level. Let x denote the optimal holding of the risky asset.

The following proposition extends and restates, in the mean-variance framework, results derived by Pratt (1964) and Arrow (1965) in the expected-utility framework.

Proposition 3 *The following statements are equivalent.*

1. *U exhibits decreasing (respectively, increasing, constant) risk aversion.*

2. *$S(\sigma, \mu)$ is decreasing (respectively, increasing, constant) as a function of μ.*

3. *$\pi^*(\sigma_0, \mu_0; \sigma)$ is decreasing (respectively, increasing, constant) as a function of μ_0.*

4. *$\pi(\sigma_0, \mu_0; \sigma)$ is decreasing (respectively, increasing, constant) as a function of μ_0.*

5. *For every $(R_f, p, \bar{R})$, x is an increasing (respectively, decreasing, constant) function of w.*

PROOF: Consider the case of decreasing risk aversion. The case of increasing risk aversion is analogous, and the case of constant risk aversion follows from the two others.

(1) equivalent to (2):

Let $k > 0$. Then U_k is less risk averse than U if and only if $S_k(\sigma, \mu) \leq S(\sigma, \mu)$, where $S_k(\sigma, \mu)$ is the slope of the indifference curve of U_k at (σ, μ). Notice that $S_k(\sigma, \mu) = S(\sigma, \mu + k)$.

(1) equivalent to (3):

Let $k > 0$ and let π_k^* be the compensating risk premium associated with U_k. Then U_k is less risk averse than U if and only if $\pi_k^*(\sigma_0, \mu; \sigma) \leq \pi^*(\sigma_0, \mu; \sigma)$ for all σ_0, σ with $0 \leq \sigma_0 \leq \sigma$ and all μ. But it is easily seen that $\pi_k^*(\sigma_0, \mu; \sigma) = \pi^*(\sigma_0, \mu + k; \sigma)$.

(1) equivalent to (4)

The proof is completely analogous to the proof above replacing π^* by π and π_k^* by π_k.

(1) equivalent to (5):

Let $k > 0$. Then by Proposition 2, U_k is less risk averse than U if and only if $x_k \geq x$, where x_k denotes the optimal number of shares of the risky asset held by an individual who has utility function U_k. Note that x_k for initial wealth w equals x for initial wealth $w + k/R_f$.

□

Proposition 4 *If U exhibits decreasing (respectively, increasing, constant) risk aversion then $\pi^*(\sigma_0, \mu_0; \sigma) \leq \pi(\sigma_0, \mu_0; \sigma)$ (respectively, $\geq$, $=$) for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$* [2].

PROOF: If U exhibits decreasing risk aversion then

$$\pi^*(\sigma_0, \mu_0; \sigma) = \pi(\sigma_0, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma); \sigma) \leq \pi(\sigma_0, \mu_0; \sigma)$$

The cases of increasing and constant risk aversion are similar. □

In the following section, we will link the mean-variance framework to the expected-utility framework by considering mean-variance preferences derived from expected utility.

[2]This result is also true in the general expected-utility case addressed in Pratt (1964). The proof would be very similar to the mean-variance case.

4 Risk Aversion: The 2-Parameter Expected Utility Case

As a special case of the above analysis we will consider mean-variance preferences derived from expected utility. Let Y be a random variable with mean zero and unit variance. For every $\sigma \geq 0$ and every μ, the random variable $\mu + \sigma Y$ has mean μ and standard deviation σ. Let u be a von Neumann-Morgenstern utility function defined on the entire real line.

Assumption 3 *For every $\sigma \geq 0$ and for every μ, the random variable $u(\mu + \sigma Y)$ is integrable.*

Define

$$U(\sigma, \mu) = Eu(\mu + \sigma Y).$$

Then

$$W(v, \mu) = U(\sqrt{v}, \mu) = Eu(\mu + \sqrt{v}Y).$$

This is one particular way of deriving a utility function for standard deviation and mean or variance and mean from expected utility. It relies on the particular parametrized family $\mu + \sigma Y$ of random variables or distributions. Bardsley (1991) studies other families parametrized by the crude moments, the central moments, or the cumulants of the distributions. His derived utility function for mean and standard deviation or variance would assume that the higher moments or cumulants stay constant as μ and σ change, whereas in our case, the higher moments and cumulants do not necessarily stay constant.

We need to verify Assumptions 1 and 2. For this purpose, make the following additional assumptions.

Assumption 4 *Y is symmetric in the sense that Y and $-Y$ have the same distribution.*

Assumption 5 *u is twice continuously differentiable with $u' > 0$ and $u'' < 0$, and u'' is non-decreasing.*

Assumption 6 *For every $\sigma \geq 0$ and for every μ, the random variable $Y^2 u''(\mu + \sigma Y)$ is integrable.*

Proposition 5 *Assumptions 3–6 imply Assumptions 1–2 and imply that the derivatives U_μ, U_σ and $U_{\sigma\sigma}$ exist, are continuous, and have these properties :*

- $U_\mu(\sigma,\mu) = Eu'(\mu + \sigma Y) > 0.$
- $U_\sigma(\sigma,\mu) = E[Yu'(\mu + \sigma Y)],$
  $U_\sigma(0,\mu) = 0$ *for* $\sigma = 0$, *and* $U_\sigma(\sigma,\mu) < 0$ *for* $\sigma > 0$.
- $U_{\sigma\sigma}(\sigma,\mu) = E[Y^2 u''(\mu + \sigma Y) < 0$ *and*
  $U_{\sigma\sigma}(0,\mu) = 2W_v(0,\mu) = u''(\mu).$

PROOF: See Appendix A. □

Note from the above proposition that the slope of the indifference curve in (σ,μ)-space at $\sigma = 0$ is

$$S(0,\mu) = -\frac{U_\sigma(0,\mu)}{U_\mu(0,\mu)} = 0$$

and that the slope in (v,μ)-space at $v = 0$ is

$$-\frac{W_v(0,\mu)}{W_\mu(0,\mu)} = -\frac{1}{2}\frac{u''(\mu)}{u'(\mu)},$$

or half of the Arrow-Pratt coefficient of absolute risk aversion. These results were shown by Chipman (1973) in the special case where Y follows a normal distribution; but the present results are more general. The results of Bardsley (1991) are not directly comparable because he uses another parametrization.

