"IDIOSYNCRATIC RISK, SHARING RULES AND THE THEORY OF RISK BEARING"

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# Idiosyncratic Risk, Sharing Rules and the Theory of Risk Bearing

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**Abstract**

It has been recognized in the literature that the choice of an agent between risky and riskless assets is complicated by the existence of other unavoidable risks [Ross (1981), Kihlstrom, Romer and Williams (1981), Nachman (1982) and Pratt and Zeckhauser (1987)]. For instance, the purchase of assets by an individual investor may be made in the context of uncertain wage income.

In this paper, we are concerned with the *response* of the agent to the existence of additional non-insurable income risk. In particular, the agent chooses state-dependent shares of aggregate marketable income (a sharing rule) to provide a partial hedge against the idiosyncratic risk. We focus on the form of the sharing rule in order to determine whether and when the agent is a buyer / seller of insurance. Since our emphasis is on the sensitivity of the optimal allocation decision with respect to the non-insurable risk, we need a measure to define the behavior of the marginal utility function. These are the concepts of absolute prudence and the precautionary premium proposed by Kimball (1990).

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The paper first derives the properties of the precautionary premium in the presence of idiosyncratic risk and different levels of marketable income. The higher the non-insurable risk, the more the agent purchases claims in states with low aggregate income and sells claims in states with high aggregate income - a type of insurance. In this context, the equilibrium is characterized by the sharing rules of the various agents in the economy. Then, the sensitivity of the equilibrium to an increase in the non-insurable risk across all investors is analyzed.

These results are made more specific by considering the special case of the Hyperbolic Absolute Risk Aversion (HARA) family of utility functions. In this case, the precautionary premium can be defined more precisely and the sharing rules can be derived explicitly.

The results in the paper have a number of applications such as in the analysis of the demand for portfolio insurance, the hedging decisions of corporations, and the impact of an aggregate "shock" in idiosyncratic risk on the allocation of risk bearing in an economy.

# 1 INTRODUCTION

The purpose of this paper is to analyze the effects of non-insurable idiosyncratic risk on the optimal sharing rules of agents in an economy. Agents are allowed to buy and sell claims contingent upon aggregate marketable income in the presence of an idiosyncratic risk which cannot be hedged. We investigate the optimal behavior of the individual agent in such a situation and, in particular, the agent's sharing rule defined in terms of claims on the aggregate marketable income. We also study the effect of the behavior of the various agents in the economy on the relative pricing of claims.

It has increasingly been recognised in the literature that an agent's choice between a risky and a riskless asset is complicated by the existence of other unavoidable risks. As Kihlstrom et al (1981) point out, it is rare for decisions on the purchase of risky assets to be taken in the absence of wage income risk, for example. This has led Ross (1981), Kihlstrom et al (1981), Nachman (1982), and Pratt and Zeckhauser (1987) to consider the robustness of the Pratt (1964) - Arrow (1965) theory of risk aversion in the presence of such an additional income risk. Essentially, these papers conclude that the agent's utility function must exhibit non-increasing risk aversion for the Pratt - Arrow results to generalize to the case of more than one risk.

In this paper, we are also concerned with the agent's *response* to the existence of an additional non-insurable income risk. Our agent chooses a state-dependent share $x=g(X)$ of the aggregate marketable income, $X$. We assume that the capital market is perfect and complete with respect to the *marketable* aggregate income. The independent income risk $\varepsilon$ faced by the agent neither can be hedged nor diversified away and, therefore, is called an idiosyncratic risk, $\varepsilon$. However, the agent can modify optimal purchases of claims on the aggregate marketable income, in the presence of the idiosyncratic risk. The focus here is on the effect of $\varepsilon$ on the sharing rule g(X). For example, the agent may choose g(X) so as to receive a greater proportion of the aggregate income in the lower states than in the higher states. A low (high) state is defined as a state in which the aggregate marketable income, X, is low (high). In contrast, the papers by Pratt-Arrow, Kihlstrom et al, Ross, and Nachman consider only the choices between a risky and a risk-free-asset. In the literature, the risky asset is not necessarily a marketable claim.

In a related paper, Kimball (1990) discusses the demand for precautionary savings employing the concept of a "precautionary risk premium". His analysis is also restricted to risk-free claims as a vehicle for saving. Kimball emphasises the degree of absolute prudence defined as $-\upsilon'''(y)/\upsilon''(y) = \eta(y)$ associated with the agent's utility function $\upsilon(y)$, where y is the consumption. He shows that the shift in the consumption function induced by an idiosyncratic future income risk depends upon absolute prudence. He also shows that risk aversion decreases if and only if absolute prudence exceeds absolute risk aversion. We apply Kimball's concepts of absolute prudence and the precautionary risk premium in our analysis. These measures characterize the optimal behavior of an agent under conditions of uncertainty by dealing with the marginal utility function. Hence, they are useful in analyzing the comparative statics of the optimal behavior of the agent. In turn, given the pricing of claims on aggregate marketable income, the sharing rule of the agent may be characterized . Therefore, our analysis may be viewed as an extension of the work of Kihlstrom et al, Ross, Nachman and Kimball. Also, since our methodology relies heavily on Kimball's work, it can be seen as an application of his results on absolute risk aversion and absolute prudence in a context where a market exists for risky claims.

The economic setting of the paper is as follows. The agent has an opportunity to buy a claim on a single marketable aggregate income, X. The agent's consumption y at the end of a single period is the chosen amount of the marketable income, $x=g(X)$, plus an independent non-marketable income $\varepsilon$, with a zero mean.[1] Hence, $y=g(X)+\varepsilon$. The agent's problem is to choose the functional form of g(X) so as to maximise a utility function $E[\upsilon(y)]$.

The agent's demand for claims on aggregate marketable income in the face of non-marketable, idiosyncratic risk depends upon the absolute risk aversion and the absolute prudence of the utility function. $a(y) = -\upsilon''(y)/\upsilon'(y)$ is the Arrow-Pratt index of absolute risk aversion. The degree of risk aversion is reflected in the risk premium which is defined by $\pi$ in $E[\upsilon(y)] = \upsilon[E(y)-\pi]$. Analogously, the precautionary risk premium, or simply, the precautionary premium, is defined as $\Psi$ in $E[\upsilon'(y)] = \upsilon'[E(y)-\Psi]$. It is the degree of absolute prudence $\eta(y) = -\upsilon'''(y)/\upsilon''(y)$, which determines the precautionary risk premium $\Psi$ and, hence, the agent's response to the existence of non-marketable risks. We show that for utility functions which exhibit positive declining absolute prudence, i.e. for which

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[1] The results of the paper do not change if the mean of $\varepsilon$ conditional on X depends on X. In a complete market, the agent can always sell these means.

$\eta(y)>0$ and $\eta'(y)<0$, the function g(X) depends on $\varepsilon$. Kimball (1989) gives many examples which suggest that agents have both, positive decreasing absolute risk aversion and positive, *decreasing absolute prudence*. These properties imply $\upsilon'(y) > 0$, $\upsilon''(y) < 0$, $\upsilon'''(y) > 0$ and $\upsilon''''(y) < 0$.[2]

In section 2 of the paper we analyze the optimal sharing rule of the agent for the general case, when the only restriction is that the utility function has positive decreasing absolute risk aversion and absolute prudence. We first characterize the precautionary risk premium, as defined by Kimball, in our setting. Then, we derive the effect of an increase in the idiosyncratic risk on the optimal sharing rule. In particular, we show that the agent buys more claims in states with low aggregate marketable income and less claims in states with high aggregate marketable income. Next, we consider the effects of changes in the idiosyncratic risk on the relative pricing of claims in different states. We show that the prices of claims in low states go up in relation to those of claims in high states in response to an aggregate shock to idiosyncratic risk if the aggregate derived risk tolerance goes up in every state.

