"THE RESIDUAL LIFE OF THE RENEWAL PROCESS: A SIMPLE ALGORITHM"

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THE RESIDUAL LIFE OF THE RENEWAL PROCESS: A SIMPLE ALGORITHM

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ABSTRACT

We develop a simple algorithm for computing the distribution of the residual life when the renewal process is discrete. The algorithm does not require convolutions of the renewal process. We define the conditions under which the distribution converges and when it does not. We also investigate a well-known approximation for the mean and variance of the residual life widely used in inventory applications, based on the limit of the excess random variable as the number of renewals approaches infinity. We describe conditions under which the approximation can be used and when the algorithm is preferred.

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under which the residual life distribution will converge. Finally, in Section 6 we present a summary and conclusions.

2 - Literature Review

Karlin (1958) defines the excess random variable for a renewal process and presents its Laplace transform. The excess random variable (or the residual life) at time t is the time until the next renewal. Likewise, the deficit random variable (or the age) at time t is the time since the last renewal. Karlin then presents the value of the excess random variable for the case of the exponential distribution. He applies the excess random variable to the case of the (s,S) inventory policy, but he restricts the application to exponential demands. (In the case of exponential renewal processes, the residual life is also exponential.) Karlin also notes that the excess and the deficit random variables of a renewal process are identical. Ross (1983, pp. 67ff) discusses the excess and deficit random variables and notes their asymptotic behavior. Tijms (1976) develops the exact and approximate distributions for the excess random variable applied to (s,S) inventory systems for continuous demand distributions. Silver and Peterson (1985, pp. 346ff) draw on this work to present approximations of the mean and variance of the undershoot of the reorder point (the residual life of a renewal process) based on the limit as the order size (time since the process started) goes to infinity.

Sahin (1990, Chapter 2) discusses the renewal function and its shape. Applying the generalized cubic splining algorithm of McConalogue (1981), he computes the renewal function for five distributions: the Gamma, the Weibull, the truncated Normal, the Inverse Gaussian, and the Lognormal. The algorithm approximates the convolution of the renewal function by a cubic spline function. Sahin suggests that the accuracy of the approximation is 4 to 6 decimal places, and he notes that the renewal density may oscillate or may be monotone as it approaches its asymptote. In the context of inventory theory, he reports the order size (time since the process started) as a multiple of mean demand (mean time between renewals) such that the relative error of the renewal function relative to its asymptote is less than or equal to some constant but

arbitrary value. The research suggests that the accuracy of the asymptotic approximation is a function of the number of multiples of the mean, perhaps of distribution type and coefficient of variation of the distribution as well. There is no information on the magnitude of the error for small order sizes (time between renewals); rather, information is given only about how large the order size must be to obtain acceptable approximations. The order size varies from one-half the mean to 15 times mean demand to obtain approximations within 1 percent of exact value. Typical values are closer to 1.5 to 2.5 times mean demand. Our work advances Sahin's research by specifically examining the excess and deficit random variables rather than renewal functions. As a result, we extend the understanding of the errors in the commonly used renewal approximation.

In another work regarding the (s,S) inventory policy, Tijms and Groenevelt (1984) suggest that, if the coefficient of variation of demand over the review period is not extremely small, the undershoot approximation is accurate when the order size (time since the process started) is greater than 1.5 times the mean demand (mean time between renewals). In an accompanying research, we tested the accuracy of that approximation against the exact distribution given by the algorithm (Baganha, Pyke and Ferrer, 1994). We found that, for certain cases, our results differ substantially from the results of both Sahin and Tijms and Groenevelt.

Finally, we note that a number of authors have included the residual life in calculations pertaining to inventory. Beside the ones already mentioned, we list this sample: Silver (1970) applies the undershoot of the reorder point to items having lumpy demand. Cohen, Kleindorfer, Lee and Pyke (1992), apply the undershoot to multi-item (s,S) policies in logistics systems with lost sales. Ernst and Pyke (1992) apply it to the problem of ordering component parts that will be assembled into a final product. Many other examples exist in the literature.