Let us consider the portfolio choice problem in the two-parameter expected-utility setting. We need to adjust it by specifying the random gross return per share of the risky asset as

$$R = \bar{R} + Y$$

The individual has a von Neumann-Morgenstern utility function u. Given $(w, R_f, \bar{R}, p)$, he chooses his asset holdings (x_f, x) so as to maximize the expected utility $Eu(x_f R_f + xR)$ subject to the budget constraint, $w = x_f + px$. Equivalently, he chooses $\sigma = x$ so as to maximize

$$Eu(wR_f + x(R - pR_f)) = U(wR_f + x(\bar{R} - pR_f), x)$$

Define the *asymptotic slopes* S^+ and S^- of u by

$$S^+ = \lim_{t \to \infty} u'(t)$$

and

$$S^- = \lim_{t \to -\infty} u'(t)$$

Define the *incomplete means* E^+ and E^- of the return to a portfolio containing $x = 1$ share of the risky asset as

$$E^+ = E[(wR_f + x(R - pR_f))1_{\{wR_f + x(R - pR_f) > 0\}}$$

and

$$E^- = E[(wR_f + x(R - pR_f))1_{\{wR_f + x(R - pR_f) < 0\}}$$

Assume that $\bar{R} \geq pR_f$. An optimal portfolio exists if and only if $S^+E^+ + S^-E^- < 0$. This follows from the analysis in Bertsekas (1974) and Nielsen (1992c). An optimal portfolio is unique if it exists.

Arrow (1965) and Pratt (1964) studied this portfolio selection problem (formulating it in terms of rates of returns). In particular, Pratt (1964, Theorem 7) showed that one individual is more risk averse than another if and only if the first individual invests more in the risky asset than the second one does at all levels of wealth and for all distributions of returns to the risky asset. In our formalization, the result can be stated as follows. Let u and $\bar{u}$ be two von Neumann-Morgenstern utility functions. If $(w, R_f, \bar{R}, p)$ is such that the portfolio selection problem has a solution for both u and $\bar{u}$, let x and $\bar{x}$ denote the optimal holdings of the risky asset chosen by the two individuals. Then u is more risk averse than $\bar{u}$ if and only if for all $(w, R_f, \bar{R}, p)$ such that the portfolio selection problem has a solution for $\bar{u}$, it also has a solution for u, and $x \leq \bar{x}$.

Proposition 6 *U is more risk averse than $\bar{U}$ if and only if u is more risk averse than $\bar{u}$.*

PROOF: According to Proposition 2, if U is more risk averse than $\bar{U}$, then $S(\sigma, \mu) \geq \bar{S}(\sigma, \mu)$ for all (σ, μ) with $\sigma \geq 0$. Since

$$S(\sigma, \mu) = -\frac{U_\sigma(\sigma, \mu)}{U_\mu(\sigma, \mu)} = -2\sigma \frac{W_v(\sigma^2, \mu)}{W_\mu(\sigma^2, \mu)},$$

it follows that

$$\frac{W_v(\sigma^2,\mu)}{W_\mu(\sigma^2,\mu)} \le \frac{\bar{W}_v(\sigma^2,\mu)}{\bar{W}_\mu(\sigma^2,\mu)}$$

for all $\sigma > 0$, and hence, by continuity, for all $\sigma \ge 0$. For $\sigma = 0$ this means that

$$-\frac{u''(\mu)}{u'(\mu)} = -\frac{2W_v(\sigma^2,\mu)}{W_\mu(\sigma^2,\mu)} \ge -\frac{2\bar{W}_v(\sigma^2,\mu)}{\bar{W}_\mu(\sigma^2,\mu)} = -\frac{\bar{u}''(\mu)}{\bar{u}'(\mu)}$$

which says that u is more risk averse than $\bar{u}$. Conversely, according to Pratt's (1964) Theorem 7, if u is more risk averse than $\bar{u}$, then $x \ge \bar{x}$ for all $(w, R_f, \bar{R}, p)$ such that both portfolio selection problems have a solution. According to Proposition 2, this implies that U is more risk averse than $\bar{U}$. □

Proposition 6 shows that the usual definition of "more risk averse" is consistent with our definition based on the utility function for standard deviation and mean. As a consequence, it gives us some new criteria for one von Neumann-Morgenstern utility function to be more risk averse than another. The usual definition is in terms of gambles: the less risk averse individual will accept any gamble accepted by the more risk averse individual. It is not necessary to check all gambles. For example, it is well known that it suffices to check the binary lotteries. Proposition 6 implies another criterion: for any given Y, it suffices to check the decision makers' behavior with respect to gambles of the form $\mu + \sigma Y$. Another criterion is based on the portfolio selection problem rather than on gambles. As mentioned above, Pratt (1964) showed that one individual is more risk averse than another if and only if he invests more in a risky asset at all wealth levels and for all distributions of returns to the risky asset. Proposition 6 implies that when mean variance preferences are derived from expected utility, it is enough to consider the particular parametrized family $\mu + \sigma Y$ of returns distributions where Y is a fixed symmetric random variable with mean zero and unit variance.

Recall that U exhibits increasing (decreasing) risk aversion if and only if for every $k > 0$, the function U_k given by $U_k(\sigma,\mu) = U(\sigma,\mu + k)$ is more (less) risk averse than U. Similarly, u exhibits increasing (decreasing) risk aversion if and only if for every $k > 0$, the function u_k given by $u_k(t) = u(t + k)$ is more (less) risk averse than u. These facts are exploited in the proof of the next proposition.

