"PORTFOLIO CHOICE AND EQUILIBRIUM WITH EXPECTED-UTILITY PREFERENCES"

by

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Printed at INSEAD,
Fontainebleau, France

Portfolio Choice and Equilibrium with Expected-Utility Preferences

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October 1992

Abstract

This paper exhibits conditions for the existence of satiation portfolios, the existence of optimal portfolios, and the existence of equilibrium in a model of asset markets where preferences for portfolios of assets are derived from expected utility based on a von Neumann–Morgenstern utility function and a joint distribution of total returns to the assets. The conditions are based on an analysis of directions of improvement (directions of recession of the preferred sets). In particular, the relation between absence of arbitrage and existence of optimal portfolios is investigated.

1 Introduction

This paper analyzes portfolio choice and equilibrium in a model of asset markets where preferences for portfolios of assets are derived from expected utility based on a von Neumann–Morgenstern utility function and a joint distribution of total returns to the assets. The main objective is to determine conditions for the existence of satiation portfolios, the existence of optimal portfolios, and the existence of equilibrium.

Satiation portfolios may exist because the expected utility of a portfolio is not necessarily an increasing function of the number of shares held of the various assets. This is so because the returns to the assets may be negative with positive probability. As the investor acquires more shares of an asset, the potential positive returns increase, but so do the potential negative returns. It may well be that the negative returns outweigh the positive returns.

Optimal portfolios may fail to exist because the investor may be allowed to short sell assets, so that his choice set will be unbounded below. For some systems of asset prices, the investor may be able to afford unlimited amounts of a portfolio which indefinitely increases his expected utility.

Finally, equilibrium may fail to exist either because of unbounded choice sets (short-selling) or because of satiation.

For the case of mean-variance preferences, the existence of satiation portfolios and optimal portfolios was analyzed in Nielsen (1987). The existence of equilibrium has been analyzed in Allingham (1991) and Nielsen (1990a, 1990b, 1992).

For the case of expected utility, Nielsen (1993) has analyzed the indirect utility function for portfolios, demonstrating under what conditions it has the properties needed in demand and equilibrium analysis, such as various kinds of differentiability, strict quasi-concavity, and differentiable strict concavity.

The present paper takes the analysis of the indirect utility function for portfolios a step further and investigates its directions of improvement (the directions of recession of the preferred sets), because they are needed in the analysis of the existence of satiation portfolios, optimal portfolios and equilibrium. A direction of improvement is a portfolio with the following property. No matter which initial portfolio the investor starts out with, his expected utility will not decrease if he adds to it a fraction or a multiple of a direction of improvement. If the expected utility increases strictly, independently of the initial portfolio, then the direction is called a direction of global strict improvement.

To appreciate the use of directions of improvement in studying the existence of satiation portfolios, consider first the standard situation where all directions of improvement are directions of global strict improvement. There exists a satiation portfolio if and only if there exists no non-zero direction of improvement, and there exists an optimal portfolio (subject to a budget constraint given by a price system and a wealth level) if and only if all non-zero directions of improvement have positive cost.

This standard situation, where all directions of improvement are directions of global strict improvement, is by far the most likely. It will prevail unless the von Neumann-Morgenstern utility function is risk neutral at the extremes, which means that it is affine at very high and at very low levels of final wealth. Even if the utility function is not risk neutral at the extremes, the standard situation prevails if all portfolios which are directions of recession of the choice set have non-negative return for sure. These and other conditions are summarized in Proposition 1.

If we are not in the standard situation, then it is possible that there exists a satiation portfolio even though there exists a non-zero direction of improvement. The set of satiation portfolios will then be unbounded and contain a half-line in the direction of this direction of improvement. Moreover, there may exist an optimal portfolio even if some non-zero direction of improvement has non-positive cost. The set of optimal portfolios will then be unbounded and contain a half-line in the direction of this direction of improvement.

Building upon earlier results by Leland (1972), Bertsekas (1974) analyzed directions of improvement in terms of the asymptotic slopes of the von Neumann–Morgenstern utility function and the incomplete means of the returns. Theorem 1 expands on Bertsekas' analysis. It contains a full characterization not only of directions of improvement but also of directions of global strict improvement. This characterization is further elaborated in Theorem 2 in the special case where the closed support of the distribution of returns is convex (which will be true in many practical examples).

