"A DYNAMIC PROGRAMMING APPROACH TO THE ECONOMIC DESIGN OF $\bar{X}$-CHARTS"

by

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A DYNAMIC PROGRAMMING APPROACH

TO THE ECONOMIC DESIGN OF X-CHARTS *

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ABSTRACT

Recent technological advances have rendered dynamic process control a viable alternative. A dynamic programming approach is proposed for the modeling and cost minimization of statistical process control activities. The decision parameters of the control chart are allowed to change dynamically as new information about the process becomes available. This general approach has been known as a theoretical possibility for many years, but its practical performance is explicitly investigated in this paper. It is shown with numerical examples that the dynamic programming solution can be much more economical than the conventional static solution with fixed control chart parameters. The substantial potential cost savings and the feasibility of a dynamic control procedure suggest that dynamic process control should replace standard statistical or economic design of control charts as the preferred method in automated production processes.

1. Introduction

A common assumption in the design of a process control procedure is that samples of constant size are taken at equal time intervals and decisions are based on a control chart with fixed control limits. Once the process operation begins, the decision variables (sample size, sampling interval, location of control limits) are treated as static, thus simplifying the mathematical analysis in the chart design stage and the actual on-line implementation. One exception to that approach is the work of Parkhideh and Case [6], where the decision variables are allowed to vary over time, but in a specific, prespecified way. In other recent contributions, Reynolds et al. [8], Reynolds and Arnold [9], Reynolds [7] and Runger and Pignatiello [10] allow variable (adaptive) sampling intervals, but keep sample sizes and control limits fixed.

While simplicity and ease of implementation were critical until a few years ago, the tremendous recent developments in factory automation, and, consequently, automated inspection and measurement and computerized data analysis have attenuated the importance of simplicity in the implementation of process control procedures. The way is now open to dynamic SPC approaches, which take all available information into account and continuously update the decision parameters of the control scheme.

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A question that arises naturally at this point is whether more sophisticated techniques are justified on an economic basis. The purpose of this paper is to initiate the investigation of this important question by proposing an alternative, dynamic approach to the on-line control of production processes, formulating a mathematical model and making comparisons with static methods. The theoretical background and past related research on the issue of optimal design of control charts are laid out in the next section. Section 3 presents a dynamic programming approach for the optimization of process control for a class of problems, where the decision variables are allowed to change dynamically during operation. Numerical examples are presented in Section 4. Comparisons with standard economic design in Section 5 indicate that adoption of a dynamic programming approach can result in significant cost savings. The paper's contributions and conclusions are summarized in Section 6.

2. Bayesian Framework for SPC

After their introduction in the 1930s, control charts were designed based on statistical considerations only, with fixed sample size, sampling interval and control limits. Cost modeling of SPC was popularized by Duncan [3], who presented a simple model for the economic design of control charts, also assuming static decision parameters. Since then, most researchers have concentrated on the optimal economic design of control charts assuming fixed parameters, although earlier Girshick and Rubin [4] had introduced a cost model for the control of production processes using a Bayesian approach, and Taylor [12] further discussed this model and proved that sampling frequency and control decisions are optimally determined from the posterior probability that the process is in an out-of-control state, implying that the common practice of fixed design parameters leads to suboptimal decisions. In the case of processes characterized by two states, an in-control state and an out-of-control state, Girshick and Rubin [4] showed that when inspection costs are ignored and 100% inspection is assumed, the minimal cost rule is of the form: "Stop and repair at time t if and only if the posterior probability at time t that the process is in the bad state exceeds a critical value λ." Taylor [13] provided a method for choosing the optimal critical value λ, when the shift in the process mean in the out-of-control state is small.

Bather [1] proposed another Bayesian model for the control of processes subject to a continuous shift in the process mean, where the sample size and sampling interval are assumed fixed, but the effective control limits are dynamic; this model was extended by Carter [2] to allow for variable sample size and sampling interval.

Despite their theoretical importance, the aforementioned models have not been used in practice because of the complicated rules some of them advocate, of unrealistic simplifying assumptions and/or extensive computational requirements. Moreover, there are only limited numerical examples of model application in those papers and there has been no attempt to compare the Bayesian models with the prevailing static designs and to illustrate the extent of superiority of the former.

3. A Dynamic Programming Formulation

The Bayesian framework will be used in this section for the modeling and optimization of a class of industrial processes, where process control procedures are allowed to have a dynamic form with design parameters changing over time. We consider here a production process subject to a single assignable cause, or to multiple assignable causes but with identical effect on the process. The following notation is used in the development of the mathematical models:

X - quality characteristic.

μ_j - mean of X when the process is in state j, $j = 0, 1$.

σ - standard deviation of X.

δ - magnitude cf the out-of-control shift in the process mean in multiples of σ. The shift is $\delta\sigma$.

$f_j(X)$ - normal probability density function of X, when the process is in state j, $j = 0, 1$.