3 - Renewal Theory and the Approximate Residual Life

In this section we briefly discuss the application of renewal models to discrete inventory system. We refer the reader for a more complete development to Heyman and Sobel (1982, Chapter 5), Ross (1983, Chapter 3), Silver and Peterson (1985, pp. 346ff) and Sahin (1990,

Chapter 2). We use the standard notation of renewal theory, with the addition of some problem-specific variables:

X_n	=	time between the (n-1)st and the nth renewal.
$F(x)$	=	cumulative distribution function (*cdf*) of the random variable X_n.
$f(x)$	=	probability density function (*pdf*) of the random variable X_n.
$F_n(x)$	=	n-fold convolution of $F(x)$ with itself.
$f_n(x)$	=	n-fold convolution of $f(x)$ with itself.
$M(j)$	=	$\sum_{n=1}^{\infty} F_n(j)$ The renewal function, the number of renewals at time j.
$m(j)$	=	$\frac{\partial M(j)}{\partial j} = \sum_{n=1}^{\infty} f_n(j) \quad j \geq 0$ The derivative of the renewal function.
μ	=	expected value of $X_n = E[X]$
σ^2	=	variance of X_n.
cv	=	coefficient of variation of $X_n = \sigma/\mu$
Δ	=	cumulative time since process started.
$Y(\Delta)$	=	random variable representing the residual life at time Δ.
$H_\Delta(y)$	=	*cdf* of $Y(\Delta)$, or $P(Y(\Delta) \leq y)$.
$h_\Delta(y)$	=	*pdf* of $Y(\Delta)$, or $P(Y(\Delta) = y)$.

The residual life will result from solving a renewal equation. The discrete case is given by

$$h_\Delta(y) = f(y+\Delta) + \sum_{j=0}^{\Delta-1} m(j) f(\Delta + y - j). \tag{1}$$

From the key renewal theorem, the limit as $\Delta \to \infty$ is computed and used as an approximation:

$$\lim_{\Delta \to \infty} H_\Delta(y) = \frac{1}{\mu} \sum_{x=0}^{y} [1 - F(x)].$$

The limit for the discrete case applies if the greatest common divisor of x for which $f(x) > 0$ is 1. We then have

$$\lim_{\Delta \to \infty} h_\Delta(y) = \frac{1 - F(y)}{\mu}. \tag{2}$$

Thus, the expected value of $Y(\Delta)$ as $\Delta \to \infty$ is given by Silver and Peterson (1985), who used the asymptotic distribution in (2) to derive its mean and variance:

$$\hat{\mu}_h = \sum_{y=0}^{\infty} y \frac{1 - F(y)}{\mu} = \frac{\sigma^2 + \mu^2}{2\mu} - \frac{1}{2} \tag{3}$$

$$\hat{\sigma}_h^2 = \sum_{y=0}^{\infty} y^2 \frac{1 - F(y)}{\mu} - \hat{\mu}_h^2 = \frac{E(X^3)}{3\mu} - \left[\frac{\sigma^2 + \mu^2}{2\mu}\right]^2 - \frac{1}{12} \tag{4}$$

Notice that this approximation assumes that Δ is very large. We will examine the robustness of this assumption in the following sections, by comparing it to the exact distribution of the residual life that we develop herein.

4 - Computation of the Residual life for the Discrete Case

In this section we develop an algorithm for computing the residual life of discrete processes, which we will use to test the approximations given by (3) and (4). We now establish basic equations for $m(0)$ and $m(\Delta)$.

Recall that

$$m(j) = \sum_{n=1}^{\infty} f_n(j) \text{ for } j \geq 0$$

Using standard results of renewal theory, we see that

$$m(j) = f(j) + \sum_{i=0}^{j} f(j-i) m(i) \tag{5}$$

Clearly, $m(0) = f(0) + f(0)m(0)$

and $$m(0) = \frac{f(0)}{1 - f(0)}. \tag{6}$$

Substituting $y = 0$ in (1), we have that

$$h_\Delta(0) = f(\Delta) + \sum_{j=0}^{\Delta-1} m(j) f(\Delta - j) \tag{7}$$

Now from (5)

$$m(\Delta) = f(\Delta) + \sum_{i=0}^{\Delta} f(\Delta - i)m(i)$$
$$= f(\Delta) + \sum_{i=0}^{\Delta-1} f(\Delta - i)m(i) + f(0)m(\Delta)$$