In order to make more precise statements about the behavior of the precautionary premium and the form of the sharing rule, we need to make more specific assumptions about the preferences of agents. In Section 3, we assume that all agents have utility functions belonging to the Hyperbolic Absolute Risk Aversion class (HARA). For any agent, the sharing rule is a linear function of

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[2] Non-satiation implies $\upsilon'(y) > 0$. Risk aversion implies $\upsilon''(y) < 0$. Decreasing absolute risk aversion and decreasing absolute prudence imply $\upsilon'''(y) > 0$ and $\upsilon''''(y) < 0$.

the agent's precautionary premium and the aggregate precautionary premium. However, the sharing rule of the agent is non-linear. For an agent without idiosyncratic risk, the sharing rule is strictly concave. A general rise in idiosyncratic risk renders claims in low states more expensive relative to those in high states if welfare is reduced more in the low states. In each state, welfare is measured by the exogeneous aggregate marketable income minus the endogeneous aggregate precautionary premium. In section 4, we conclude with an interpretation of the results.

## 2 PRECAUTIONARY PREMIUM AND THE SHARING RULE: THE GENERAL CASE

In this section, we assume that agents have utility functions such that the absolute risk aversion $a(y) = -\upsilon''(y)/\upsilon'(y)$ and the absolute prudence $\eta(y) = -\upsilon'''(y)/\upsilon''(y)$ are positive and decreasing.[3] First, some properties of the precautionary premium will be discussed which are helpful in analyzing the sharing rules. These properties do not depend on the sharing rules.

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[3] This requirement is closely related to the notion of proper utility functions as defined by Pratt and Zeckhauser [1987]. Proper risk aversion is a property of utility functions whereby "an undesirable lottery can never be made desirable by the presence of an undesirable independent lottery" (p. 143). A subset of these functions has positive odd derivates and negative even derivatives over some interval of wealth. The requirement of this paper is that the first four derivatives have alternating signs everywhere.

## A SOME PROPERTIES OF THE PRECAUTIONARY RISK PREMIUM

Suppose now that $y = x + \sigma\varepsilon$ where $\varepsilon$ is a zero mean risk with standard deviation 1; $\varepsilon$ is independent of X and hence of $x = g(X)$ and $\sigma$ is a positive scalar denoting the standard deviation of $\sigma\varepsilon$. Hence, an increase in $\sigma$ denotes the addition of a mean-preserving spread to the income y, conditional on the marketable income x. Kimball (1990) defines the precautionary risk premium $\Psi$ conditional on x by the relation $E_x(\upsilon'(y)) = \upsilon'(x-\Psi)$ where $\Psi = \Psi(x,\sigma)$. $\upsilon'(x-\Psi)$ may be interpreted as a derived marginal utility function in x, similarly to Nachman's (1982) definition of a derived utility function. The difference is that we start from marginal utility instead of utility.[4]

In particular, $\Psi$ is positive and a strictly decreasing function of x if the absolute prudence is positive and decreasing, i.e.

$$\Psi > 0 \quad \leftrightarrow \quad \eta(y) > 0 \quad ;$$

$$\frac{\partial\Psi}{\partial x} < 0 \quad \leftrightarrow \quad \eta'(y) < 0 \quad .$$

This is discussed by Kimball (1990, p.62) and follows directly by applying the argument of Pratt (1964).

We now look at the effect of an increase in idiosyncratic income risk ($\sigma\varepsilon$) on the

[4] In Kimball's context, x is period 1 income and $\sigma\varepsilon$ represents a risk to income in period 2. For the problem that we address here, $\sigma\varepsilon$ represents a contemporaneous income risk which is idiosyncratic to the agent. Kimball shows that the properties of the precautionary premium $\Psi$ are analogous to those of the Pratt - Arrow risk premium.

precautionary risk premium in order to determine the effect on the optimal allocation of claims. We have $\partial \Psi / \partial \sigma > 0$ [5]. In fact, for a small income risk,

$$\Psi = \frac{1}{2} \eta(x) \sigma^2$$

so that

$$\frac{\partial \Psi}{\partial \sigma^2} = \frac{1}{2} \eta(x) > 0 \quad .$$

The precautionary premium $\Psi = \Psi(x,\sigma)$ is the crucial determinant of the agent's demand for risky assets in the face of idiosyncratic income risk. It is relevant, therefore, to consider further characteristics of $\Psi$. If the coefficient of absolute prudence is declining in y, it follows from the last equation for a small income risk that

$$\frac{\partial^2 \Psi}{\partial \sigma^2 \partial x} = \frac{1}{2} \eta'(x) < 0.$$

For large risks, these equations do not hold. But, it is possible to derive some statements on the derived marginal utility, $\upsilon'(x - \Psi)$, which are relevant for the

[5] Rothschild and Stiglitz (1970) have shown for utility functions with positive absolute risk aversion that the addition of a mean preserving spread raises the risk premium. Using the Kimball analogy, it follows that an increase in $\sigma$ raises the idiosyncratic income risk, and hence the precautionary risk premium, if $\eta(y) > 0$.

sharing rules. It is convenient to analyze the growth rates of $\upsilon'(x - \Psi)$ with respect to income x and idiosyncratic risk $\sigma$, i.e. we analyze the sensitivities of $\ln \upsilon'(x - \Psi)$.

Lemma 1:

a) The derived marginal utility function exhibits risk aversion.

$$\hat{a}(x) = -\frac{\partial \ln \upsilon'(x-\Psi)}{\partial x} = -\frac{\upsilon''(x-\Psi)}{\upsilon'(u-\Psi)}\left[1-\frac{d\Psi}{dx}\right] > 0.$$

b) The rate of change of the derived marginal utility increases with the idiosyncratic risk.

$$\frac{\partial \ln \upsilon'(x-\Psi)}{\partial \sigma} > 0.$$

Proof:

a)

$$\hat{a}(x) = -\frac{\partial \ln \upsilon'(x-\Psi)}{\partial x} = -\upsilon''(x-\Psi)(1-\partial\Psi/\partial x)/\upsilon'(x-\Psi) > 0 \quad \text{as}$$

$$\upsilon''(\cdot) < 0,\ \upsilon'(\cdot) > 0 \quad \text{and} \quad \partial\Psi/\partial x < 0.$$

b)

$\partial \ln \upsilon'(x - \Psi) / \partial\sigma = -\upsilon''(x - \Psi) (\partial\Psi/\partial\sigma) / (\upsilon'(x - \Psi) > 0$ as

$$\partial\Psi/\partial\sigma > 0 \text{ and } \upsilon'(\cdot) > 0 \blacksquare$$

Lemma 1a) says that the negative rate of change of the derived utility function which we call the derived absolute risk aversion is positive since the absolute risk aversion is positive.

1b) says that the rate of change of the derived marginal utility increases with idiosyncratic risk since an increase in idiosyncratic risk raises the precautionary premium and, thus, has the same effect as a decline in income.

In order to determine the effects of an increase in idiosyncratic risk on the sharing rule, we also need to know the second derivatives of $\ln \upsilon'(x - \Psi)$ with respect to income x and idiosyncratic risk $\sigma$.

<u>Lemma 2:</u>

a) The derived marginal utility function exhibits decreasing derived absolute risk aversion.

$$\frac{\partial \hat{a}(x)}{\partial x} = -\frac{\partial^2 \ln \upsilon'(x - \Psi)}{\partial x^2} < 0$$

b) The rate of change of the derived marginal utility, due to an increase in idiosyncratic risk, declines if marketable income increases.

$$\frac{\partial^2 \ln \upsilon'(x-\Psi)}{\partial\sigma\ \partial x} < 0,$$

c) The marginal "rate of substitution" between idiosyncratic risk and marketable income increases with income.

$$\partial\left[\frac{\partial \ln \upsilon'(x-\Psi)/\partial\sigma}{\partial \ln \upsilon'(x-\Psi)/\partial x}\right]\partial x > 0.$$

Lemma 2 is proved in appendix A. Lemma 2a) states that ln $\upsilon'(x - \Psi)$ is a convex function of x; in addition, it is decreasing as shown in lemma 1a). This is in line with the behavior of marginal utility in the absence of idiosyncratic risk. Lemma 2b) states in conjunction with lemma 1b) that ln $\upsilon'(x - \Psi)$ grows with idiosyncratic risk, but this effect is smaller, the higher the income x is.[6] Lemma 2c) considers the marginal rate of substitution between changes in idiosyncratic risk and income x which leave derived marginal utility unaffected. Lemma 2c) says that the marginal rate of substitution (which is negative) is higher, the higher is the income x.