ν - rate of occurrence of assignable causes.

H - length of production run in time units of continuous operation.

n - sample size.

h - length of sampling interval in the static model.

m - number of sampling intervals in the static model. $m = H/h$.

k - parameter of control limit location in the static model. The control limit is located at a distance of $k\sigma/\sqrt{n}$ from the central line.

h_i – length of sampling interval after stage i of the dynamic programming model.

k_i – parameter of control limit location at stage i of the dynamic programming model.

S – inspection cost per measurement.

L_0 – cost of false alarm.

L_1 – cost of restoration from state 1 (out of control) to state 0 (in control).

M – loss rate during operation in the out-of-control state.

α_0 – probability of false alarm.

α_1 – probability of detecting a shift, when there is an assignable cause.

p_i – prior probability that the process is out of control, before the potential i^{th} inspection of the dynamic programming model.

$\tilde{p}_i$ – posterior probability that the process is out of control, after the potential i^{th} inspection but before possible restoration.

p_i' – posterior probability that the process is out of control, after the potential i^{th} inspection and, possibly, restoration.

$p'_{i,max}(p'_{i-1},h_{i-1},k_i)$ – maximum possible value of p_i' for given values of p'_{i-1}, h_{i-1}, k_i.

$q(h_i)$ – probability of occurrence of assignable cause in the time interval h_i.

$\Phi(.)$ – cumulative probability function of a standard normal variable.

$E[.]$ – expected value.

There is a single continuous quality characteristic of interest, denoted by X, which can be viewed as a characteristic of the process output or as a characteristic of a process itself. When the process is in the in-control state 0, X is assumed to be normally distributed with mean μ_0 and standard deviation σ. The process operates in control until the occurrence of the assignable cause, which has the effect of instantaneously increasing the process mean to $\mu_1 = \mu_0 + \delta\sigma$ ($\delta>0$) without changing σ. The time of occurrence of the assignable cause is assumed to be an exponentially distributed random variable with mean 1/v. It should be noted that the assumptions of constant σ and exponential distribution of the failure time are not necessary for the development of the dynamic programming formulation. They are maintained here so that direct comparisons with existing static models can be made. Unlike the usual assumption in the process control literature, the process does not operate indefinitely, but it is set up for a production run of length H in time units of continuous operation, that will produce a prespecified lot size. At the end of the production run the machine is re-set for the next production run of the same or a different item. This model represents many practical situations and it is similar to that formulated by Ladany [5], with the difference that the quality characteristic here is continuous and a variables control chart is called for, while in [5] a fraction-defective chart was employed.

The control chart design suggested by Ladany [5] assumes constant sample size n, constant sampling interval h, and fixed location of the control limits. For simplicity of notation and exposition, it has been assumed here that the assignable cause can only result in an increase in the process mean by $\delta\sigma$, so that only a single upper control limit is required. Moreover, it is also assumed that the sample size is always $n=1$, i.e., a single measurement is taken every h time units. This measurement can come directly from the machine or process, or indirectly from the process output. The single control limit is then located at a distance $k\sigma$ above the central line and an alarm is issued when the value of the measurement X exceeds $\mu_0 + k\sigma$. The assumption of constant $n=1$ is made here to facilitate direct correspondence and comparison with the dynamic control chart design, which will be proposed in the following paragraphs. This assumption, though, can be easily relaxed; in such a case it is convenient to express the position of the single upper control limit as $\mu_0 + k\sigma/\sqrt{n}$.

The inspection cost is S per sample measurement. When an alarm is issued the process is stopped and a restoration procedure is initiated. The restoration cost is L_0 when the alarm is false and L_1 when the process is actually out of control. The cost of downtime is included in L_0 and L_1. The loss rate when operating in the out-of-control state is M per time unit. If $n=1$, the single upper control limit is at $\mu_0 + k\sigma$ and $m-1$ samples are taken during the production run, so that the sampling interval is $h = H/m$. Then, similar to [5], the total quality control related cost during a production run, $TC(k,m)$, can be written as

$$TC(k,m) = (m-1)S$$

$$+ [Mh(1-e^{-vh})/2] \sum_{i=0}^{m-1} F(i)\{1+2(1-\alpha_1)[1-(1-\alpha_1)^{m-1-i}]/\alpha_1\}$$

$$+ L_0\alpha_0 e^{-vh} \sum_{i=0}^{m-2} F(i)$$

$$+ L_1(1-e^{-vh}) \sum_{i=0}^{m-2} F(i)[1-(1-\alpha_1)^{m-1-i}] \qquad (1)$$

where $\alpha_0 = \Phi(-k)$ is the probability of false alarm, $\alpha_1 = \Phi(\delta-k)$ is the probability of correct indication of an out-of-control state and

$$F(i) = [(1-\alpha_1)^{i+1}(1-e^{-vh})e^{-vih}+\alpha_1]/[1-(1-\alpha_1)e^{-vh}], \qquad i=0,1,\ldots,m-1$$

is the probability that the sampling interval after the i^{th} sample will start with the process in the in-control state. Since the initial set-up is assumed correct, $F(0) = 1$. The derivation of (1) is outlined in the Appendix.