Proposition 7 *U exhibits increasing (decreasing, constant) risk aversion if and only if u exhibits increasing (decreasing, constant) risk aversion.*

PROOF: U is increasingly (decreasingly) risk averse if and only if U_k is more (less) risk averse than U for all $k > 0$. By the previous proposition, this is the case if and only if u_k is more (less) risk averse than u for all $k > 0$. But this is true if and only if u is increasingly (decreasingly) risk averse. The case of constant risk aversion is straightforward from the other two. □

In the following sections, we will study the concepts of prudence and precautionary saving, the above results on risk aversion will be applied to the "utility functions" $-U_\mu$ and $-u'$.

5 Prudence and Precautionary Saving

In this section we will consider a two-period model. The investor has some current wealth and will receive random future income. In the first period, he decides how much to consume and how much to save, and in the second period he will spend his savings plus his random income. It is likely that the more uncertain his future income is, the more he will save in order to insure against the possibility of low future income.

Kimball (1990) considers this issue in an expected-utility framework and introduces the concepts of prudence and precautionary saving. He defines prudence as

> the strength of the precautionary saving motive which induces individuals to prepare and forearm themselves against uncertainty they can not avoid, in contrast to risk aversion which suggests how much one dislikes uncertainty and wants to avoid it.

The theory of prudence is isomorphic to the Arrow-Pratt theory of risk aversion as applied to minus the derivative of the utility function. Prudence is measured by

$$\eta = -\frac{u'''}{u''}$$

i.e. the Arrow-Pratt coefficient of absolute risk aversion as applied to $-u'$.

Here, we define and study the concepts of prudence and precautionary saving in the mean-variance framework. This theory of precaution is isomorphic to the theory of risk aversion, as applied to minus the derivative of the utility function with respect to the mean.

Results analogous to ours hold also in the expected-utility framework and many of them have been stated by Kimball.

For simplicity, we follow Kimball in assuming that the investor can borrow and lend at a riskfree interest rate of zero. His total resources are then the sum of his current wealth, which is known, and his future income, which is random. Let σ and w denote the standard deviation and mean of these total resources: σ is the standard deviation of future income, and w is the sum of current wealth and expected future income.

If the investor decides to consume c now, then his future consumption has mean $w-c$ and standard deviation σ. Assuming that he has an additive, time-separable utility function, he maximizes an objective function of the form

$$v(c) + U(\sigma, w - c).$$

We shall need to impose restrictions on W_μ and U_μ similar to those imposed on W and U in Assumption 1.

Assumption 7 *W_μ is continuously differentiable (also at $v=0$) with $W_{\mu v} > 0$ and $W_{\mu\mu} < 0$, and U_μ is strictly quasi-concave.*

These restrictions will allow us to draw an analogy between our theory of precautionary saving and that of risk aversion developed in the previous section as applied to minus the derivative of the utility function with respect to μ. The economic interpretation of the signs assumed for the second derivatives of W is that consumption is a decreasing function of σ and an increasing function of μ. This will become clear shortly.

We also need the following assumption about the utility function v.

Assumption 8 *v is continuously differentiable and v' is decreasing.*

The first-order condition for the optimal choic of c says

$$v'(c) = U_\mu(\sigma, w - c).$$

Since v' is strictly decreasing and U_μ is stricly decreasing in its second argument, if a solution exists it is unique. The following conditions will ensure that a solution does exist.

$$\lim_{c \to +\infty} v'(c) < \lim_{\mu \to -\infty} U_\mu(\sigma, \mu) \text{ for all } \sigma \geq 0$$

and

$$\lim_{c \to -\infty} v'(c) > \lim_{\mu \to +\infty} U_\mu(\sigma, \mu) \text{ for all } \sigma \geq 0.$$

For example, these conditions will hold in the typical case where

$$\begin{aligned}
\lim_{c \to -\infty} v'(c) &= \lim_{\mu \to -\infty} U_\mu(\sigma, \mu) = +\infty \\
\lim_{c \to +\infty} v'(c) &= \lim_{\mu \to +\infty} U_\mu(\sigma, \mu) = 0
\end{aligned}$$

The optimization problem defines a consumption function $c = c(\sigma, w)$. Since Assumption 7 implies that $U_{\mu\sigma} > 0$ and $U_{\mu\mu} < 0$, c is a strictly decreasing function of σ and a strictly increasing function of w. Indeed, this is the economic interpretation of the signs of the derivatives of W imposed in Assumption 7.

So, if the riskiness of future income increases, reflected in an increase in σ, then the investor decreases his current consumption and saves more in order to provide for the future. This is what is referred to as prudent behavior, and the strength of the effect will reflect the investor's degree of prudence.

To formalize this idea, we first define comparative prudence. Let U and $\bar{U}$ be two utility functions for standard deviation and mean. Say that U is *more prudent* than $\bar{U}$ if for all μ_0 and μ_1 and all $\sigma \geq \sigma_0 \geq 0$, $U_\mu(\sigma_1, \mu_1) \leq U_\mu(\sigma_0, \mu_0)$ implies $\bar{U}_\mu(\sigma_1, \mu_1) \leq \bar{U}_\mu(\sigma_0, \mu_0)$.

This definition is somewhat abstract, but we will show below how it relates to consumption behavior.

It is immediately seen from the definition that U is more prudent than $\bar{U}$ if and only if $-U_\mu$ is more risk averse than $-\bar{U}_\mu$. This fact is what leads to the analogy between the theory of prudence and the theory of risk aversion.

The *compensating precautionary premium* $\psi^* = \psi^*(\sigma_0, \mu_0; \sigma)$ is defined for any $\sigma \geq \sigma_0 \geq 0$ and any μ_0 by

$$U_\mu(\sigma, \mu_0 + \psi^*(\sigma_0, \mu_0; \sigma)) = U_\mu(\sigma_0, \mu_0).$$

So, the compensating precautionary premium for U corresponds to the compensating risk premium for $-U_\mu$. Since U_μ is strictly decreasing in μ, ψ^* will be unique if it exists.