## B IDIOSYNCRATIC RISK AND THE OPTIMAL SHARING RULE

We assume that the capital market is perfect and complete with respect to marketable aggregate income X. An agent i in the economy solves the following

[6] From lemmas 1b) and 2b), it follows that an investor who can buy a risk-free and a risky asset, invests less in the risky asset when his idiosyncratic risk increases.

maximization problem

$$\max_{x_i = g_i(X)} \quad E\,[\upsilon(x_i + \sigma_i\,\varepsilon)] \tag{1}$$

subject to

$$w_i = E\,[\Phi(X)\,x_i]$$

where $w_i$ is the agent's initial endowment and $\Phi(X)$ is the market pricing function for state contingent claims on X.[7] The first order condition for a maximum is

$$E_X\,[\upsilon'(x_i + \sigma_i\varepsilon)] = \lambda_i\,\Phi(X) \tag{2}$$

where $E_x(\cdot)$ is the conditional expectation given X, and $\lambda_i$ is a positive state independent Lagrange multiplier reflecting the tightness of the budget constraint. Notice that the derived marginal utility of the agent is proportional to the price $\Phi(X)$. If all agents are risk averse, it follows immediately that $\Phi'(X) < 0$. To see this, if we differentiate (2) with respect to X we have (dropping the agent index i)

$$\frac{dx}{dX} = \frac{dg(X)}{dX} = \frac{\lambda\,\Phi'(X)}{E_x\,[\upsilon''(x + \sigma\,\varepsilon)]}$$

Thus, dg(X)/dX has the same sign for each agent. Since it must be positive, it follows immediately that $\Phi'(X) < 0$. This establishes our first result regarding the optimal sharing rule for the individual agent in this economy. Agents buy increasing claims on the aggregate income X. The sharing rule has a positive

[7] Note that the market pricing function gives the market price of a state contingent claim divided by the probability of the state occuring.

slope as it does in the absense of idiosyncratic risk (see Rubinstein (1974)).

Equation (2) can be written, using the precautionary premium, as

$$\upsilon' [x_i - \Psi_i (x_i, \sigma_i)] = \lambda_i \Phi(X) \tag{3}$$

In (3), $\upsilon'(\cdot)$ is the derived marginal utility of the agent.

Totally differentiating the left hand side of equation (3) with respect to $\sigma$ yields

$$\frac{d\upsilon'(\bullet)}{d\sigma} = \frac{\partial\upsilon'(\bullet)}{\partial\sigma} + \frac{dg(X)}{d\sigma}\frac{\partial\upsilon'(\bullet)}{\partial g(X)} \quad .$$

It follows from equation (3) that $d\upsilon'(\cdot) \,/\, d\sigma = \upsilon'(\cdot)\, d\ln\lambda \,/\, d\sigma$ . Hence the effect of the idiosyncratic risk on the sharing rule is given by

$$\frac{dg(X)}{d\sigma} = \frac{\frac{d\ln\lambda}{d\sigma}\upsilon'(\bullet)}{\partial\upsilon'(\bullet)/\partial g(X)} - \frac{\frac{\partial\upsilon'(\bullet)}{\partial\sigma}}{\frac{\partial\upsilon'(\bullet)}{\partial g(X)}} \,. \tag{4}$$

The effect on the sharing rule in equation (4) is the sum of two effects. The first is analogous to an "income effect". The second term is the pure "substitution effect" of the change in $\sigma$. If we were to compare the sharing rules of two agents, cross-sectionally within an equilibrium, with differing $\sigma_i$ but with the same $\lambda_i$, then consideration of only the "substitution effect" would be relevant.

An alternative question to ask is the following: How does an agent react if idiosyncratic risk increases? In this case, both the "income effect" and the "substitution effect" are relevant since we are talking about a comparative statics shift. In order to distinguish these two questions, we establish Proposition 1a and 1b.

Proposition 1:

Assume that $\upsilon(y)$ has the properties of positive and declining absolute risk aversion and prudence, i.e. $a(y) > 0$, $a'(y) < 0$, $\eta(y) > 0$, $\eta'(y) < 0$.

a) Consider two agents i and j such that $\sigma_i < \sigma_j$ but $\lambda_i = \lambda_j$. Then

$$\frac{dg_i(X)}{dX} > \frac{dg_j(X)}{dX}$$

b) Consider an increase in $\sigma$ for a given agent. Then $\dfrac{d^2g(X)}{d\sigma\ dX} < 0.$

Also $\exists\ X^*$ such that $\dfrac{dg(X)}{d\sigma} \ >=<\ 0$ for $X <=> X^*$.

Proof:

a) If $\lambda$ is constant, only the "substitution effect" is relevant. From (4), in this case

$$\frac{dg(X)}{d\sigma} = - \frac{\dfrac{\partial \upsilon'(\bullet)}{\partial \sigma}}{\dfrac{\partial \upsilon'(\bullet)}{\partial g(X)}} \tag{5}$$

and

$$\frac{d^2g(X)}{d\sigma\ dX} = \frac{d}{dg(X)}\left[ - \frac{\dfrac{\partial \upsilon'(\bullet)}{\partial \sigma}}{\dfrac{\partial \upsilon'(\bullet)}{\partial g(X)}} \right] \frac{dg(X)}{dX} \tag{6}$$

From Lemma 2c, the first term on the right hand side of equation (6) is negative. The second term is positive and hence the derivative in (6) is negative. The statement in proposition 1a then follows.

b) Here we consider both, the "income" and the "substitution effect" in (4). However, since the "substitution effect" is positive (which follows from $\frac{\partial \upsilon'}{\partial g(X)} < 0$ ), the "income effect" must be negative since otherwise $\frac{dg(X)}{d\sigma}$ would be positive in all states, violating the budget constraint. Hence $\frac{d\ln \lambda}{d\sigma} > 0$. To establish (b) note that in this case

$$\frac{d^2 g(X)}{d\sigma\, dX} = \frac{d}{dg(X)} \left[ \frac{\frac{d\ln \lambda}{d\sigma}\, \upsilon'(\bullet)}{\frac{\partial \upsilon'(\bullet)}{\partial g(X)}} - \frac{\frac{\partial \upsilon'(\bullet)}{\partial \sigma}}{\frac{\partial \upsilon'(\bullet)}{\partial g(X)}} \right] \frac{dg(X)}{dX} \qquad (7)$$

From proposition 1a we know that the derivative of the second term in the bracket is negative. From Lemma 2b the derivative of the first term in the bracket is also negative and proposition 1b follows directly. Finally, since the purchase and sales of state contingent claims must be self financing, it follows that there must exist a critical level $X^*$ such that $dg(X)/d\sigma$ is positive for $X < X^*$ and negative for $X > X^*$ ♦

The significance of proposition 1a is that it tells us, <u>within an equilibrium</u>, how the sharing rules of agents differ in response to differences in idiosyncratic risk. Given the utility function and tightness of the budget constraint, 1a says that agents with high idiosyncratic risk will tend to buy relatively more contingent claims in low states and relatively less claims in high states. If we think of agents as holding linear shares in X plus "insurance" against low states, we could say that the agents with high $\sigma_i$ tend to buy "insurance" from those with low $\sigma_i$.

In contrast, Proposition 1b tells us the effect of an increase in $\sigma_i$ on an agent taking into account the effects of $\sigma_i$ on the budget constraint and on $\lambda_i$. Again the effect is unambiguous. The agent increases the purchase of claims in the low states of X. The implications of this behaviour by agents in general will be looked at in the following section.