The cost function (1) can be easily minimized using a direct search procedure that identifies the optimal values of k and m, and consequently h. For example, if $S=1$, $M=30$, $L_0=100$, $L_1=150$, $H=40$, $v=0.02$ and $\delta=1$, the optimal static solution yields $k=1.7$ and $m=13$ ($h=3.08$) at a cost of 299.42 per production run.

It has already been stated that the optimal SPC scheme for the above problem must be dynamic, based on the posterior probability that the process is in the out-of-control state 1. In the remainder of this section a dynamic programming formulation will be presented, similar in spirit to that of Carter [2]. The inspection epochs will constitute the stages, while the state of the dynamic program will be the probability that the process is out of control.

In this framework, the assumption $n=1$ is not limiting in many practical situations, since the sampling interval is variable and dynamically determined. In fact, we have a more accurate representation of the inspection process in the usual cases where measurements are made one by one, as, for instance, in continuous flow processes. The variable sampling interval, meanwhile, allows essentially for any (variable) sample size, as we may require consecutive measurements in consecutive inspection epochs, which may be very close to each other (provided that the minimum sampling interval is chosen small enough relative to the production rate). Moreover, by not examining a sample of $n>1$ as a whole, we exclude from consideration the troublesome possibility of occurrence of the assignable cause during the collection of a sample, in which case some of the items in the sample are produced with the process in control, while the rest are produced with the process out of control.

The derivation of the out-of-control probability follows the analysis in Taylor [13]. Let $f_0(X)$ and $f_1(X)$ be the normal probability density functions of the quality characteristic X, when the process is in states 0 and 1 respectively, i.e.

$$f_j(x) = (1/\sigma\sqrt{2\pi})\exp(-(1/2)[(x-\mu_j)/\sigma]^2) \quad , \quad j = 0,\ 1. \qquad (2)$$

Suppose that before measurement i is taken, the prior probability that the process is in state 1 is p_i and that the value of measurement i is x. Since the likelihood of the latter is $p_i f_1(x) + (1-p_i)f_0(x)$, from Bayes' rule we conclude that the posterior probability is $\tilde{p}_i = p_i f_1(x)/[p_i f_1(x)+(1-p_i)f_0(x)]$. Let the control limit be located at $\mu_0 + k_i\sigma$ at that instant. The decision rule: "Restore if $x > \mu_0 + k_i\sigma$" is proven below to be equivalent to a decision rule of the form suggested by Girshick and Rubin [4] and Taylor [13]: "Restore if $\tilde{p}_i > \lambda$" for some critical value λ. Using (2) $\tilde{p}_i > \lambda$ can be written as

$$\frac{p_i \exp(-(1/2)[(x-\mu_1)/\sigma]^2)}{p_i \exp(-(1/2)[(x-\mu_1)/\sigma]^2) + (1-p_i)\exp(-(1/2)[(x-\mu_0)/\sigma]^2)} > \lambda$$

and after some algebraic manipulations $\tilde{p}_i > \lambda$ is found to be equivalent to $x > \mu_0 + k_i\sigma$, where

$$k_i = \delta/2 + \delta^{-1}\ln([\lambda(1-p_i)]/[(1-\lambda)p_i]). \qquad (3)$$

Thus the control limit parameter corresponding to λ is the parameter k_i given by (3).

As expected, k_i is increasing in λ and decreasing in p_i, since a large prior probability of the process being in state 1 induces a tight control limit to ensure that restoration will be initiated if the posterior probability exceeds the critical value. For a given k_i it is easy to show that the corresponding λ is computed from

$$\lambda = \frac{[p_i/(1-p_i)]\exp(k_i\delta-\delta^2/2)}{1 + [p_i/(1-p_i)]\exp(k_i\delta-\delta^2/2)} \quad . \qquad (4)$$

The posterior probability that the process will be in state 1 after measurement i and, possibly, restoration (if $x > \mu_0 + k_i\sigma$) is denoted by p_i'. After measurement x is obtained, a process adjustment will take place with probability $p_i\alpha_1 + (1-p_i)\alpha_0$, where $\alpha_1 = \Phi(\delta-k_i)$ and $\alpha_0 = \Phi(-k_i)$. For simplicity and consistency with Ladany's model, it is assumed that the adjustment is perfect, so that after restoration $p_i' = 0$. However, this assumption can be easily relaxed in the dynamic programming formulation. If there is no out-of-control signal after measurement i ($x < \mu_0 + k_i\sigma$), then $p_i' = \tilde{p}_i > 0$.