So, $\quad m(\Delta) = h_{\Delta}(0) + f(0)m(\Delta)$

and $\quad m(\Delta) = \dfrac{h_{\Delta}(0)}{1 - f(0)}$ (8)

We now develop an expression for $h_{\Delta+1}(y\text{-}1)$ in terms of $h_{\Delta}(y)$, $h_{\Delta}(0)$, $f(y)$, and $f(0)$. First, by equation (1), we can say that

$$h_{\Delta+1}(y-1) = f(y+\Delta) + \sum_{j=0}^{\Delta} m(j)f(\Delta + y - j)$$
$$= f(y+\Delta) + \sum_{j=0}^{\Delta-1} m(j)f(\Delta + y - j) + m(\Delta)f(y)$$
$$= h_{\Delta}(y) + m(\Delta)f(y)$$

which implies, by equation (8), that

$$h_{\Delta+1}(y-1) = h_{\Delta}(y) + \frac{h_{\Delta}(0)f(y)}{1 - f(0)}. \tag{9}$$

When $\Delta = 1$, using equation (1),

$$h_1(y) = f(y+1) + m(0)f(y+1)$$

which implies, using equation (6),

$$h_1(y) = \frac{f(y+1)}{1 - f(0)} \tag{10}$$

The algorithm for computing the residual life begins with $\Delta = 1$, using equation (10) to compute $h_1(y)$ for all y. It then repeats for all $\Delta = 2, 3, 4, \ldots$ using equation (9). More formally:

<u>Discrete Process Residual Life Algorithm</u>

1. Let $\Delta = 1$. Compute for all $y \le y_{max}$, where y_{max} is the largest value of residual life we are willing to consider:

$$h_1(y) = \frac{f(y+1)}{1-f(0)}.$$

2. Compute for all $y \le y_{max}$:

$$h_{\Delta+1}(y-1) = h_\Delta(y) + \frac{h_\Delta(0) f(y)}{1-f(0)}.$$

3. Let $\Delta = \Delta + 1$. If Δ is less than largest desired value, go to 2; otherwise stop.

This algorithm is simple. It can be easily implemented on a spreadsheet, requiring just $O(\Delta y_{max})$ basic operations to provide the exact probability distribution of the residual life for the values of Δ and y examined. The choice of y_{max} is clear when the support of the renewal process is bounded. When $f(y)$ has an unbounded support, like in Poisson processes, the algorithm will provide the exact distribution just for the range studied. The expected value of the residual life can also be obtained by means of the recursive equation which we develop below:

$$\begin{aligned}
E[Y(\Delta+1)] &= \sum_{y=0}^{\infty} y h_{\Delta+1}(y) \\
&= \sum_{y=0}^{\infty} y \left[h_\Delta(y+1) + \frac{h_\Delta(0) f(y+1)}{1-f(0)} \right] \\
&= \sum_{y=0}^{\infty} y h_\Delta(y+1) + \frac{h_\Delta(0)}{1-f(0)} \sum_{y=0}^{\infty} y f(y+1) \\
&= \sum_{y=0}^{\infty} (y+1) h_\Delta(y+1) + \frac{h_\Delta(0)}{1-f(0)} \sum_{y=0}^{\infty} (y+1) f(y+1) \\
&\quad - \sum_{y=0}^{\infty} h_\Delta(y+1) - \frac{h_\Delta(0)}{1-f(0)} \sum_{y=0}^{\infty} f(y+1) \\
&= E[Y(\Delta)] + \frac{h_\Delta(0)}{1-f(0)} E[X] - [1 - h_\Delta(0)] - h_\Delta(0).
\end{aligned}$$

Therefore,

$$E[Y(\Delta+1)] = E[Y(\Delta)] + \frac{h_\Delta(0)}{1-f(0)} E[X] - 1.$$

Note that, as $\Delta \to \infty$, the expected value of the residual life will converge if

$$h_\Delta(0) \to \frac{1-f(0)}{E[X]} \text{ as } \Delta \to \infty. \tag{11}$$

Equation 11 is a corollary to the Key Renewal Theorem.