The von Neumann-Morgenstern utility function is said to be asymptotically risk neutral if its graph has a finite asymptotic slope both at large positive and large negative wealth levels. If the utility function is not asymptotically risk neutral, then the only directions of improvement are portfolios that have non-negative return for sure. Furthermore, there exists a satiation portfolio if and only if all non-zero portfolios (that are directions of recession of the choice set) have positive probability of negative return.

The existence of optimal portfolios subject to a budget constraint is closely related to whether the price system is arbitrage-free. A system of asset prices is said to be arbitrage-free if every portfolio which is a direction of recession of the choice set and has non-negative return for sure has positive cost. If an optimal portfolio exists, then the price system is arbitrage-free. The opposite implication holds if all portfolios which are directions of recession of the choice set have non-negative return for sure, and it also holds if the von Neumann-Morgenstern utility function is not asymptotically risk neutral. In other cases, it is possible that no optimal portfolio exists even if the price system is arbitrage-free.

Be aware that Werner (1987) used the term "arbitrage" somewhat differently from the way it is used here. He defined it in terms of the utility of portfolios (or commodity bundles) rather than the returns to portfolios.

The analysis of the existence of equilibrium uses the sufficient conditions identified in the existence result of Nielsen (1989). The question is under what circumstances these conditions are satisfied in the expected-utility model.

The first condition is the condition of positive semi-independence of directions of improvement. It says that if a set of directions of improvement, one for each investor, add up to zero, then each of the directions is zero. It is equivalent to boundedness of the set of individually rational allocations, to existence of a boundedly viable price system, and to the existence of a bounded Pareto optimum. A boundedly viable price system is one such that all investors have a non-empty and bounded set of optimal portfolios. A bounded Pareto optimum is a Pareto optimum such that the set of allocations which Pareto dominate it is bounded.

If the investors are not asymptotically risk neutral, then the condition can be interpreted in terms of arbitrage across individuals, and it is equivalent to the existence of a price system which is arbitrage-free for all investors and to a condition of "overlapping expectations." If the investors agree on the expected returns to all the assets, and none of them is risk neutral, then the condition holds.

The second condition says that there is no satiation at Pareto attainable portfolios. It holds if there exists a direction of global strict improvement. Even if there is no direction of global strict improvement, the condition will hold if the investors agree on the expected returns to all the assets, if the certain equivalent of each investor's initial portfolio is at least zero, and if the returns distributions and and utility functions are such that all satiation portfolios have an expected return which is larger than the expected return to the market.

Existence of equilibrium in the expected-utility model has previously been analyzed by Hart (1974), Hammond (1983), and Page (1987).

Hart (1974) showed the existence of equilibrium when the returns are bounded and non-negative (so that satiation is not an issue). He used a version of the condition of positive semi-independence of directions of improvement which is equivalent to the one used here when there are no redundant assets. He assumed that the investors' choice sets are intersections of half-spaces; that is not necessary here.

Hammond (1983) elaborated on Hart's analysis. He introduced a condition of "overlapping expectations" and showed that under certain additional assumptions it is equivalent Hart's version of the condition of positive semi-independence of directions of improvement. Also, under his assumptions, he could do away with Hart's assumption that the choice sets are intersections of half-spaces.

Page (1987) constructed an alternative proof of Hart's existence result under slightly different assumptions. Like Hart, Page assumed that the returns are non-negative and bounded.

The organization of the paper is as follows. Section 2 describes the two models used: one where the utility function is defined only on the non-negative real line, and one where it is defined on the entire real line. Section 3 analyzes the directions of improvement and directions of global strict improvement and introduces the concepts of risk neutrality at the extremes and asymptotic risk neutrality. Section 4 analyzes the existence of satiation portfolios, and Section 5 analyzes the existence of optimal portfolios. Definitions and previous results related to equilibrium are reviewed in Section 6. The condition of positive semi-independence of directions of improvement is studied in Section 7, and the condition of non-satiation at Pareto attainable portfolios is studied in Section 8.