## <u>C</u> <u>PRICING EFFECTS OF CHANGES IN IDIOSYNCRATIC RISK</u>

In the preceding section, sharing rules of agents have been investigated with prices of contingent claims being given. From proposition 1b, it follows that an agent demands more claims in the states with low aggregate marketable income and less claims in the states with high aggregate marketable income if his idiosyncratic risk increases. Suppose now that there is an aggregate shock such that idiosyncratic risk increases for every agent in the economy. Then, every agent wishes to buy more claims in the "low" states at the <u>old</u> prices. But this is impossible since the agents' additional demand for marketable claims must sum to zero for every state. Hence, the prices of all claims must change to reflect the change in demand. Proposition 1b suggests that claims in "low" states become relatively more expensive compared to claims in "high" states. A somewhat weak qualification is necessary, however, for this implication to go through. This qualification is a restriction on the agents' derived risk tolerance. Agent i's derived risk tolerance $\hat{\varrho}_i\,(x_i)$ is the inverse of his derived risk aversion $\hat{a}_i\,(x_i)$, i.e. $\hat{\varrho}_i\,(x_i) = 1/\hat{a}_i\,(\hat{x}_i)$. As $x_i = g_i\,(X)$, $\hat{\varrho}_i\,(x_i) = \hat{\varrho}_i(X)$. We now state the equilibrium implication as

Proposition 2: A general rise in idiosyncratic risk across many agents in the economy leads to an increase in the price of claims contingent upon marketable aggregate income in state s, $X_s$, relative to the price of claims contingent upon a higher marketable aggregate income in state t, $X_t$, if and only if the aggregate derived risk tolerance decreases in every state. More precisely, for a small general rise in idiosyncratic risk,

$$\frac{d(\Phi_s/\Phi_t)}{d\sigma} > 0 \quad \forall\, [(s,t){:}X_s < X_t] \quad \leftrightarrow \quad \frac{d\sum_i \hat{\varrho}_i(X)}{d\sigma} < 0 \quad \forall\, X .$$

Proof: The first order condition for an optimal sharing rule of investor i is

$$\upsilon_i^{/}\,(g_i(X) - \Psi_i(X,\sigma_i)) = \lambda_i\,\Phi\,(X)$$

Take logarithms of this equation, then differentiating with respect to X, we have

$$\frac{dg_i(X)}{d(X)} = -\,\hat{\varrho}_i(X)\,\frac{d\ln\Phi\,(X)}{dX} \quad ; \quad \forall\, X\ ,$$

where $\hat{a}_i(X) = 1/\hat{\varrho}_i(X)$ is defined in Lemma 1.

Then, aggregation over all investors yields

$$1 = -\frac{d\ln\Phi(X)}{dX} \sum_i \hat{\varrho}_i(X) .$$

An increase in idiosyncratic risk affects both factors on the right hand side of the equation in an offsetting manner: an increase in idiosyncratic risk across many agents decreases $\frac{d\ln\Phi(X)}{dX}$ if and only if it decreases $\sum_i \hat{\varrho}_i(X)$. In other words, $d^2\ln\Phi(X)/dXd\sigma < 0 \quad \forall X$ means that the growth rate of $\Phi(X_t)$ is less than that of $\Phi(X_s)$ for <u>every</u> pair (s,t) with $X_s < X_t$. $d^2\ln\Phi(X)/dXd\sigma < 0 \ \forall X$ is true if and only if $d\sum_i \hat{\varrho}_i(X)/d\sigma < 0 \ \forall X$.

Hence, the proposition follows♦

The intuition behind proposition 2 is straightforward. Since the excess demand in state s is higher than that in state t, the price relative $\Phi_s/\Phi_t$ must change in order to make these excess demands disappear. Clearly, we would expect the price relative to increase. But price changes can have various feedback effects, for example, on the agents' initial endowments. Moreover, the price change of one state interacts with the price changes in other states. Therefore these feedback effects need to be constrained in order to get a unique answer [Deaton/Muellbauer (1981)].

The necessary and sufficient condition established in the proposition is that aggregate derived risk tolerance is a decreasing function of idiosyncratic risk. This condition is quite plausible since, without trade and without endowment changes, an increase in an agent's idiosyncratic risk raises the precautionary premium and, thus, the derived risk aversion. Hence the derived risk tolerance of the agent is reduced. If all agents are "reasonably" similar in terms of endowments, risk aversion and idiosyncratic risk, then trade and endowment effects cannot overturn this. But if agents are very different in terms of endowments or idiosyncratic risk, it is possible that after trade, aggregate derived risk tolerance might increase in some states. This possibility is ruled out by the condition that aggregate derived risk tolerance decreases in every state.

## 3 FURTHER RESULTS IN THE HARA-CASE

In this section, we assume that agents have utility functions belonging to the class of functions with hyperbolic absolute risk aversion (HARA-class). This allows us to derive more specific results and to provide some illustration for the propositions of the previous section. As in the general case, we first derive some additional properties of the precautionary premium.

## A FURTHER PROPERTIES OF THE PRECAUTIONARY PREMIUM

The HARA-class of utility functions is defined by

$$\upsilon(y) = \frac{1-\gamma}{\gamma}\left(A+\frac{y}{1-\gamma}\right)^{\gamma} > 0$$

with A being a constant and $A + y/(1-\gamma) > 0$.[8] Then

$$\upsilon'(y) = \left(A+\frac{y}{1-\gamma}\right)^{\gamma-1} > 0 \quad ,$$

$$a(y) = \left(A+\frac{y}{1-\gamma}\right)^{-1} \quad ,$$

$$\eta(y) = \frac{\gamma-2}{\gamma-1}\left(A+\frac{y}{1-\gamma}\right)^{-1} = \frac{\gamma-2}{\gamma-1}\, a(y).$$

If $1>\gamma>-\infty$, then $\eta(y) > 0$ and $\eta'(y)<0$. Then, as shown by Pratt and Zeckhauser, the utility function exhibits proper risk aversion. $\gamma > 1$ implies utility functions with negative marginal utility for high values of y, and thus , is irrelevant for our analysis.

For the HARA-class, stronger results hold than for the class of all utility functions with $\eta(y) > 0$ and $\eta'(y) < 0$. For the HARA-class with $1>\gamma>-\infty$, the precautionary premium $\Psi(x,\sigma)$ has the following properties besides $\frac{\partial\Psi}{\partial x} < 0$ (for the proof see appendix B):

$$\frac{\partial^2\Psi}{\partial x^2} > 0 \quad ,$$

$$\frac{\partial^2\Psi}{\partial\sigma\partial x} < 0 \quad ,$$

8 This constraint can be satisfied only if $\varepsilon$ is bounded from below.

$$\frac{\partial^3 \Psi}{\partial \sigma \partial x^2} > 0 .$$

The first property says that the precautionary premium is a decreasing, convex function in x. The second and the third properties characterize the effects of an increase in the idiosyncratic risk on the precautionary premium. The increase in the premium is the smaller, the higher the marketable income x. Moreover, the convexity of the premium in x increases with the idiosyncratic risk. This is significant when we consider optimal sharing rules for the HARA-class in section 4. Figure 1 illustrates these properties.[9]

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[9] For exponential utility $\gamma = -\infty$, absolute risk aversion and absolute prudence are constant. Hence, the precautionary premium is independent of x.

For $\gamma = 2$ (quadratic utility), absolute prudence is zero and, therefore, the precautionary premium is zero. Generally speaking, idiosyncratic risk which is independent of the marketable risk, X, is completely irrelevant for the sharing rule in a mean-variance world. This follows from the derivation of efficient sharing rules. The objective is to minimize the variance. In this objective function idiosyncratic risk is an additive constant and, thus, cannot affect the efficient sharing rule. Therefore, mean-variance theory cannot capture the effects of idiosyncratic risk as discussed in this paper.