If we set

$$c_0 = c(\sigma_0, w_0)$$

and

$$\psi^* = \psi^*(\sigma_0, w_0 - c_0; \sigma),$$

then

$$U_\mu(\sigma, w_0 - c_0 + \psi^*) = U_\mu(\sigma_0, w_0 - c_0) = v'(c_0).$$

Thus,

$$c(\sigma, w_0 + \psi^*) = c(\sigma_0, w_0).$$

Hence, the compensating precautionary premium is the addition to expected wealth required by the individual in order to be willing to keep current consumption unchanged while taking a higher risk σ.

In the situation in Figure 1, where the indifference curve is reinterpreted as the level curve for the function $-U_\mu$, the compensating precautionary premium does not exist for (s, μ). We need an assumption similar to Assumption 2 in order to ensure that the compensating precautionary premium always exists.

Assumption 9 *For every $\sigma \geq \sigma_0 \geq 0$, and every μ, there exists h such that $U_\mu(\sigma, \mu + h) \leq U_\mu(\sigma_0, \mu)$.*

By analogy to the case of the compensating risk premium, it follows from the assumption that the compensating precautionary premium always exists.

The *equivalent precautionary premium* $\psi = \psi(\sigma_0, \mu_0; \sigma)$ is defined for any $\sigma \geq \sigma_0 \geq 0$ and any μ_0 by

$$U_\mu(\sigma, \mu_0) = U_\mu(\sigma_0, \mu_0 - \psi(\sigma_0, \mu_0; \sigma)).$$

So, the equivalent precautionary premium for U corresponds to the equivalent risk premium for $-U_\mu$. Since U_μ is strictly decreasing in μ, ψ will be unique.

If we set

$$c = c(\sigma, w_0)$$

$$\psi = \psi(\sigma_0, w_0 - c; \sigma),$$

then

$$U_\mu(\sigma_0, w_0 - c - \psi) = U_\mu(\sigma, w_0 - c) = v'(c).$$

Thus,

$$c(\sigma, w_0) = c(\sigma_0, w_0 - \psi(\sigma_0, w_0 - c; \sigma)).$$

Hence, the equivalent precautionary premium is the reduction in wealth required for the individual to keep his consumption constant when risk decreases from σ to σ_0.

Note that the equivalent precautionary premium is equivalent to the compensating precautionary premium at a lower level of consumption. In fact, ψ can be written as

$$\psi(\sigma_0, \mu_0; \sigma) = \psi^*(\sigma_0, \mu_0 - \psi(\sigma_0, \mu_0; \sigma); \sigma)$$

Similarly, the compensating precautionary premium is equivalent to the equivalent precautionary premium at a higher level of consumption. In fact, ψ^* can be written as

$$\psi^*(\sigma_0, \mu_0; \sigma) = \psi(\sigma_0, \mu_0 + \psi^*(\sigma_0, \mu_0; \sigma); \sigma)$$

Observe that, like in the case of the risk premia, our definitions of the precautionary premia are more general than those normally used in the expected-utility framework, in the sense that they refer to the comparison of two risky situations, whereas the usual definitions refer to the comparison of one risky and one riskless situation. So, in this sense our results about the precautionary premia below are also more general than the usual ones in the expected-utility framework.

Let U and $\bar{U}$ be two utility functions for standard deviation and mean, and let $T(\sigma, \mu)$ and $\bar{T}(\sigma, \mu)$ be the slopes of the level curves of the functions $-U_\mu$ and $-\bar{U}_\mu$ in (σ, μ)-space.

Proposition 8 *The following statements are equivalent:*

1. *U is more prudent than $\bar{U}$.*
2. *$-U_\mu$ is more risk averse than $-\bar{U}_\mu$.*
3. *$T(\sigma, \mu) \geq \bar{T}(\sigma, \mu)$ for all (σ, μ) with $\sigma \geq 0$.*
4. *$\psi^*(\sigma_0, \mu_0; \sigma) \geq \bar{\psi}^*(\sigma_0, \mu_0; \sigma)$ for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$.*
5. *$\psi(\sigma_0, \mu_0; \sigma) \geq \bar{\psi}(\sigma_0, \mu_0; \sigma)$ for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$.*

PROOF: The equivalence of 1 and 2 is immediate. The equivalence of 2, 3, 4 and 5 follows from Proposition 2. □

Now say that U exhibits

- decreasing prudence if and only if for all h, all μ_0, μ_1 with $\mu_0 \leq \mu_1$, and all σ and σ_0 with $\sigma \geq \sigma_0 \geq 0$, $U_\mu(\sigma_0, \mu_0) \geq U_\mu(\sigma, \mu_0 + h)$ implies $U_\mu(\sigma_0, \mu_1) \geq U_\mu(\sigma, \mu_1 + h)$;

- increasing prudence if and only if for all h, all μ_0, μ_1 with $\mu_0 \leq \mu_1$, and all σ and σ_0 with $\sigma \geq \sigma_0 \geq 0$, $U_\mu(\sigma_0, \mu_1) \geq U_\mu(\sigma, \mu_1 + h)$ implies $U_\mu(\sigma_0, \mu_0) \geq U_\mu(\sigma, \mu_0 + h)$.
- constant prudence if and only if U exhibits both increasing and decreasing prudence.

So, U exhibits decreasing (increasing, constant) prudence if and only if $-U_\mu$ exhibits decreasing (increasing, constant) risk aversion.

Note that U exhibits increasing prudence if and only if for every $k > 0$, the function U_k given by $U_k(\sigma, \mu) = U(\sigma, \mu + k)$ is more prudent than U; and U exhibits decreasing prudence if and only if for every $k > 0$, the function U_k is less prudent than U.

Proposition 9 *The following statements are equivalent:*

1. *U exhibits decreasing (respectively, increasing, constant) prudence.*
2. *$-U_\mu$ exhibits decreasing (respectively, increasing, constant) risk aversion.*
3. *$T(\sigma, \mu)$ is decreasing (respectively, increasing, constant) as a function of μ.*
4. *$\psi^*(\sigma_0, \mu_0; \sigma)$ is decreasing (respectively, increasing, constant) as a function of μ_0.*
5. *$\psi(\sigma_0, \mu_0; \sigma)$ is decreasing (respectively, increasing, constant) as a function of μ_0.*

PROOF: Cf. Proposition 3. □

In the present framework, the slope $T(\sigma, \mu)$ measures the degree of prudence of an individual who has utility function U (recall that $S(\sigma, \mu)$ measures the degree of risk aversion). In a sense, it is more general than Kimball's coefficient of prudence, $\eta = -u'''/u''$, since the latter measures prudence only when taking on risk starting from a riskless situation while our measure is also defined for risky initial situations.