2 The Models

There are n *assets* or *securities*, $j = 1, \ldots, n$. A *portfolio* is represented by an n-vector x, where the j'th entry x_j indicates the number of shares or units of the j'th asset in the portfolio. A portfolio x may well have some negative entries x_j, interpreted as *short-selling* or borrowing.

The investor has a *choice set* X, which is a subset of $\mathbb{R}^n$. It represents possible restrictions on his portfolio choices, such as full or partial prohibitions against short-selling. Typically X has the form $[0,\infty)^n$, $\mathbb{R}^{n-h} \times [0,\infty)^h$ or $\mathbb{R}^n$, reflecting short-selling constraints for all assets, for the last h assets, or no constraints. The choice set corresponds to the consumption set in microeconomic demand and equilibrium analysis. We assume that X is closed and convex.

The *total returns* per share of the assets are given by a random vector R where the j'th entry R_j indicates the (random) total (gross) return per share of asset j. The total return to a portfolio x is $x'R$. The investor considers the returns R to be distributed in $\mathbb{R}^n$ according to a *probability distribution* π.

Assumption 1 *Existence of expectations: R has a finite mean.*

The investor has a *von Neumann–Morgenstern utility function* u which is a function of end-of-period wealth or total portfolio return $x'R$. Like in Nielsen (1993), we distinguish two models, depending on whether u is defined on a limited or an unlimited domain. If the domain is limited to the non-negative real line, then it is necessary that all portfolios in the choice set have non-negative return for sure.

Model 1 *Limited domain, non-negative returns and no short-selling:*

1. *Limited domain: u is defined on $[0, +\infty)$ and takes finite real values except possibly the value $u(0) = -\infty$.*
2. *Non-negative returns: For all assets j, $R_j \geq 0$ with probability one.*
3. *No short-selling: X is contained in $[0, +\infty)^n$.*

Model 2 *Unlimited domain: u is defined on the entire real line (and takes finite real values).*

The assets in Model 2 may have negative return with positive probability. This specification allows for the classical case of normal distributions.

Model 1 is a special case of Model 2 except for the possibility that $u(0) = -\infty$. One of these two models and its corresponding assumptions will always be in force. If neither is explicitly specified, either will do.

Assumption 2 *u is strictly increasing, concave and continuous (also at zero).*

Define the *extended choice set* $\bar{X}$ as $\bar{X} = [0, +\infty)^n$ in Model 1 and $\bar{X} = \mathbb{R}^n$ in Model 2. Then X is contained in $\bar{X}$. Define the *extended utility function* $\bar{V}$ on the extended choice set $\bar{X}$ by

$$\bar{V}(x) = Eu(x'R).$$

Strictly speaking, the utility function V for portfolios is the restriction of the extended utility function $\bar{V}$ to the choice set X.

The expectations symbol "E" refers to the mathematical expectation with respect to the probability distribution π. Write $u(x'R)^+$ for the positive part of $u(x'R)$. More generally, the positive part of a variable Y is $Y^+ = \max\{0, Y\}$ and the negative part is $Y^- = -\min\{0, Y\} = Y^+ - Y$. Since u is concave (Assumption 2) and expectations exist (Assumption 1), it is easily seen that $Eu(x'R)^+ < 0$ for all x in $\bar{X}$, so that $\bar{V}$ is well defined and $\bar{V}(x) < +\infty$. It also follows that $\bar{V}$ is concave.

Assumption 3 *Integrability (below): $Eu(x'R)^- < +\infty$ for all x in $\bar{X}$ such that $\pi(u(x'R) = -\infty) = 0$.*

This assumption implies that $\bar{V}$ is continuous, cf. Nielsen (1993). It also follows from Assumption 3 that $\bar{V}(x) = -\infty$ if and only if $\pi(u(x'R) = -\infty) > 0$. A portfolio has utility negative infinity only if it has positive probability of zero return and the utility of zero return is negative infinity. Furthermore, $\bar{V}$ has the property that if $\bar{V}(y) > -\infty$ and $\bar{V}(x) = -\infty$, then $\bar{V}(tx + (1-t)y) > -\infty$ for all t with $0 < t < 1$.