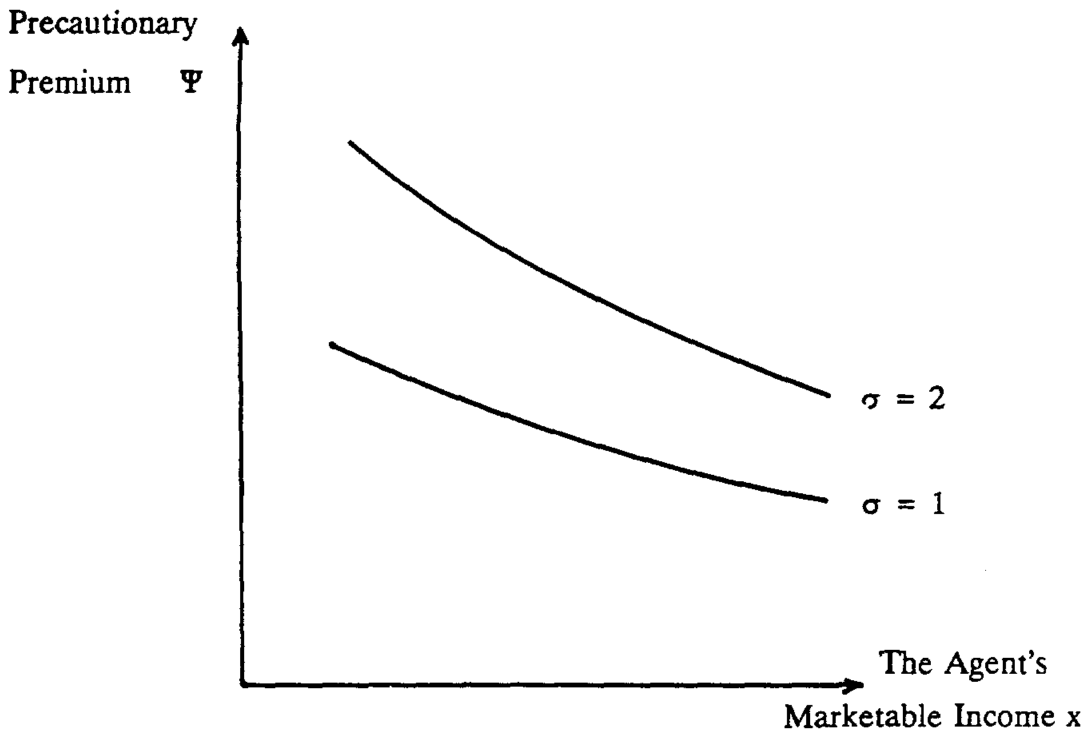


Figure 1: The precautionary premium is a decreasing, convex function of the agent's marketable income x if the utility function belongs to the HARA-class with $-\infty < \gamma < 1$.

A further result is important in the subsequent analysis. It follows from the monotonicity of the HARA-functions. If idiosyncratic risk grows by the factor q and x changes from $x_0$ to $x_1$ such that $q[A(1-\gamma)+x_0] = A(1-\gamma)+x_1$, then the precautionary premium grows by the factor q, $\Psi(x_1,q\sigma) = q\Psi(x_0,\sigma)$. This is also proved in appendix B.

## B IDIOSYNCRATIC RISK AND THE OPTIMAL SHARING RULE

In this section we investigate the optimal sharing rules of agents for the HARA-case. Consider, the first order condition (3) for the optimal sharing rule and insert the HARA-marginal utility,

$$\left(A + \frac{g(X) - \Psi(X,\sigma)}{1-\gamma}\right)^{\gamma-1} = \lambda\Phi(X) \quad ; \forall X .$$

Taking roots yields

$$x - \Psi(X,\sigma) = g(X) - \Psi(g(X),\sigma) = (\lambda^{\frac{1}{\gamma-1}} (\Phi(X))^{\frac{1}{\gamma-1}} - A)(1-\gamma) \; ; \forall X . \quad (8)$$

As in proposition 1, we now investigate two questions. First, consider two agents with the same $\gamma$ and the same tightness of the budget constraint (i.e. $\lambda$ is the same for both), but the constant A need not be the same. Suppose that idiosyncratic risk is higher for the second agent ($\sigma_2 > \sigma_1$). Then equation (8) implies that the convexity or concavity of $[x - \Psi(x,\sigma)]$ measured by the second derivative with respect to X, is the same for both agents since $\lambda\,\Phi(X)$ is the same for both. In section A, we saw already that the precautionary premium $\Psi(x,\sigma)$ is convex and the convexity increases with $\sigma$. Hence the convexity of $\Psi_2(x,\sigma_2)$ exceeds that of $\Psi_1(x,\sigma_1)$ which implies that the convexity of $g_2(X)$ exceeds that of $g_1(X)$ since the convexity of $g(X) - \Psi(X,\sigma)$ is the same for both.

Second, we consider one agent only and ask the question how his sharing rule changes when his idiosyncratic risk increases. The difference between the preceding question and this question is that now $\lambda$ changes, too. It has been shown before that $d\lambda/d\sigma > 0$ and hence $d\lambda^{\frac{1}{\gamma-1}} / d\sigma < 0$. Hence from equation (8) it follows that $d^2g(X)/dX^2$ increases due to the increase in $\lambda$ if $[\Phi(X)]^{\frac{1}{\gamma-1}}$ is concave, i.e. $\Phi(X)$ is convex. This establishes

Propositon 3:

a) Consider two agents with HARA-utility functions such that $\gamma$ is the same for both ($1 > \gamma > -\infty$) and the tightness of the budget constraint, $\lambda$, is the same for both. Then the convexity of the sharing rule, i.e. the second derivative of the sharing rule with respect to the aggregate marketable income, X, is higher for the agent with the higher idiosyncratic risk, measured by $\sigma$.

b) Consider an agent with HARA-utility. Then the convexity of his sharing rule increases with his idiosyncratic risk if the pricing function $\Phi(X)$ is convex♦

This is an interesting result since it says that a higher idiosyncratic risk not only implies portfolio adjustment as stated in proposition 1, but also greater convexity of the sharing rule.[10]

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[10] This relates our findings to Leland's (1980) who defines portfolio insurance as convexity of the sharing rule. Note, however, that we do not prove convexity of the sharing rule. Proposition 3 shows that higher

Although proposition 3 relates the increase in convexity of the sharing rule to the increase in idiosyncratic risk, it remains open whether the sharing rule is convex, concave or of a mixed nature. More specific results can be obtained in a market equilibrium where every agent has a HARA-utility function such that $\gamma$ is the same for every agent and all agents share the same expectations. It is well known from the work of Cass and Stiglitz (1970) and Rubinstein (1974) that, under these assumptions and in the absence of idiosyncratic risk, all agents have a linear sharing rule since the HARA-class implies separation, i.e. the sharing rules of different agents differ only through their level and their constant slope. In the following, we investigate the effects of idiosyncratic risk on the sharing rule. The main result is contained in proposition 4.

Proposition 4: Assume that every agent has a utility function belonging to the HARA-class with $1 > \gamma > -\infty$ and $\gamma$ is the same for every agent. Moreover, assume homogeneous expectations.

a) Then investor i's sharing rule in equilibrium is

$$g_i(X) = [\alpha_i A - A_i](1-\gamma) + \alpha_i X + [\Psi_i(X,\sigma_i) - \alpha_i \Psi(X)] \quad ; \qquad \forall\ i; \qquad (9)$$

with

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idiosyncratic risk raises the convexity although the sharing rule need not be convex. For a related discussion of portfolio insurance see also Brennan and Solanki (1981) and Benninga and Blume (1985).

$$\alpha_i \equiv \lambda_i^{\frac{1}{\gamma-1}} / \sum_j \lambda_j^{\frac{1}{\gamma-1}}, \text{ so that } \sum_i \alpha_i = 1,$$

$$A \equiv \sum_i A_i \quad ; \quad \Psi(X) \equiv \sum_i \Psi_i(X,\sigma_i) \quad ,$$

b) the pricing function $\Phi(X)$ is a decreasing, convex function, i.e. $d\Phi(X) / dX < 0$, $d^2 \Phi(X) / d X^2 > 0$,

c) the aggregate precautionary premium $\Psi(X)$ is a decreasing, convex function with $\Psi(X) \to 0$ for $X \to \infty$,

d) the agent's standardized "tightness" of the budget constraint, $\alpha_i$ is a function of $A_i$, $\sigma_i$, $w_i$ such that

$$\alpha_i > 0, \ \frac{\partial \alpha_i}{\partial \sigma_i} < 0, \ \frac{\partial \alpha_i}{\partial A_i} > 0, \ \frac{\partial \alpha_i}{\partial w_i} > 0 \ \blacklozenge$$

Proposition 4 is proved in Appendix C.