Decreasing prudence may be considered to be the natural case (like the case of decreasing risk aversion), cf. Kimball (1990b).

Proposition 10 *If U exhibits decreasing (respectively, increasing, constant) prudence then $\psi^*(\sigma_0, \mu_0; \sigma) \leq \psi(\sigma_0, \mu_0; \sigma)$ (respectively, $\geq$, $=$)* [3].

PROOF: Cf. Proposition 4 □

In the general-expected utility case, Kimball (1990) has shown that

$$\frac{a'}{a} = a - \eta,$$

where $a = -u''(\mu)/u'(\mu)$ is the Arrow-Pratt coefficient of absolute risk aversion and $\eta = -u'''(\mu)/u''(\mu)$ is the coefficient of prudence.

We can derive a similar relationship that links risk aversion and prudence in the mean-variance framewok. Recall that

$$S(\sigma, \mu) = -\frac{U_\sigma}{U_\mu}$$

and that

$$T(\sigma, \mu) = -\frac{U_{\mu\sigma}}{U_{\mu\mu}}.$$

This implies that

$$S_\mu(\sigma, \mu) = \partial(-\frac{U_\sigma}{U_\mu})/\partial\mu = -\frac{U_{\sigma\mu}U_\mu - U_\sigma U_{\mu\mu}}{U_\mu^2}$$

which can be rearranged as

$$-\frac{U_\mu}{U_{\mu\mu}} S_\mu(\sigma, \mu) = S(\sigma, \mu) - T(\sigma, \mu).$$

Proposition 11 *The following statements are equivalent:*

1. *U exhibits decreasing (respectively, increasing, constant) risk aversion.*
2. *$T(\sigma, \mu) \geq S(\sigma, \mu)$ (respectively, $\leq$, $=$) for all (σ, μ) with $\sigma \geq 0$.*
3. *$\psi^*(\sigma_0, \mu_0; \sigma) \geq \pi^*(\sigma_0, \mu_0; \sigma)$ (respectively, $\leq$, $=$) for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$.*

[3] This result is also true in the general expected-utility case. The proof would be similar.

4. $\psi(\sigma_0, \mu_0; \sigma) \geq \pi(\sigma_0, \mu_0; \sigma)$ *(respectively, $\leq$, $=$) for all* $(\sigma_0, \mu_0; \sigma)$ *with* $\sigma \geq \sigma_0 \geq 0$.

PROOF: We will consider the case of decreasing risk aversion (and the corresponding inequalities in statements 2–4). The case of increasing risk aversion is similar and the case of constant risk aversion follows directly from the previous two.

(1) equivalent to (2):

In Proposition 3, we have shown that U exhibits decreasing risk aversion if and only if $S(\sigma, \mu)$ is decreasing in μ. This is equivalent to

$$S_\mu(\sigma, \mu) \leq 0.$$

Since

$$-\frac{U_\mu}{U_{\mu\mu}} S_\mu(\sigma, \mu) = S(\sigma, \mu) - T(\sigma, \mu)$$

and

$$-\frac{U_\mu}{U_{\mu\mu}} > 0,$$

decreasing risk aversion is equivalent to

$$T(\sigma, \mu) \geq S(\sigma, \mu).$$

(1) implies (3):

Again using Proposition 3, U exhibits decreasing risk aversion if and only if $\pi^*(\sigma_0, \mu_0; \sigma)$ is decreasing as a function of μ_0, for all $\sigma \geq \sigma_0 \geq 0$. It follows from the relation

$$U(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0, \sigma)) = U(\sigma_0, \mu_0)$$

that π^* is continuously differentiable with respect to μ_0 with

$$U_\mu(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma))(1 + \pi^*_{\mu_0}) = U_\mu(\sigma_0, \mu_0)$$

Since $U_\mu > 0$, it follows that $1 + \pi^*_{\mu_0} > 0$. In addition, since U exhibits decreasing risk aversion, $\pi^*_{\mu_0} \leq 0$, thus $1 + \pi^*_{\mu_0} \leq 1$. So,

$$U_\mu(\sigma, \mu_0 + \pi^*(\sigma_0, \mu_0; \sigma)) \geq U_\mu(\sigma_0, \mu_0) = U_\mu(\sigma, \mu_0 + \psi^*(\sigma_0, \mu_0; \sigma))$$

Therefore, since U_μ is strictly decreasing in μ,

$$\psi^*(\sigma_0, \mu_0; \sigma) \geq \pi^*(\sigma_0, \mu_0; \sigma).$$

(3) implies (2):
Note that

$$S(\sigma_0, \mu_0) = \lim_{\sigma \to \sigma_0} \frac{\pi^*(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

Similarly

$$T(\sigma_0, \mu_0) = \lim_{\sigma \to \sigma_0} \frac{\psi^*(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

Since

$$\psi^*(\sigma_0, \mu_0; \sigma) \geq \pi^*(\sigma_0, \mu_0; \sigma).$$

for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$, it follows that

$$\lim_{\sigma \to \sigma_0} \frac{\psi^*(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0} \geq \lim_{\sigma \to \sigma_0} \frac{\pi^*(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

i.e.

$$T(\sigma_0, \mu_0) \geq S(\sigma_0, \mu_0).$$

(1) implies (4):
U exhibits decreasing risk aversion if and only if $\pi(\sigma_0, \mu_0; \sigma)$ is decreasing as a function of μ_0, for all $\sigma \geq \sigma_0 \geq 0$. It follows from the relation

$$U(\sigma_0, \mu_0 - \pi(\sigma_0, \mu_0, \sigma)) = U(\sigma, \mu_0)$$

that π is continuously differentiable with respect to μ_0 with

$$U_\mu(\sigma_0, \mu_0 - \pi(\sigma_0, \mu_0; \sigma))(1 - \pi_{\mu_0}) = U_\mu(\sigma, \mu_0)$$