5 - Conditions for Convergence of the Residual Life Distribution

Now the Key Renewal Theorem (KRT) requires that the distribution f not be lattice. A distribution is lattice if the random variable X, with density f, is such that there exists $k \geq 0$ such that $\sum_{n=0}^{\infty} P(X = nk) = 1$, where k is the periodicity of f. Strictly speaking, any discrete distribution is lattice. We generalize this application of the KRT showing that only if $k > 1$ will the density not converge. To do this, we begin with two useful Lemmas. The first we state without proof.

LEMMA 1. *If f is lattice, with positive probability only at values that are multiples of k, then $m(j) = \sum_{l=0}^{\infty} f_l(j)$ will be positive only for $j = nk$ for some integer $n \geq 1$.*

For example, a random variable that is of the two-mass-point type has positive probability only for two values. Denote these kn_1 and kn_2. The convolution will yield positive probability at values that are linear combinations of the type $\lambda_1 n_1 k$ and $\lambda_2 n_2 k$ that are also multiples of k, where λ_1 and λ_2 are integers.

LEMMA 2. *If f is lattice, with positive probability only at values that are multiples of k, then $h_\Delta(0) > 0 \Rightarrow \Delta = nk$ for some integer n.*

Proof: If $\Delta \neq nk$, then $f(\Delta) = 0$. For $h_\Delta(0) > 0$, by (7) we must have $m(j) > 0$ and $f(\Delta - j) > 0$ for some j. Thus, $j = n_1 k$ and $\Delta - j = n_2 k$. But this implies that $\Delta = j + \Delta - j = k(n_1 + n_2)$ which contradicts $\Delta \neq nk$.

QED

We now introduce a useful Proposition. We assume throughout that $f(0) < 1$.

PROPOSITION 1. *If $h_\Delta(y) = 0$ and $\Delta + y$ is not a multiple of k, then $h_{\Delta+1}(y - 1) = 0$, where k is the period of the lattice density, f.*

Proof: Since $h_{\Delta+1}(y-1) = h_\Delta(y) + \dfrac{h_\Delta(0) f(y)}{1 - f(0)}$ by (9), we require that $h_\Delta(0) > 0$ and $f(y) > 0$ for $h_{\Delta+1}(y - 1) > 0$. For $h_\Delta(0) > 0$, $\Delta = n_1 k$ by Lemma 2 for some integer n_1. For $f(y) > 0$, $y =$

n_2k, for some integer n_2, by the fact that f is lattice with period k. But $y = n_2k$ and $\Delta = n_1k$ imply that $\Delta + y = k(n_1 + n_2)$ which contradicts $\Delta + y \neq nk$. Thus $h_{\Delta+1}(y - 1) = 0$.

QED

In words, as Δ increases, the zeros in $h_\Delta(y)$ cycle. Figure 2 illustrates. Thus, $h_\Delta(y)$ and the expected value $E[Y(\Delta)]$ will not converge, if $k > 1$. We have shown that if $k \neq 1$, $h_\Delta(y)$ will not converge; i.e., different Δ may yield different probabilities for a given y, which is captured by the algorithm. Clearly, if $h_\Delta(y)$ does not converge, the two-moment approximation given by (3) and (4) is not appropriate, regardless of how long the process has been taking place.

If f is lattice with period k, Blackwell's theorem says (Ross, 1983)

$$\lim_{\Delta\to\infty} E\left[\text{number of renewals at } \Delta k\right] = \frac{k}{\mu}.$$

Hence, if $k = 1$, $m(\Delta) \to \frac{1}{\mu}$ as $\Delta \to \infty$ and $E[Y(\Delta)]$ converges. We now show that $h_\Delta(y)$ converges if $k = 1$.