3 Directions of Improvement

The analysis of the existence of satiation portfolios, optimal portfolios and equilibrium will rely on the concept of directions of improvement of the indirect utility function V. First, we need to recall the concept of a direction of recession.

Suppose B is a non-empty, closed and convex set in $\mathbb{R}^n$. A vector e in $\mathbb{R}^n$ is a *direction of recession* of B (cf. Rockafellar (1970)) if there is a point b in B such that $b + te$ belongs to B for all $t \geq 0$. If so, then $b + te$ belongs to B for all b in B and all $t \geq 0$. The set of directions of recession of B is a closed convexd cone. The set B is bounded if and only if it has no non-zero direction of recession.

If e is a direction of recession of X, say that e is a *direction of improvement* for the investor if there is a portfolio x in X such that $V(x+te) \geq V(x)$ for all $t \geq 0$. Equivalently, e is a direction of recession of the *preferred set*

$$P(x) = \{y \in X : V(y) \geq V(x)\}$$

(which is closed and convex). Because V is concave, all preferred sets have the same directions of recession. So, if e is a direction of improvement for the investor, then $V(x + te) \geq V(x)$ for all x in X and all $t \geq 0$.

A direction of improvement e of V is a *direction of global strict improvement* of V if $V(x + te) > V(x)$ for all portfolios x in X with $V(x) > -\infty$ and all $t > 0$.

Either all non-zero directions of improvement are directions of global strict improvement or else some indifference curve contains a half-line.

Assumption 4 *No redundant assets: For all x in $\mathbb{R}^n$, if $x'R = 0$ for sure according to π, then $x = 0$.*

It is helpful to relate the directions of improvement to the closed support of π. The negative polar cone of the closed support $\text{supp}(\pi)$ of π is

$$\text{supp}(\pi)^+ = \{R \in \mathbb{R}^n : e'R \geq 0 \text{ for all } R \text{ in } \text{supp}(\pi)\ \}.$$

Let I denote the set of directions of improvement of V. Let D denote the set of directions of recession of X. The sets $\text{supp}(\pi)^+$, I, and D are closed, convex cones. In general,

$$D \bigcap \text{supp}(\pi)^+ \subset I \subset D.$$

It may happen that all directions of recession of X have non-negative return for sure. Formally,

$$D \subset \mathrm{supp}(\pi)^{+}.$$

It follows that

$$D \bigcap \mathrm{supp}(\pi)^{+} = I = D.$$

In this case, every direction of recession of X is a direction of improvement of V, and every non-zero direction of improvement of V is a direction of global strict improvement.

In Model 1, it is always true that all directions of recession of X have non-negative return for sure. Therefore, it is fairly simple to identify the directions of improvement and the directions of global strict improvement. In the rest of this section, we consider only Model 2 (where $\bar{V}(x) > -\infty$ for all portfolios x).

The directions of improvement and directions of global strict improvement of $\bar{V}$ can be described in terms of the von Neumann–Morgenstern utility function u and the probability distribution π. Define the *asymptotic slopes* S^{+} and S^{-} of u by

$$S^{+} = \lim_{t \to +\infty} u'^{-}(t) = \lim_{t \to +\infty} u'^{+}(t)$$

and

$$S^{-} = \lim_{t \to -\infty} u'^{-}(t) = \lim_{t \to -\infty} u'^{+}(t).$$

Clearly,

$$0 \leq S^{+} \leq S^{-} \leq +\infty$$

and

$$S^{+} < +\infty \text{ and } 0 < S^{-}.$$

If u is risk neutral, then $S^{+} = S^{-}$; otherwise, $S^{+} < S^{-}$. If e is a portfolio, define the *incomplete means* of the returns to e (computed according to the distribution π) by

$$E^{+}(e) = E[(e'R)1_{\{e'R>0\}}]$$

and

$$E^{-}(e) = E[(e'R)1_{\{e'R<0\}}].$$

Then $E^+(e) \geq 0$, $E^-(e) \leq 0$, and $E^+(e) + E^-(e) = E(e'R)$.