Since U exhibits decreasing risk aversion, $\pi_{\mu_0} \leq 0$, thus $1 - \pi_{\mu_0} \geq 1$. So,

$$U_\mu(\sigma_0, \mu_0 - \pi(\sigma_0, \mu_0; \sigma)) \leq U_\mu(\sigma, \mu_0) = U_\mu(\sigma_0, \mu_0 - \psi(\sigma_0, \mu_0; \sigma))$$

Therefore, since U_μ is strictly decreasing in μ,

$$\psi(\sigma_0, \mu_0; \sigma) \geq \pi(\sigma_0, \mu_0; \sigma).$$

(4) implies (2):

Note that

$$S(\sigma, \mu_0) = \lim_{\sigma_0 \to \sigma} \frac{\pi(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

Similarly

$$T(\sigma, \mu_0) = \lim_{\sigma_0 \to \sigma} \frac{\psi(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

Since

$$\psi(\sigma_0, \mu_0; \sigma) \geq \pi(\sigma_0, \mu_0; \sigma).$$

for all $(\sigma_0, \mu_0; \sigma)$ with $\sigma \geq \sigma_0 \geq 0$, it follows that

$$\lim_{\sigma_0 \to \sigma} \frac{\psi(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0} \geq \lim_{\sigma_0 \to \sigma} \frac{\pi(\sigma_0, \mu_0; \sigma)}{\sigma - \sigma_0}$$

i.e.

$$T(\sigma, \mu_0) \geq S(\sigma, \mu_0).$$

for all (σ, μ_0) with $\sigma \geq 0$. □

Proposition 12 *If U exhibits constant risk aversion then it also exhibits constant prudence and $S(\sigma, \mu) = T(\sigma, \mu)$.*

Proof: In Proposition 3, we have shown that U exhibits constant risk aversion if and only if $S(\sigma, \mu)$ is constant as a function of μ. In addition, we have just shown in Proposition 11 that U exhibits constant risk aversion if and only if $T(\sigma, \mu) = S(\sigma, \mu)$. Thus $T(\sigma, \mu)$ is constant as a function of μ and by Proposition 9 U exhibits constant prudence. □

In the following section, we will consider a special case of the above analysis where utility of mean and variance is derived from expected utility.

6 Prudence and Precautionary Saving : The 2-Parameter Expected Utility Case

In this section, we will study the concept of prudence in the two-parameter expected-utility framework as a special case of the previous analysis in the mean-variance framework. So, as in Section 4, we consider the case where

$$U(\sigma, \mu) = Eu(\mu + \sigma Y)$$

We need to verify Assumptions 7 and 9. For this purpose, we impose the same assumptions on u and Y made in Section 4, except that the following assumptions replace Assumptions 5 and 6, respectively.

Assumption 10 *u is three times continuously differentiable with $u' > 0$, $u'' < 0$ and $u''' > 0$ and u''' is non-increasing.*

Assumption 11 *The random variable $Y^2 u'''(\mu + \sigma Y)$ is integrable for every $\sigma \geq 0$ and every μ.*

Proposition 13 *Assumptions 3, 4, 10 and 11 imply Assumptions 7 and 9 and imply that the derivatives $U_{\mu\mu}$, $U_{\sigma\mu}$ and $U_{\sigma\sigma\mu}$ exist, are continuous, and have the following properties:*

- $U_{\mu\mu}(\sigma, \mu) = Eu''(\mu + \sigma Y) < 0$.
- $U_{\sigma\mu}(\sigma, \mu) = E[Yu''(\mu + \sigma Y)]$,
  $U_{\sigma\mu}(0, \mu) = 0$ *and* $U_{\sigma\mu}(\sigma, \mu) > 0$ *for* $\sigma > 0$.
- $U_{\sigma\sigma\mu}(\sigma, \mu) = E[Y^2 u'''(\mu + \sigma Y)] > 0$ *and*
  $U_{\sigma\sigma\mu}(0, \mu) = 2W_{v\mu}(0, \mu) = u'''(\mu)$.

PROOF: The proof is analogous to the proof of Proposition 5 with $-u'$ and $-U_\mu$ substituted for u and U, respectively. □

Note from the above proposition that the slope of the indifference curve of $-U_\mu$ in (σ, μ)-space at $\sigma = 0$ is

$$T(0, \mu) = -\frac{U_{\sigma\mu}(0, \mu)}{U_{\mu\mu}(0, \mu)} = 0$$

and that the slope in (v,μ)-space at $v=0$ is

$$-\frac{W_{v\mu}(0,\mu)}{W_{\mu\mu}(0,\mu)}=-\frac{1}{2}\frac{u'''(\mu)}{u''(\mu)},$$

or half the coefficient of prudence.

Proposition 14 *U is more prudent than $\bar{U}$ if and only if u is more prudent than $\bar{u}$.*

PROOF: The proof follows directly the proof of Proposition 6. We just need to notice that U is more prudent than $\bar{U}$ is equivalent to $-U_\mu$ being more risk averse than $-\bar{U}_\mu$. □

Proposition 15 *U exhibits increasing (decreasing, constant) prudence if and only if u exhibits increasing (decreasing, constant) prudence.*

PROOF: Cf. Proposition 7. □

Finally, we shall investigate the conditions under which W is concave. Concavity of W is a stronger requirement than concavity of U. For example, when W is concave, the limiting slopes of all the indifference curves of U in (σ,μ)-space are infinity. This implies that the portfolio selection problem always has an optimal solution. In the CAPM without a riskless asset, it also implies that there necessarily exists a satiation portfolio for the investor. Allingham (1991), in his study of existence of equilibrium in CAPM, assumed that W is concave, whereas Nielsen (1990b, 1992) assumed only that U is quasi-concave (or in some cases concave). Chipman (1973) found a sufficient condition for W to be concave in the expected-utility case with nomal distributions, but he did not give an economic interpretation. We shall provide the economic interpretation here. We shall show that in the case of normal distributions, W is concave if and only if U exhibits decreasing prudence.