PROPOSITION 2: *If f is lattice with period $k = 1$,* $\lim_{\Delta\to\infty} h_\Delta(y) - h_{\Delta-1}(y) = 0$. In other words, for any y, $h_\Delta(y)$ converges as $\Delta \to \infty$.

Proof: Recall that

$$h_{\Delta-1}(y-1) = h_\Delta(y) + \frac{h_\Delta(0)f(y)}{1-f(0)} \tag{9}$$

implying

$$h_\Delta(y-1) = h_{\Delta-1}(y) + \frac{h_{\Delta-1}(0)f(y)}{1-f(0)}.$$

Subtracting both expressions, we have

$$h_{\Delta-1}(y-1) - h_\Delta(y-1) = h_\Delta(y) - h_{\Delta-1}(y) + \frac{f(y)}{1-f(0)}\left(h_\Delta(0) - h_{\Delta-1}(0)\right)$$

By (8), $h_\Delta(0) = m(\Delta)(1-f(0))$. Thus, after appropriate substitution,

$$h_{\Delta+1}(y-1) - h_\Delta(y-1) = h_\Delta(y) - h_{\Delta-1}(y) + \left(m(\Delta) - m(\Delta-1)\right)f(y).$$

Using Blackwell's theorem for lattice distributions (see Ross 1983, p. 63), we have $\lim_{\Delta\to\infty} m(k\Delta) = \frac{k}{\mu}$. Hence, when k=1, $\lim_{\Delta\to\infty} m(\Delta) - m(\Delta-1) = \frac{1}{\mu} - \frac{1}{\mu} = 0$. Then, we have

$$\begin{aligned}\lim_{\Delta\to\infty}\left[h_{\Delta+1}(y-1) - h_{\Delta}(y-1)\right] &= \lim_{\Delta\to\infty}\left[h_{\Delta}(y) - h_{\Delta-1}(y) + \left(m(\Delta) - m(\Delta-1)\right)f(y)\right] \\ &= \lim_{\Delta\to\infty}\left[h_{\Delta}(y) - h_{\Delta-1}(y)\right]\end{aligned} \tag{12}$$

Because f is lattice, y can only take on integer values. In particular, let $y = 1$. Then,

$$\lim_{\Delta\to\infty}\left[h_{\Delta+1}(0) - h_{\Delta}(0)\right] = \lim_{\Delta\to\infty}\left[h_{\Delta}(1) - h_{\Delta-1}(1)\right]$$

Since $h_{\Delta}(0) = m(\Delta)\left(1 - f(0)\right)$, we also have that

$$\lim_{\Delta\to\infty}\left[h_{\Delta-1}(0) - h_{\Delta}(0)\right] = \lim_{\Delta\to\infty}\left[\left(m(\Delta+1) - m(\Delta)\right)\left(1 - f(0)\right)\right] = 0$$

Hence, $\lim_{\Delta\to\infty}\left[h_{\Delta}(1) - h_{\Delta-1}(1)\right] = 0$.

Using (12) recursively, for $y = 1, 2, \ldots, y_{max}$, we find that, for any integer y,

$$\lim_{\Delta\to\infty}\left[h_{\Delta}(y) - h_{\Delta-1}(y)\right] = 0$$

QED

Proposition 2 implies that if f is lattice with period k=1, the residual life distribution converges and the use of the approximations in (3) and (4) is appropriate for large order sizes.

6 - Summary and Conclusions

In this paper we have developed a simple algorithm for computing the distribution of the residual life of discrete renewal processes. We also provide a recursive formula to determine the expected value of the residual life as a function of cumulative time. We examine a commonly used renewal approximation for the mean and the variance of the undershoot of the reorder point in (s,S) inventory systems, indicating conditions under which the approximation is asymptotically correct or not. If the distribution is lattice of period 1, then the approximation is correct and may be used if the cumulative time is large. If the period of the lattice distribution is greater than 1, then the distribution of the residual life will not converge asymptotically, and the use of the algorithm is recommended.

With the computational algorithm suggested in Section 4, the use of the approximation may not be necessary. However, we do not expect that practitioners will always want to use an algorithm when a closed form approximation exists. In addition, the case of continuous demand still presents computational problems.

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