Proposition 4 says that every agent's sharing rule is composed of three elements, the risk-free asset, a constant fraction of the marketable aggregate income X plus a nonlinear term. This generalizes Rubinstein's results (1974). If no idiosyncratic risk exists in the economy, then the last term is zero so that a linear sharing rule follows. With idiosyncratic risk, the investor without idiosyncratic risk has a strictly concave sharing rule. Thus, if there are only two agents, one of whom has no idiosyncratic risk, then the agent with idiosyncratic risk must have a strictly convex sharing rule.[11]

The sharing rule (9) also reveals that for high values of marketable aggregate

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[11] One might conjecture that under appropriate conditions there exists an agent with a linear sharing rule. This is very doubtful, however. The following example shows a situation in which such an agent cannot exist. There exist three agents. Agent 1 has no idiosyncratic risk. The other two agents have small idiosyncratic risks so that

$$\Psi_i(x_i,\sigma_i) = (1/2)\ \eta_i(x_i)\ \sigma_i^2; \quad i= 2,3.$$

Now suppose that agent 2 has a linear sharing rule. Then

$$\Psi_2(x_2,\sigma_2) = \alpha_2 \left(\Psi_2(x_2,\sigma_2) + \Psi_3(x_3,\sigma_3)\right)$$

follows from his sharing rule, or

$$\Psi_2\ (x_2,\sigma_2)\ (1-\alpha_2) = \alpha_2\ \Psi_3\ (x_3,\sigma_3).$$

For small risks it follows

$$\eta_2(x_2)\ \sigma_2^2(1-\alpha_2) = \alpha_2\ \eta_3(x_3)\ \sigma_3^2.$$

In the HARA-case, this yields

$$\frac{\sigma_2^2(1-\alpha_2)}{A_2(1-\gamma) + x_2} = \frac{\alpha_2\ \sigma_3^2}{A_3(1-\gamma) + x_3},$$

so that $x_3$ is linear in $x_2$. Hence linearity of $x_2 = g_2(X)$ implies linearity of $x_3 = g_3(X)$. But then agent 1 must also have a linear sharing rule in equilibrium which contradicts proposition 4. Therefore, in this example a representative investor, i.e. an agent with a linear sharing rule, cannot exist.

income, all sharing rules become approximately linear, since the precautionary premia approach 0. This means that an agent who purchases many claims in the low states because of high idiosyncratic risk pays for these claims by accepting a relatively low slope ($\alpha_i$) of his sharing rule in the high states as evidenced by $\partial\alpha_i/\partial\sigma_i < 0$.

Comparing two agents i and j who differ only in their constants $A_i$ and $A_j$ such that $A_i < A_j$ means that agent j is less risk averse and less prudent. Hence, agent j is affected less by idiosyncratic risk. The consequence is that agent j buys a larger share $\alpha$ of marketable aggregate income X as indicated by $\partial\alpha/\partial A > 0$. As a consequence, agent j invests less in purchasing other claims.

Finally, if the two agents differ only in their initial endowments such that $w_i < w_j$, then agent j buys more of the risk-free asset and of the marketable aggregate income as evidenced by $\partial\alpha/\partial w > 0$. In addition, agent j is less risk averse and less prudent so that the sharing rule adjustment for own idiosyncratic risk is reduced and in consequence this agent is more inclined to sell claims in the low states.

## C PRICING EFFECTS OF CHANGES IN IDIOSYNCRATIC RISK

The effect of an aggregate shock that changes the idiosyncratic risk of many agents in the economy can be studied more precisely for the HARA-class of preferences. Since we now have a specific expression for the pricing function $\Phi(X)$, we can provide a more intuitive, sufficient condition for the changes in relative prices of state contingent claims.

Proposition 5: Under the assumptions of Proposition 4, an increase in idiosyncratic risk for many agents raises the price relative $\Phi_s/\Phi_t$ for any states s and t with $X_s < X_t$ if it raises the aggregate precautionary premium at least as much in state s as in state t♦

Proof: In the aggregate, we have

$$\frac{A + \dfrac{X_s - \Psi(X_s)}{1 - \gamma}}{A + \dfrac{X_t - \Psi(X_t)}{1 - \gamma}} = \left[\frac{\Phi(X_t)}{\Phi(X_s)}\right]^{\frac{1}{1 - \gamma}}$$

Hence, if $\Psi(X_s)$ increases at least as much as $\Psi(X_t)$ with idiosyncratic risk, then the left hand side decreases, and, therefore, $\Phi(X_t)/\Phi(X_s)$ does■

It is instructive to compare Proposition 2 and Proposition 5. Both are similar in spirit although Proposition 2 gives a necessary and sufficient condition whereas Proposition 5 gives a sufficient condition.

The condition in Proposition 5 is that "Benthamite" welfare[12], measured by $X - \Psi(X)$, is at least as much reduced by the increase in idiosyncratic risk in state s as in state t. Again, this condition appears to be innocuous as long as there are no large differences among investors in terms of endowments, risk attitudes as determined by A and idiosyncratic risks.

[12] This is the modified Benthamite criterium of social welfare with compensation across agents based upon the precautionary premium.

## 4 CONCLUSIONS

The general approach in the paper has been to consider the optimal decision making by an agent who faces idiosyncratic risk but can buy/sell state-contingent claims on aggregate marketable income. This framework has several applications to problems in financial economics. The common feature of all these applications is that there are some risks that are non-marketable, but the agent is able to manage the overall risk through other tradeable claims.

Consider the case of the owner of a firm whose shares are not traded, but whose cash flow is dependent on several economy-wide variables such as interest rates, foreign exchange rates and commodity prices. In addition to these economy-wide risks, the cash flow of the firm is also affected by firm-specific factors which cannot be perfectly hedged. The question is how the entrepreneur should optimally hedge against the economy-wide risks, given the exposure to idiosyncratic risk.

An additional dimension can be introduced into this example by considering the behavior of the manager of a firm whose compensation is based on the cash flow of the firm as well as the cash flows of competing firms. The manager can alter the cash flow of the firm by buying claims on the marketable cash flow in the economy, but he is prohibited from trading in the firm's shares (no insider trading). The manager of the firm, therefore, chooses the firm's hedging policy to maximize his own expected utility given the risks of the firm's cash flows.

In all the above cases, idiosyncratic risk induces the agent to buy a type of insurance to optimize risk bearing: the insurance involves the purchase of claims that pay off in "low" states of aggregate marketable income financed by the sale of claims that pay off in the "high" states. However, this type of insurance differs from conventional insurance in an important sense. Conventional insurance is defined as the purchase of a perfectly negatively correlated risk. Here, insurance means the purchase of claims so as to balance the effects of some independent risk on conditional expected marginal utility.

Idiosyncratic risk lowers the agent's risk taken in the market. When idiosyncratic risk increases for many agents, then prices of marketable claims must adjust. Under a mild condition, prices of claims in the "low" states of aggregate marketable income increase relative to the prices of claims in the "high" states. This price change motivates agents whose idiosyncratic risk has not increased, to take more risk in the market.

When all agents have utility functions of the HARA class with the same exponent, then agents without idiosyncratic risk have a concave sharing rule. But the other agents' sharing rules need not be linear, convex or concave. The convexity of an agent's sharing rule increases, however, with his idiosyncratic risk.

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## APPENDIX A

### Proof of Lemma 2

Lemma 2a)

$$\frac{\partial \hat{a}(x)}{\partial x} = - \frac{\partial^2 \ln \upsilon'(x - \Psi)}{\partial x^2} < 0. \qquad (A.\ 1)$$

Proof: Inequality (A.1) is equivalent to

$$\frac{\partial \left( \frac{\partial \upsilon'(.)/\partial x}{\upsilon'(.)} \right)}{\partial x} > 0. \qquad (A.2)$$

υ'(.) is defined by

$$\upsilon'(.) \equiv \upsilon' \left[ x - \Psi(x, \sigma) \right] \equiv E_x[\upsilon'(x+\sigma\varepsilon)] \qquad (A.3)$$

Hence

$$\frac{\frac{\partial \upsilon'(.)}{\partial x}}{\upsilon'(.)} = \frac{E[\upsilon''(x+\sigma\varepsilon)]}{E[\upsilon'(x+\sigma\varepsilon)]} \qquad (A.4)$$

where the expectation is conditional on x, but the subscript is dropped for notational simplicity.

In order to prove Lemma 2a), we need to show that the term in equation (A.4) strictly increases with x. We differentiate the fraction in equation (A.4) with respect to x.