First, let us show an example which illustrates that concavity of U does not imply concavity of W: consider the utility function $U(\sigma,\mu)=\mu-\sigma$. It is linear, hence concave. The corresponding W is

$$W(v,\mu)=U(\sqrt{v},\mu)=\mu-\sqrt{v}.$$

This function is not concave because the function

$$v \mapsto W(v,0) = -\sqrt{v}$$

is not concave [4].

On the other hand, concavity of W implies concavity of U. This can be seen as follows:

$$\begin{aligned}
U(t\sigma_1 + (1-t)\sigma_2, t\mu_1 + (1-t)\sigma_2) &= W((t\sigma_1 + (1-t)\sigma_2)^2, t\mu_1 + (1-t)\sigma_2) \\
&\geq W(t\sigma_1^2 + (1-t)\sigma_2^2, t\mu_1 + (1-t)\sigma_2) \\
&\geq tW(\sigma_1^2, \mu_1) + (1-t)W(\sigma_2^2, \mu_2) \\
&= tU(\sigma_1, \mu_1) + (1-t)U(\sigma_2, \mu_2),
\end{aligned}$$

The first inequality follows from the fact that

$$(t\sigma_1 + (1-t)\sigma_2)^2 \leq t\sigma_1^2 + (1-t)\sigma_2^2,$$

and the second inequality follows from the concavity of W.

We need to make the following assumption:

Assumption 12 *u is four times continuously differentiable with $u'''' \leq 0$.*

Note that in the case where Y follows a normal distribution, Chipman (1973) has shown that the second derivative of W with respect to v is the following:

$$W_{vv} = \frac{1}{4}Eu''''(\mu + \sqrt{v}Y).$$

Thus, Assumption 12 ensures that $W_{vv} \leq 0$. Recall that Proposition 13 ensures that $W_{\mu\mu} < 0$.

Proposition 16 *In the expected-utility case with normal distributions, the following conditions are equivalent :*

1. W is concave.

[4] The utility function in this example does not satisfy Assumption 7, but nevertheless it illustrates the point. Proposition 16 below can be used to generate examples which are in strict accordance with our assumptions.

2. U exhibits decreasing prudence.

3. The matrix

$$Q = \begin{pmatrix} u'' & u''' \\ u''' & u'''' \end{pmatrix}$$

is negative semi-definite.

PROOF:

(1) equivalent to (2) :

The utility function U satisfies the following differential equation (see Chipman (1973)) :

$$U_\sigma = \sigma U_{\mu\mu},$$

and the derivatives of U and W are related by

$$U_\sigma = 2\sigma W_v$$

$$U_\mu = W_\mu.$$

So,

$$2W_v = W_{\mu\mu}.$$

According to Proposition 9, the utility function U exhibits decreasing prudence if and only if $T(\sigma,\mu)$ is a decreasing function of μ. Since

$$T(\sigma,\mu) = -\frac{U_{\mu\sigma}}{U_{\mu\mu}} = -2\sigma\frac{W_{\mu v}}{W_{\mu\mu}},$$

$T(\sigma,\mu)$ is decreasing in μ if and only if $-W_{\mu v}/W_{\mu\mu}$ is decreasing in μ. The derivative with respect to μ is

$$-\frac{W_{\mu v\mu}W_{\mu\mu} - W_{\mu v}W_{\mu\mu\mu}}{W_{\mu\mu}^2} = -\frac{2W_{vv}W_{\mu\mu} - W_{\mu v}2W_{v\mu}}{W_{\mu\mu}^2} = -\frac{W_{vv}W_{\mu\mu} - W_{\mu v}W_{v\mu}}{W_{\mu\mu}^2}.$$

This derivative is non-positive if and only if

$$W_{vv}W_{\mu\mu} - W_{\mu v}W_{v\mu} \geq 0,$$

which means that the determinant of the matrix

$$\begin{pmatrix} W_{vv} & W_{v\mu} \\ W_{\mu v} & W_{\mu\mu} \end{pmatrix}$$

is non-negative.

Since $W_{vv} \leq 0$ and $W_{\mu\mu} < 0$, the determinant is non-negative if and only if the matrix is negative semi-definite, which is the case if and only if W is concave.

(2) equivalent to (3) :

The matrix Q is negative semi-definite if and only if u exhibits decreasing prudence. According to Proposition 16, this is the case if and only if U exhibits decreasing prudence. □

Chipman (1973, Theorem 3(c)) showed that statement 3 implies concavity of W. Our proposition says that they are equivalent.

A Proof of Proposition 5

First Let us show that Assumption 2 is satisfied. It is easily seen that $U(\sigma,\mu) \to \sup u$ as $\mu \to \infty$, and so the assumption is satisfied.

Now let us show that the derivatives of U exist and are continuous, and that Assumption 1 is satisfied.

The following derivatives:

$$U_\mu(\sigma,\mu) = Eu'(\mu + \sigma Y)$$

$$U_\sigma(\sigma,\mu) = E[Yu'(\mu + \sigma Y)]$$

follow from Propostion 5 of Nielsen (1993).Thus, $u'(\mu + \sigma Y)$ and $Yu'(\mu + \sigma Y)$ are integrable for all $\sigma \geq 0$ and all μ.

The existence and continuity of the second derivative with respect to σ

$$U_{\sigma\sigma} = E[Y^2 u''(\mu + \sigma Y)]$$

follows from Proposition 6 of Nielsen (1993), which requires u'' to be non-decreasing.