Suppose that the time interval between inspections i-1 and i is h_{i-1}, and let $q(h_{i-1})$ be the probability of occurrence of the assignable cause in that time interval if the process starts in state 0. Then

$$p_i = p_{i-1}' + (1-p_{i-1}')q(h_{i-1}) \tag{5}$$

where $q(h_{i-1}) = 1 - \exp(-vh_{i-1})$, since the time of occurrence of the assignable cause is an exponentially distributed random variable. The state transition mechanism in the dynamic programming formulation is described by the relationship between p_{i-1}' and p_i'. For specified p_{i-1}', h_{i-1} and k_i at stage i, the posterior probability that the system is restored to state 0, is

$$\Pr\{p_i'=0 \mid p_{i-1}', h_{i-1}, k_i\} = [p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]\alpha_1 + [1-p_{i-1}'-(1-p_{i-1}')q(h_{i-1})]\alpha_0 \quad . \tag{6}$$

A strictly positive value of p_i' for given p_{i-1}', h_{i-1}, k_i will be realized for that value of x at measurement i which satisfies $x < \mu_0 + k_i\sigma$ and

$$p_i' = \frac{[p_{i-1}' + (1-p_{i-1}')q(h_{i-1})]f_1(x)}{[p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]f_1(x) + [1-p_{i-1}'-(1-p_{i-1}')q(h_{i-1})]f_0(x)} .$$

Using (2) and taking logarithms, the above expression can be written as

$$x = \mu_0 + \left[\delta/2 + \delta^{-1} \ln \frac{p_i'[1-p_{i-1}' - (1-p_{i-1}')q(h_{i-1})]}{(1-p_i')[p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]} \right] \sigma . \quad (7)$$

From (7) and $x < \mu_0 + k_i\sigma$ it follows that

$$\delta/2 + \delta^{-1} \ln \frac{p_i'[1-p_{i-1}' - (1-p_{i-1}')q(h_{i-1})]}{(1-p_i')[p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]} < k_i .$$

Therefore, the allowable values of p_i' are

$$p_i' < \frac{\{[p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]/[1-p_{i-1}'-(1-p_{i-1}')q(h_{i-1})]\}\exp(k_i\delta-\delta^2/2)}{1+\{[p_{i-1}'+(1-p_{i-1}')q(h_{i-1})]/[1-p_{i-1}'-(1-p_{i-1}')q(h_{i-1})]\}\exp(k_i\delta-\delta^2/2)}$$

$$= p_{i,max}'(p_{i-1}', h_{i-1}, k_i) . \quad (8)$$

Expressions (4), (5) and (8) make obvious that larger values of p_i' signify that the equivalent critical value λ is exceeded and a restoration is performed resulting in $p_i' = 0$.

In practice the dynamic programming computations require a discretization of the state variable p_i'. Assume that the simplest approximation is adopted, according to which the cost functions are always evaluated at the nearest quantized state. Then, dynamic programming state p_i at stage i includes all the values of p_i' in an interval $(\underline{p}_i, \overline{p}_i)$. The transition probability from quantized state

p^j_{i-1} to quantized state $p^j_i > 0$ for given h_{i-1} and k_i is therefore computed from

$$\Pr\{p^j_i | p^j_{i-1}, h_{i-1}, k_i\} = \Pr\{\mu_0 + A\sigma < x < \mu_0 + B\sigma\}, \quad \underline{p}^j_i < p^j_{i,max}(p^j_{i-1}, h_{i-1}, k_i)$$

where

$$A = \delta/2 + \delta^{-1} \ln \frac{\underline{p}^j_i [1 - p^j_{i-1} - (1 - p^j_{i-1}) q(h_{i-1})]}{(1 - \underline{p}^j_i)[p^j_{i-1} + (1 - p^j_{i-1}) q(h_{i-1})]}$$

$$B = \begin{cases} \delta/2 + \delta^{-1} \ln \dfrac{\overline{p}^j_i [1 - p^j_{i-1} - (1 - p^j_{i-1}) q(h_{i-1})]}{(1 - \overline{p}^j_i)[p^j_{i-1} + (1 - p^j_{i-1}) q(h_{i-1})]}, & \overline{p}^j_i \le p^j_{i,max}(p^j_{i-1}, h_{i-1}, k_i) \\ \delta/2 + \delta^{-1} \ln \dfrac{p^j_{i,max} [1 - p^j_{i-1} - (1 - p^j_{i-1}) q(h_{i-1})]}{(1 - p^j_{i,max})[p^j_{i-1} + (1 - p^j_{i-1}) q(h_{i-1})]}, & \overline{p}^j_i > p^j_{i,max}(p^j_{i-1}, h_{i-1}, k_i) \end{cases}$$

and

$$\Pr\{p^j_i | p^j_{i-1}, h_{i-1}, k_i\} = 0 \quad , \quad \underline{p}^j_i \ge p^j_{i,max}(p^j_{i-1}, h_{i-1}, k_i) \ .$$

Note that $\Pr\{p^j_i | p^j_{i-1}, h_{i-1}, k_i\}$ is essentially a shorthand notation for $\Pr\{\underline{p}^j_i \le p^j_i \le \overline{p}^j_i | p^j_{i-1}, h_{i-1}, k_i\}$, namely for the probability that, after measurement i, the posterior probability p^j_i is between $\underline{p}^j_i$ and $\overline{p}^j_i$, given that the respective probability was p^j_{i-1} after the previous measurement and that decisions h_{i-1} and k_i were made at that time.