This differential is positive if and only if

$$E[\upsilon'(.)]\,E[\,\upsilon'''(.)] > \left(E[\upsilon''(.)]\right)^2 \tag{A.5}$$

Decreasing absolute risk aversion implies that

$$[\,\upsilon'''(.)]\,[\,\upsilon'(.)] > [\,\upsilon''(.)]^2 \tag{A.6}$$

Define

$$\hat{\upsilon}'(.) \equiv \frac{[\,\upsilon''(.)]^2}{\upsilon'''(.)} \tag{A.7}$$

Then, from (A.6) and (A.7),

$$\hat{\upsilon}'(.) < \upsilon'(.) \tag{A.8}$$

Hence, a sufficient condition for (A.5) to hold is

$$E[\hat{\upsilon}'(.)]\,E[\,\upsilon'''(.)] \geq \left(E[\upsilon''(.)]\right)^2 = \left(E[\hat{\upsilon}'(.)\,\upsilon'''(.)]\right)^2 \tag{A.9}$$

Cauchy's inequality implies that (A.9) is true since $\hat{\upsilon}'(.)$ and $\upsilon'''(.)$ are positive.

<u>Lemma 2b)</u>

$$\frac{\partial^2 \ln \upsilon'(x - \Psi)}{\partial\sigma\,\partial x} < 0 \tag{A.10}$$

Proof:

We first differentiate $\ln \upsilon'(x - \Psi)$ with respect to $\sigma$ and obtain

$$\frac{\partial \ln \upsilon'(x - \Psi)}{\partial \sigma}$$

$$= \frac{\frac{\partial E[\upsilon'(x+\sigma\varepsilon)]}{\partial \sigma}}{\upsilon'[x-\Psi]} = \frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon'(x+\sigma\varepsilon)]}$$

$$= \frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[-\upsilon''(x+\sigma\varepsilon)]} \cdot \frac{E[-\upsilon''(x+\sigma\varepsilon)]}{E[\upsilon'(x+\sigma\varepsilon)]} \qquad \text{(A.11)}$$

Note that both fractions in equation (A.11) have to be positive, since the left-hand side is positive because $\frac{\partial \Psi}{\partial \sigma} > 0$, and the second fraction on the right-hand side is positive, due to the assumption of risk aversion. By Lemma 2a), the second fraction (see A.4) was shown to decrease as x increases. Hence the product decreases as x increases if the first fraction also decreases as x increases. The latter will be shown now.

Differentiate the first fraction in (A.11) with respect to x. The differential is negative if and only if

$$E[\upsilon''(x+\sigma\varepsilon)]\,E[\upsilon'''(x+\sigma\varepsilon)\varepsilon] \;>\; E[\upsilon''(x+\sigma\varepsilon)\,\varepsilon]\,E[\upsilon'''(x+\sigma\varepsilon)] \qquad \text{(A.12)}$$

which is the same as

$$\frac{E[\upsilon'''(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon'''(x+\sigma\varepsilon)]} < \frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon''(x+\sigma\varepsilon)]} \qquad \text{(A.13)}$$

since $E[\upsilon''(x+\sigma\varepsilon)] < 0$ and $E[\upsilon'''(x+\sigma\varepsilon)] > 0.$

Decreasing absolute prudence implies equation (A.13). In order to see this, consider the analogous problem where decreasing absolute risk aversion implies

$$\frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon''(x+\sigma\varepsilon)]} < \frac{E[\upsilon'(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon'(x+\sigma\varepsilon)]} \qquad \text{(A.14)}$$

Consider an investor facing the choice between a riskless and a risky asset, where the excess return on the risky asset is equal to $\mu + \varepsilon$, and $\mu$ is the expected excess return of the risky asset over the riskless rate. Then, if $\sigma$ denotes the optimal dollar investment in the risky asset, the optimality condition is that the right hand side of inequality (A.14) equals $-\mu$, with x being the expected total income. If the initial wealth increases, decreasing absolute risk aversion implies a higher dollar investment in the risky asset [Theorem 7 of Pratt (1964)]. Hence, the right hand side of inequality (A.14) evaluated at the new level of wealth is higher. This is true if and only if inequality (A.14) holds. By analogy, (A.13) holds.

Lemma 2c)

$$\partial\left[\frac{\partial \ln \upsilon'(x-\Psi)/\partial\delta}{\partial \ln \upsilon'(x-\Psi)/\partial x}\right]/\partial x > 0 \tag{A.15}$$

Proof

Inequality (A.15) is equivalent to

$$\frac{\partial\left[\dfrac{\partial\upsilon'(.)/\partial\sigma}{\partial\upsilon'(.)/\partial x}\right]}{\partial x} > 0 \tag{A.16}$$

Using (A.3), the term within square brackets in inequality (A.16) is equal to

$$\frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[\upsilon''(x+\sigma\varepsilon)]}.$$

This term equals the first fraction in (A.11), multiplied by -1. Hence it follows from the proof of lemma 2b) that (A.16) holds.

## Appendix B

### Properties of the Precautionary Premium for the HARA Class of Preferences with $\gamma < 1$

1) Consider the case of $\gamma = -\infty$ , where the agents have exponential utility.

Then, the absolute risk aversion and the absolute prudence are constants, so that the risk premium and the precautionary premium are independent of marketable income x.

2) Assume that $1 > \gamma > -\infty$ . We now prove

$$\frac{\partial^2 \Psi(x,\sigma)}{\partial x \, \partial \sigma} < 0 \ . \tag{B.1}$$

Differentiate the definitional equation

$$\upsilon'[x-\Psi(x,\sigma)] \equiv E[\upsilon'(x+\sigma\varepsilon)] \tag{B.2}$$

with respect to σ and obtain

$$\frac{\partial \Psi(x,\sigma)}{\partial \sigma} = \frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{-\upsilon''[x-\Psi(x,\sigma)]}$$

$$= \frac{E[\upsilon''(x+\sigma\varepsilon)\varepsilon]}{E[-\upsilon''(x+\sigma\varepsilon)]} \cdot \frac{E[-\upsilon''(x+\sigma\varepsilon)]}{-\upsilon''[x-\Psi(x,\sigma)]} \tag{B.3}$$

The second term on the right hand side of equation (B.3) is positive, given the assumption of risk aversion. Since the left hand side is positive, both fractions on the right hand side of (B.3) are positive.

As shown in the proof of lemma 2b), the first fraction decreases as x increases. Hence, a sufficient condition for inequality (B.1) to hold is that the second fraction decreases as x increases.

Define

$$\upsilon''[x-\varphi(x,\sigma)] = E[\upsilon''(x+\sigma\varepsilon)] \tag{B.4}$$

where $\varphi$ is the premium defined by the second derivative of the utility function. [$\pi$ is the premium defined by the utiliy function (risk premium) and $\Psi$ is the premium defined by the first derivative (precautionary premium)]. Then, the second fraction in (B.3) can be rewritten as

$$\frac{E[-\upsilon''(x+\sigma\varepsilon)]}{-\upsilon''[x-\Psi(x,\sigma)]} = \frac{-\upsilon''[x-\varphi(x,\sigma)]}{-\upsilon''[x-\Psi(x,\sigma)]} \tag{B.5}$$

For the HARA class of preferences, the right hand side of (B.5) can be written as

$$\frac{-\upsilon''[x-\varphi(x,\sigma)]}{-\upsilon''[x-\Psi(x,\sigma)]} = \left(\frac{A+\frac{x-\varphi(x,\sigma)}{1-\gamma}}{A+\frac{x-\Psi(x,\sigma)}{1-\gamma}}\right)^{\gamma-2} \tag{B.6}$$

Differentiate the right hand side of (B.6) with respect to x. The differential is negative (since $\gamma < 1$), if

$$\left[A+\frac{x-\varphi}{1-\gamma}\right]^{-1}\left(1-\frac{\partial\varphi}{\partial x}\right) > \left[A+\frac{x-\Psi}{1-\gamma}\right]^{-1}\left(1-\frac{\partial\Psi}{\partial x}\right) \tag{B.7}$$