The sign of U_σ is as follows: $U_\sigma(0,\mu) = 0$ for $\sigma = 0$, while $U_\sigma(\sigma,\mu) < 0$ for $\sigma > 0$. The latter follows from the assumptions that Y is symm tric and $u'' < 0$. Formally,

$$U(\sigma,\mu) = E(1_{\{Y \geq 0\}}[u(\mu + \sigma Y) + u(\mu - \sigma Y)])$$

Thus,

$$U_\sigma(\sigma,\mu) = E(1_{\{Y \geq 0\}}[Y(u'(\mu + \sigma Y) - u'(\mu - \sigma Y))])$$

Since u' is strictly decreasing, $u'(\mu+\sigma Y) \leq u'(\mu-\sigma Y)$, with strict inequality whenever $Y \neq 0$. Hence, $U_\sigma(\sigma,\mu) < 0$.

From the above reasoning, it follows that W_μ exists and is continous everywhere,

$$W_\mu(v,\mu) = U_\mu(\sqrt{v},\mu) > 0,$$

and $W_v(v,\mu)$ exists and is continuous for $v > 0$:

$$W_v(v,\mu) = \frac{1}{2\sqrt{v}} U_\sigma(\sqrt{v},\mu) < 0.$$

Observe that $U_\sigma(0,\mu) = 0$ and $U_{\sigma\sigma}(0,\mu) = u''(\mu)$. Therefore,

$$W_v(0,\mu) = \frac{1}{2}\frac{U_\sigma(\sqrt{v},\mu) - U_\sigma(0,\mu)}{\sqrt{v}} \to \frac{1}{2}U_{\sigma\sigma}(0,\mu) = \frac{1}{2}u''(\mu)$$

as $v \to 0$. This shows that $W_v(0,\mu)$ exists and equals

$$W_v(0,\mu) = u''(\mu)/2 < 0.$$

It remains to show that W_v is continuous at $v = 0$. Let (v_n, μ_n) be a sequence such that $(v_n, \mu_n) \to (0, \mu)$. Then

$$W_v(v_n,\mu_n) - W_v(v_n,\mu) = \frac{1}{2\sqrt{v_n}}[U_\sigma(\sqrt{v_n},\mu_n) - U_\sigma(\sqrt{v_n},\mu)] \to 0.$$

Since we have already shown that

$$W_v(v_n,\mu) \to W_v(0,\mu)$$

it follows that

$$W_v(v_n,\mu_n) \to W_v(0,\mu)$$

Finally, the function U will be strictly concave because u is strictly concave. □

B References

1. Allingham, M.: "Existence Theorems in the Capital Asset Pricing Model," *Econometrica* 59 (1991), 1169–1174.

2. Arrow, K. J. :"Aspects of a Theory of Risk Bearing,"Yrjo Jahnson Lectures, Helsinki (1965). Reprinted in *Essays in the Theory of Risk Bearing* (1971). Chicago: Markham Publishing Co.

3. Bardsley, P: "Global Measures of Risk Aversion," *Journal of Economic Theory* 55 (1991), 145–160.

4. Bertsekas, D. P.: "Necessary and Sufficient Conditions for Existence of an Optimal Portfolio," *Journal of Economic Theory* 8 (1974), 235–247.

5. Chamberlain, G.: "A Characterization of the Distributions that Imply Mean-Variance Utility Functions," *Journal of Economic Theory* 29 (1983), 185–201.

6. Chipman, J. S.: "The Ordering of Portfolios in Terms of Mean and Variance," *Review of Economic Studies* 40 (1973), 167–190.

7. Drèze, J. H. and Modigliani F.: "Consumption Decisions under Uncertainty," *Journal of Economic Theory* 5 (1972), 308–335.

8. Elmendorf, W. D. and Kimball, M. S.: "Taxation of Labor Income and the Demand for risky assets," unpublished, University of Michigan (1988).

9. Epstein, L. G.: "Decreasing Risk Aversion and Mean-Variance Analysis," *Econometrica* 53 (1985), 945–961.

10. Kimball, M. S.: "Precautionary Saving in the Small and in the Large," *Econometrica* 58 (1990a), 53–73.

11. Kimball, M. S. : "Precautionary Saving and the Marginal Propensity to Consume," *NBER Working Papers Series*, Working Paper No. 3403 (1990b).

12. Kimball, M. S.: "Standard Risk Aversion," forthcoming in *Econometrica* (1993).

13. Kimball, M. S. and Mankiw, N. G.: "Precautionary Saving and the Timing of Taxes," *Journal of Political Economy* 97 (1989), 863-879.

14. Lintner, J.: "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics* 47 (1965), 13–37.

15. Leland, H. E.: "Saving Under Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics* 82 (1968), 465–473.

16. Markowitz, H.: "Portfolio Selection," *Journal of Finance* 7 (1952), 77–91.

17. Mehra, R. and Prescott, E.: "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 10 (1985), 335–339.

18. Nielsen, L. T.: "Portfolio Selection in the Mean-Variance Model: A Note," *Journal of Finance* 42 (1987), 1371–1376.

19. Nielsen, L. T.: "Existence of Equilibrium in CAPM," *Journal of Economic Theory* 52 (1990b), 223–231.

20. Nielsen, L. T.: "Existence of Equilibrium in CAPM: Further Results," *Nationaløkonomisk Tidsskrift* 130 (1992), 189–197.

21. Nielsen, L. T.: "Portfolio Choice and Equilibrium with Expected-Utility Preferences," unpublished, Insead (1992c),

22. Nielsen, L. T.: "The Expected Utility of Portfolios of Assets," forthcoming in the *Journal of Mathemathical Economics* (1993).

23. Pratt, J. W.:"Risk Aversion in the Small and in the Large," *Econmetrica* 32 (1964), 122–136.

24. Pratt, J. W. and R. J. Zeckhauser: "Proper Risk Aversion," *Econometrica* 55 (1987), 143–154.

25. Sandmo, A.: "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies* 37 (1970), 353–360.

26. Sharpe, W. F.: "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance* 19 (1964), 425–442.

27. Tobin, J.: "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies* 25 (1957), 65–86.

28. Weil, P.: "Equilibrium Asset Prices with Undiversifiable Labor Income Risk," *Journal of Economic Dynamics and Control* 16 (1992), 769–790.