For p^j_i such that $\underline{p}^j_i < p^j_{i,max}(p^j_{i-1}, h_{i-1}, k_i)$, after some rearrangement we obtain

$$\Pr\{p_i' \mid p_{i-1}', h_{i-1}, k_i\} = [p_{i-1}' + (1-p_{i-1}')q(h_{i-1})][\Phi(B-\delta) - \Phi(A-\delta)]$$

$$+ [1-p_{i-1}' - (1-p_{i-1}')q(h_{i-1})][\Phi(B) - \Phi(A)] \ . \quad (9)$$

The dynamic programming formulation can now be formally laid out. The stage i in that formulation will be defined as the <u>potential</u> i^{th} inspection. The total production run interval H is divided into m equal subintervals of length $h=H/m$, where h is the time unit (the minimum interval between inspections). Stage 0 signifies the beginning of the production run while stage m signifies its end. The actual interval between two successive inspections is allowed to be any multiple of h and it may vary within the same production run. Thus, the stage definition differs from that of the conventional dynamic programming formulation, since only a subset of the possible m stages may be visited starting from stage 0.

At stage $i < m$ two decisions must be made based on the state p_i'. The first is the time of the next measurement, defined by $h_i = jh$, $j \in \{1,2,\ldots,m-i\}$; $j = m-i$ implies that there will not be any other measurement in the production run. The second decision is the location of the control limit k_{i+j} at the next time that a measurement is taken and an adjustment in the process may be performed (stage i+j, if $h_i = jh$). At stage m it can be assumed that the terminal cost for any state is zero, if the process is re-set anyway for the next production run. This assumption can be easily relaxed and a terminal cost dependent on the final stage can be incorporated.

The iterative functional equation of the dynamic programming expresses the minimum expected cost from state p_i' at stage i until the end of the production run as

$$f(p_i',i) = \min_{\substack{h_i=jh \\ k_{i+j}}} \{C(p_i',\ h_i=jh,\ k_{i+j},\ i) + E[f(p_{i+j}',\ i+j)]\} \qquad \begin{array}{l} i=0,1,\ldots,m-1, \\ j=1,2,\ldots,m-i \end{array} \qquad (10)$$

where $C(p_i',\ h_i=jh,\ k_{i+j},\ i)$ is the expected cost incurred between stage i (after adjustment at stage i, if any) and stage i+j (after adjustment at stage i+j, if any) for a given state p_i' and combination of decision variables $h_i=jh$ and k_{i+j} at stage i. Since the inspection/measurement cost S and the potential restoration cost L_0 or L_1 at stage i+j are included in $C(p_i',\ h_i=jh,\ k_{i+j},\ i)$, we can compute the expected cost between stages i and i+j from

$$C(p_i',\ h_i=jh,\ k_{i+j},\ i) = \begin{cases} p_i'[Mjh + \alpha_1(k_{i+j})L_1 + S] \\ \quad + (1-p_i')[M(vjh-1+e^{-vjh})/v + S \\ \qquad + q(h_i)\alpha_1(k_{i+j})L_1 \\ \qquad + (1-q(h_i))\alpha_0(k_{i+j})L_0]\ , & j < m-i \\ p_i'[Mjh] + (1-p_i')[M(vjh-1+e^{-vjh})/v]\ , & \\ & j = m-i \end{cases} \qquad (11)$$

with $\alpha_1(k_{i+j}) = \Phi(\delta-k_{i+j})$, $\alpha_0(k_{i+j}) = \Phi(-k_{i+j})$. The term $[vjh-1+\exp(-vjh)]/v$ is the expected time the process is out of control in an interval of length jh that starts with the process in control and is derived as in Duncan [3], taking into account the probability $q(h_{i=jh})$ that there will be a shift in the process mean within jh time units.

4. Implementation Issues and Examples

Since there are no analytical results directly determining optimal decision parameters for a given state p_i' at stage i, the optimal combination of h_i, k_{i+j} must be selected among all possible solutions through direct comparison. The state space and decision space should therefore be appropriately quantized so that a near-optimal solution can be obtained with reasonable computational requirements. While a finer quantization of the state space contributes mainly to a better approximation of the true total cost for a given policy, the major effect of a finer quantization of the decision space is to obtain a solution that is closer to the true optimum.

Extensive experimentation with a number of numerical examples provided many insights that can be useful guidelines in the actual computer implementation of the dynamic programming formulation.