We substitute for $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\Psi}{\partial x}$ by differentiating (B.2) and (B.4) to obtain

$$\left[1-\frac{\partial\Psi(x,\sigma)}{\partial x}\right] = \frac{E[\upsilon''(x+\sigma\varepsilon)]}{\upsilon''[x-\Psi(x,\sigma)]} \tag{B.8a}$$

$$\left[1-\frac{\partial\varphi(x,\sigma)}{\partial x}\right]. = \frac{E[\upsilon'''(x+\sigma\varepsilon)]}{\upsilon'''[x-\varphi(x,\sigma)]} \tag{B.8b}$$

We substitute (B.8a) and (B.8b) in (B.7) to yield

$$\frac{E\left[\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-3}\right]}{\left[A+\frac{x-\varphi}{1-\gamma}\right]^{\gamma-2}} > \frac{E\left[\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-2}\right]}{\left[A+\frac{x-\Psi}{1-\gamma}\right]^{\gamma-1}} \tag{B.9}$$

Substitute for the denominators in the two sides of the inequality from (B.2) and (B.4) and obtain

$$E\left[\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-3}\right] E\left[\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-1}\right] > \left[E\left\{\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-2}\right\}\right]^2 \tag{B.10}$$

Since

$$\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-3}\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-1} = \left[\left(A+\frac{x+\sigma\varepsilon}{1-\gamma}\right)^{\gamma-2}\right]^2 \quad , \tag{B.11}$$

it follows from Cauchy's inequality that (B.10) holds. Hence (B.1) follows.

3) Now we prove the convexity of $\Psi$ relative to x. From equation (B.8a), it follows that

$$\frac{\partial^2\Psi(x,\sigma)}{\partial x^2} > 0 \tag{B.12}$$

if and only if the right-hand side in equation (B.8a) decreases as x increases. We have already shown this to be true in equations (B.5) through (B.11).

4) Lastly, we show that

$$\frac{\partial^3\Psi}{\partial\sigma\,\partial x^2} > 0 \quad . \tag{B.13}$$

First, note that convexity of $\Psi$ approaches 0 as $\sigma \to 0$. Since $\Psi$ is convex for any positive value of $\sigma$, it follows that convexity increases with $\sigma$ for small changes from $\sigma=0$. We now use a monotonicity result to show that convexity increases with $\sigma$ for any value of $\sigma$.

We rewrite equation (B.2) for the HARA class and multiply throughout by

$\left[\frac{(1-\gamma)}{\sigma}\right]^{\gamma-1}$ to obtain

$$\left[\frac{[A(1-\gamma)+x]}{\sigma} - \frac{\Psi(x,\sigma)}{\sigma}\right]^{\gamma-1} = E\left[\left(\frac{[A(1-\gamma)+x]}{\sigma}+\varepsilon\right)^{\gamma-1}\right] \tag{B.14}$$

Multiply and divide equation (B.14) throughout by q, where $q>0$, to yield

$$\left[\frac{q[A(1-\gamma)+x]}{q\sigma} - \frac{q\Psi(x,\sigma)}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A(1-\gamma)+x]}{q\sigma}+\varepsilon\right)^{\gamma-1}\right] \tag{B.15}$$

Define $x_1$ such that

$$q[A(1-\gamma)+x_0] \equiv A(1-\gamma)+x_1 \quad .$$

Then, using subscript 0 for x in equation (B.15) yields

$$\left[\frac{[A(1-\gamma)+x_1]}{q\sigma} - \frac{q\Psi(x_0,\sigma)}{q\sigma}\right]^{\gamma-1} = E\left[\left(\frac{q[A(1-\gamma)+x_0]}{q\sigma}+\varepsilon\right)^{\gamma-1}\right] \tag{B.16}$$

In words, if $\sigma$ changes from $\sigma$ to $q\sigma$ and x changes from $x_0$ to $x_1$, then the new precautionary premium $\Psi(x_1,q\sigma) = q\Psi(x_0,\sigma)$.

In order to show that the convexity of $\Psi$ grows with $\sigma$, suppose that $\sigma$ is raised from a level arbitrarily close to 0. Then, the convexity of $\Psi(x_0,\sigma)$ increases.

Hence, the convexity of $\Psi(x_1,q\sigma)$ increases by the factor q. As q can be arbitrarily large, the convexity of $\Psi(x_1,q\sigma)$ increases monotonically with $q\sigma$.

## Appendix C

### Proof of Proposition 4

a) Aggregation of the individual agents' sharing rules (8) yields

$$X - \sum_i \Psi_i(X,\sigma_i) = \left[ \left(\sum_i \lambda_i^{\frac{1}{\gamma-1}}\right) \Phi(X)^{\frac{1}{\gamma-1}} - \sum_i A_i \right] (1-\gamma) \tag{C.1}$$

or, substituting for $\Phi(X)$ and simplifying the sharing rule for agent j yields

$$x_j - \Psi_j(X,\sigma_j) = \frac{\lambda_j^{\frac{1}{\gamma-1}}}{\sum_i \lambda_i^{\frac{1}{\gamma-1}}} \left[ \sum_i A_i(1-\gamma) + X - \sum_i \Psi_i(X,\sigma_i) \right] - A_j(1-\gamma) \tag{C.2}$$

which, on simplification, and using $x_i = g_i(X)$, yields Proposition 4 a.

b) Now we prove convexity of the pricing function. Differentiating the sharing rule for agent j with respect to X yields

$$\frac{dg_j(X)}{dX} \left[ 1 - \frac{\partial \Psi_j}{\partial g_j(X)} \right] = (1-\gamma)\, \lambda_j^{\frac{1}{\gamma-1}} \frac{d[\Phi(X)]^{\frac{1}{\gamma-1}}}{dX} \tag{C.3}$$

Aggregating the previous equation across all agents leads to

$$1 = \sum_i \frac{dg_i(X)}{dX} = (1-\gamma)\,\frac{d\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]}{dX}\sum_i \lambda_i^{\frac{1}{\gamma-1}}\left[1-\frac{\partial\Psi_i}{\partial g_i(X)}\right]^{-1} \qquad \text{(C.4)}$$

since the sum of the changes in the sharing rules across agents has to net to the change in the aggregate marketable income. Rearranging the above expression yields

$$\frac{1}{\dfrac{d\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]}{dX}} = (1-\gamma)\sum_i \lambda_i^{\frac{1}{\gamma-1}}\left[1-\frac{\partial\Psi_i}{\partial g_i(X)}\right]^{-1} \qquad \text{(C.5)}$$

Differentiating again with respect to X, we obtain

$$\frac{-1}{\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]^2}\,\frac{d^2\left[\Phi(X)^{\frac{1}{\gamma-1}}\right]}{dX^2} = (1-\gamma)\sum_i \lambda_i^{\frac{1}{\gamma-1}}\,\frac{\left[\dfrac{\partial^2\Psi_i}{\partial[g_i(X)]^2}\right]\left[\dfrac{dg_i(X)}{dX}\right]}{\left[1-\dfrac{\partial\Psi_i}{\partial g_i(X)}\right]^2} \qquad \text{(C.6)}$$

Since $\frac{dg_i(X)}{dX} > 0$ $\Psi_i$ is convex in $g_i(X)$ as shown in Appendix B, the right-hand side of the equation is positive. Hence, $\Phi(X)^{\frac{1}{\gamma-1}}$ is a concave function or $\Phi(X)$ is a convex function.

c) Rearranging equation (C.1) yields

$$A + \frac{X - \Psi(X,\sigma)}{1-\gamma} = \left[\sum_i \lambda_i^{\frac{1}{\gamma-1}}\right] \Phi(X)^{\frac{1}{\gamma-1}} \tag{C.7}$$

Hence strict convexity of $\Phi(X)$ [ or concavity of $\Phi(X)^{\frac{1}{\gamma-1}}$ ] implies strict convexity of $\Psi(X)$.

d) In order to avoid obvious but tedious calculations, we provide only a verbal argument for the comparative statics of $\alpha_i$. Note that the sign of the change in $\alpha_i$ is opposite to that of $\lambda_i$ because of the negative exponent of $\lambda_i$ in the definition of $\alpha_i$. First, it has already been shown in section 2 B that $d\lambda/d\sigma > 0$. Second, an increase in $A_i$ and $w_i$ increases expected utility and, thus, reduces the increase in expected utility from an additional unit of endowment■