Since at each stage i there is a possibility of restoration resulting in $p_i'=0$, the state space should be discretized so that there is one state corresponding to $p_i'=0$ and N_p additional states dividing the (0,1) interval in N_p equal subintervals of length equal to the step $s_p=1/N_p$. Thus state 0 corresponds to $p_i'=0$, and state n_p to $p_i' \in ([n_p-1]s_p,\ n_p s_p)$ with quantized value $(n_p-1)s_p+s_p/2$, $n_p=1,2,\ldots,N_p$. Quantization with $N_p=100$ ($s_p=0.01$) provides satisfactory compromise between accuracy and computational burden. More specifically, with the adopted discretization of the decision space that is described below, a dynamic programming solution required 4-8 minutes of CPU time (depending mainly on the number of stages) on a VAX 6410, when N_p=100. A finer quantization with N_p=125 increased the required CPU time to 28-180 minutes, while improving the accuracy of the solutions by less than 0.5% .

The optimal static solution can form the basis for the discretization of the decision space. Assuming N_h discrete values of h_i=jh (j=1,2,...,N_h) and N_k discrete values of k_{i+j}, there is a total of N_hN_k alternative decisions at each stage i. The total number of stages m+1 is related to the minimum h and the quantization of h_i. In practice, a relatively small value of N_h can be chosen and h be defined so that the optimal sampling interval of the static solution is an allowable h_i in the middle of the range of quantized h_i. The quantization of k_{i+j} should adequately cover the range (0,4). Numerical experience suggests that for fixed, sufficiently large N_k (e.g. N_k=14) the choice of actual values of k has negligible effect on the optimal solution. Therefore, an even spacing of k values in (0,4), independent of the optimal static k is recommended for ease of implementation.

A comparison between static and dynamic solutions was performed through 24 numerical examples. In all examples the inspection cost S=1 was used as the base cost. The values of the other cost parameters M, L_0, L_1, the length of the production run H, the rate of occurrence of assignable causes v and the magnitude of the shift δ are given in Table 1, together with the optimal static solution, the dynamic programming solution and the cost improvement resulting from the adoption of the latter approach rather than the former.

Table 1 about here

In each of the 24 cases the optimal parameters k and m of the static formulation were obtained by minimizing the cost function (1), while for the dynamic programming solution the state space was quantized with $N_p=100$ and the decision space with $N_h=8$ and $N_k=14$. The minimum sampling interval h was taken equal to H/4m, where m is the number of sampling intervals in the optimal static solution. There were two exceptions to that quantization rule: (a) when m=2 (Cases 1,3,5,6,7,8,21,23,24) the minimum sampling interval was h=H/8m=H/16; Case 15 (m=3) also used h=H/16 for consistency in the minimum h employed; (b) when m>20 (Cases 11,12,16,20), h=H/80. The first of the above modifications was dictated by the need for finer quantization in cases where there is only one inspection (in the middle of the production run) or two in the static solution. Reducing h to H/16 in those cases resulted in significant cost reductions (e.g. from 199.22 to 186.30 in Case 21). The second adjustment was made to keep the number of stages small (at most 80) and the computational requirements low. In the quantization of k, 14 evenly spaced values were used for k_{i+1} at every stage i, namely: 0.1, 0.4, 0.7, 1.0, 1.3, 1.6, 1.9, 2.2, 2.5, 2.8, 3.1, 3.4, 3.7, 4.0. Different sets of k values were also tried but the differences in the cost of the solutions were insignificant.

5. Analysis of Results

The most important conclusion from Table 1 is undoubtedly that the potential cost savings from the adoption of the dynamic programming approach and solution are large. The average cost improvement over the optimal static solution in the 24 Examples of Table 1 is 14.5%, with many cases exhibiting savings over 20% and as high as 25.8% . The lowest observed cost improvement was 2.9%, but in only 2 of the 24 cases it was lower than 5% . It must be noted that the observed savings are not the highest possible for the particular examples. Even better solutions can be derived and even greater savings can be realized with a finer quantization of the decision space.

A number of other interesting observations may also be made on Table 1. The cost improvement due to the DP approach in general increases as the duration of the production run H increases (compare Cases 1-8 where H=40, with Cases 17-24, where H=80). The implication is that the DP improvement is not due to the existence of terminal conditions, but it extends to situations of long production runs approaching steady state. From the same cases it can be observed that the cost per time unit for an optimal policy increases with H, as it approaches that of the steady-state solution. The superiority of the DP solution is also more pronounced as the failure rate v decreases and as the cost of out-of-control operation M increases relative to the restoration costs L_0, L_1. This can be explained by the longer periods of incomplete information when v is small. In such cases, the value of maintaining all previous information in p_i' is larger, and the unavoidable restoration cost is not dominant, so that the benefit realized from the use of the more complete information is greater. Finally, Table 1 indicates that the cost reduction is in general larger when δ increases. Since conventional $\overline{X}$-charts are known to perform very well when δ is large rather than small, while if δ is small other control charts are preferred (e.g., cusum charts, charts with warning lines), it can be concluded that the dynamic chart outperforms static control charts by a wider margin, when the shift in the process mean is large. Even when δ is small, the dynamic control chart is still superior to static charts like cusum charts, since in addition to maintaining all relevant information it dynamically updates the chart parameters. It is understood, though, that the results of Table 1 are only indicative of the potential cost reductions from the use of dynamic charts. An accurate analysis of the magnitude of cost savings with respect to different types of static control charts in various situations (combinations of values of model parameters) would require a much more extensive and detailed experimental design with a large number of numerical examples.

The detailed DP solutions to the Examples of Table 1, which are not presented here due to space limitations, reveal some intuitive and practical useful properties of the optimal solution. Specifically, it was observed that in all cases and at all stages i, the optimal interval h_i until the following inspection is non-increasing in the state p_i'. In other words, the higher the probability that the process is out of control, the sooner the next sample measurement should be taken to avoid long periods of out-of-control operation. Moreover, for a given h_i at stage i, the optimal control limit width k_{i+1} is also non-increasing in p_i'. Thus, if the next inspection is after a fixed time, the highest the out-of-control probability, the tighter the control limit should be to avoid a Type II error in the next inspection. These intuitively appealing observations can be stated as conjectures whose proofs remain elusive, and can be of tremendous importance in reducing the computational requirements of the dynamic program. If these conjectures are assumed to hold and are integrated in the computational procedure, at every stage and state a large part of the decision space can be eliminated as consisting of dominated solutions. The computational savings realized in that manner may then allow a finer quantization of the undominated part of the decision space and, consequently, may result in more accurate and economical solutions.

6. Conclusions

A dynamic programming formulation has been presented as an alternative way of studying the SPC problem of automated production processes. In contrast to the static economic design, which dictates constant sampling intervals, sample sizes and control limits, the DP approach allows a dynamic adjustment of those decision parameters based on a continuously updated Bayesian estimate of the state of the process. As all the available information is used for decision-making and the decision parameters are not constrained to stay constant, the DP method may result in substantial improvement over the static SPC procedure, as it has been demonstrated through numerical examples. Moreover, the DP framework facilitates the incorporation of features like imperfect restoration after an alarm, which are more difficult to include in the conventional static models.

Certain simplifying assumptions have been made in the course of developing the DP formulation, with the purpose of clarifying the exposition. These assumptions can be relaxed without invalidating the approach but increasing the notational complexity and/or the computational requirements. The case of both positive and negative shifts in the process mean, for instance, can be addressed within the DP conceptual framework, but a more general definition of the state of the dynamic program is needed. Specifically, at every stage i the knowledge about the state of the system is described by two posterior probabilities that the process is out-of-control (due to an upward or a downward shift in the process mean) and the complementary in-control probability. Hence, the state space is larger and similarly the decision space, since the locations of two control limits, not necessarily symmetric around the central line [11], must be determined.

The analysis in this model pertained to the control of a process during a finite production run. A linkage to the next production run can be accomplished through an amendment in terminal conditions, which may accommodate a state dependent reset cost. In the majority of the process control literature it is assumed that the process operates indefinitely without periodic scheduled set-ups but with overhauls only after a signal is issued from the control chart. The DP approach to that infinite horizon problem would entail the solution of an infinite stage dynamic program. Under mild conditions that are satisfied in the present context, Taylor [12] has proved the existence of an optimal stationary non-randomized solution in the cases where discounted cost and average cost per time unit are used as performance measures. In that same paper it was also shown that the average cost solution is the limit, in some sense, of the discounted cost solution as the discount factor approaches 1. In practice, the infinite horizon solution can be obtained as the converging solution in the early stages of the finite horizon problem, since the number of possible stages is often very large. For example, in Case 10 of the previous section with 4*17+1=69 stages, once the terminal effect disappears, beyond stage 22 (moving backwards), the solution stabilizes and remains constant in stages 21 through 0. This solution constitutes the optimal steady-state solution to the infinite horizon problem, and is depicted in Figure 1.

Although the discussion of extensions and implementation issues brings into focus many interesting technical questions, it should not obscure the main general thesis of this paper, which is to be reiterated at this point. With the advent of sophisticated automatic inspection, measurement and control devices, the quality control engineer can afford to resort to more complicated dynamic statistical control techniques for more effective and cost efficient control of production processes. The adoption of dynamic methods of continuous on-line adjustment of the decision parameters based on all the available information is feasible and has the potential of significantly reducing quality related costs.

APPENDIX

Derivation of cost function (1)

The cost function (1) follows easily, once the parallels with Ladany [5] are established. Therefore, the correspondence in notation is explained first. The following quantities are defined in Ladany's model:

f – number of samples taken during the production run; in this paper's notation $f = m-1$.

P_a – probability of not detecting the occurrence of the assignable cause at the first sample after the change; in this paper's notation $P_a = 1-\alpha_1 = 1 - \Phi(\delta-k)$.

α – probability of false alarm; in this paper's notation $\alpha = \alpha_0$.

P_f – probability of occurrence of the assignable cause during a sampling interval h, given that the interval starts with the process in state 0; in this paper's notation $P_f = 1 - e^{-\nu h}$.

C_2 – cost of operation in the out-of-control state 1 for the duration of the production run; in this paper's notation $C_2 = HM$.

C_3 – cost of a false alarm; in this paper's notation $C_3 = L_0$.

C_4 – cost of a restoration; in this paper's notation $C_4 = L_1$.

Following Ladany [5], let

$TC(k,m) = A + B + C + D$

where

A : cost of sampling,

B : cost of operation in the out-of-control state 1,

C : cost of false alarms,

D : cost of restorations.

Since m-1 measurements are taken during the production run, A = (m-1)S.

For the computation of B, denote the probability that the sampling interval after the i^{th} sample starts with the process in state 0 by F(i), i=0,1,...,m-1, with F(0)=1 as initial condition. If P_{α} and P_{f} in expression (12) of Ladany [5] are replaced by their equivalent $1-\alpha_1$ and $1-e^{-\nu h}$ respectively, it immediately follows that

$$F(i) = \{(1-\alpha_1)^{i+1}(1-e^{-\nu h})e^{-\nu i h}+\alpha_1\}/[1-(1-\alpha_1)e^{-\nu h}]$$

for i = 0,1,...,m-1. Then, replacing P_{α}, P_{f}, C_2 and f by their equivalent quantities in expression (13) of [5], leads to

$$B = [Mh(1-e^{-\nu h})/2] \sum_{i=0}^{m-1} F(i)\{1+2(1-\alpha_1)[1-(1-\alpha_1)^{m-1-i}]/\alpha_1\}.$$

Similarly, replacing C_3, α and P_{f} in expression (18) of [5] by L_0, α_0, $1-e^{-\nu h}$ respectively, yields

$$C = L_0\alpha_0 e^{-\nu h} \sum_{i=0}^{m-2} F(i) \ .$$

Finally, replacing C_4, P_{f}, P_{α} and f in expression (19) of [5] by L_1, $1-e^{-\nu h}$, $1-\alpha_1$, m-1 respectively, leads to

$$D = L_1(1-e^{-\nu h}) \sum_{i=0}^{m-2} F(i)[1-(1-\alpha_1)^{m-1-i}] \ .$$

Adding A, B, C and D results in expression (1) for the total cost during the production run.

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Table 1: Numerical Examples

Case	H	M	$L_0=L_1$	v	δ	Optimal Static Solution			Optimal DP Solution	
						k	m	Cost	Cost	Improvement
1	40	10	50	0.01	1	1.0	2	65.92	58.82	10.8%
2	40	10	50	0.01	2	1.9	6	52.33	43.71	16.5%
3	40	10	50	0.05	1	0.0	2	196.62	178.08	9.4%
4	40	10	50	0.05	2	1.8	10	162.21	143.44	11.6%
5	40	10	100	0.01	1	2.2	2	70.76	68.12	3.7%
6	40	10	100	0.01	2	1.9	2	65.78	57.29	12.9%
7	40	10	100	0.05	1	1.0	2	225.55	219.03	2.9%
8	40	10	100	0.05	2	1.2	2	219.14	203.87	7.0%
9	40	30	50	0.01	1	1.5	12	138.88	104.61	24.7%
10	40	30	50	0.01	2	2.1	17	92.23	70.31	23.8%
11	40	30	50	0.05	1	1.5	28	357.41	290.25	18.8%
12	40	30	50	0.05	2	2.0	34	250.16	213.27	14.7%
13	40	30	100	0.01	1	1.7	9	174.23	130.93	24.9%
14	40	30	100	0.01	2	2.3	17	120.77	89.60	25.8%
15	40	30	100	0.05	1	0.0	3	507.64	443.84	12.6%
16	40	30	100	0.05	2	2.3	36	359.31	302.72	15.7%
17	80	10	50	0.01	1	1.3	10	168.63	135.54	19.6%
18	80	10	50	0.01	2	2.0	17	116.73	93.93	19.5%
19	80	10	50	0.05	1	0.0	5	433.68	401.68	7.4%
20	80	10	50	0.05	2	2.0	31	341.41	305.45	10.5%
21	80	10	100	0.01	1	0.4	2	218.47	186.29	14.7%
22	80	10	100	0.01	2	2.3	18	161.38	127.29	21.1%
23	80	10	100	0.05	1	0.0	2	558.37	524.50	6.1%
24	80	10	100	0.05	2	0.0	2	549.80	477.59	13.1%

Figure 1: Optimal Steady-State Solution for Case 10

($M = 30$, $L_0 = L_1 = 50$, $\nu = 0.01$, $\delta = 2$, $H = 40$, $h = 0.59$)