The following theorem characterizes the directions of improvement and the directions of global strict improvement. Statement 1 of the theorem characterizes the directions of improvement and is identical to Proposition 2 of Bertsekas (1974). That paper also has a version of Statement 2. What is new is Statement 3, which together with Statement 2 yields a characterization of the directions of global strict improvement.

The characterization is this. Essentially, all directions of improvement e are directions of global strict improvement, except in the following very special case. There exist finite numbers $r \leq s$ such that the von Neumann-Morgenstern utility function u is affine below r and above s, and such that the return to e is non-negative whenever the return to x is higher than r, and the return to e is non-positive whenever the return to x is below s. This case is further analyzed in Theorem 2 below in the situation where $\text{supp}(\pi)$ is convex.

Theorem 1 *Let e be a portfolio.*

1. $S^+E^+(e) + S^-E^-(e) \geq 0$ *if and only if e is a direction of improvement of $\bar{V}$.*
2. *If* $S^+E^+(e) + S^-E^-(e) > 0$ *then e is a direction of global strict improvement of $\bar{V}$.*
3. *Suppose* $S^+E^+(e) + S^-E^-(e) = 0$ *and let x be a portfolio. Then $\bar{V}(x + te)$ is constant as a function of $t \geq 0$ if and only if*

$$\pi(x'R > r \text{ and } e'R < 0) = 0 \textit{ and } \pi(x'R < s \text{ and } e'R > 0) = 0,$$

where $r = \sup\{a : u'^-(a) = S^-\}$ *and* $s = \inf\{a : u'^+(a) = S^+\}$.

PROOF: Given portfolios y and e, let $\bar{V}(y; e)$ denote the directional derivative of $\bar{V}$ at y in the direction of e. By Proposition 3 of Nielsen (1993), for $t \geq 0$,

$$\begin{aligned} \bar{V}'(x + te; e) &= E[1_{\{e'R>0\}}(e'R)u'^+((x + te)'R)] + \\ &\quad E[1_{\{e'R<0\}}(e'R)u'^-((x + te)'R)], \end{aligned}$$

where both terms are finite. Since

$$1_{\{e'R>0\}}(e'R)u'^+(x'R) \geq 1_{\{e'R>0\}}(e'R)u'^+((x + te)'R) \searrow 1_{\{e'R>0\}}(e'R)S^+$$

almost surely as $t \to +\infty$, and since the bounds are integrable,

$$E[1_{\{e'R>0\}}(e'R)u'^{+}((x+te)'R)] \searrow E[1_{\{e'R>0\}}(e'R)S^{+}] = S^{+}E^{+}(e).$$

Since

$$0 \geq 1_{\{e'R<0\}}(e'R)u'^{-}((x+te)'R) \searrow 1_{\{e'R<0\}}(e'R)S^{-}$$

as $t \to +\infty$,

$$E[1_{\{e'R<0\}}(e'R)u'^{-}((x+te)'R)] \searrow E[1_{\{e'R<0\}}(e'R)S^{-}] = S^{-}E^{-}(e).$$

Consequently,

$$\bar{V}'(x+te;e) \searrow S^{+}E^{+}(e) + S^{-}E^{-}(e)$$

as $t \to +\infty$. So, $\bar{V}'(x+te;e)$ is non-negative for all $t \geq 0$ if and only if $S^{+}E^{+}(e) + S^{-}E^{-}(e) \geq 0$, and $\bar{V}'(x+te;e)$ is positive for all $t \geq 0$ if $S^{+}E^{+}(e) + S^{-}E^{-}(e) > 0$. This proves 1 and 2. To prove 3, suppose that $S^{+}E^{+}(e) + S^{-}E^{-}(e) = 0$ and let x be a portfolio. Then $\bar{V}(x+te)$ is constant as a function of $t \geq 0$ if and only if $\bar{V}'(x+te;e) = 0$ for all $t \geq 0$. It follows from the monotone limit results above that this holds if and only if

$$E[1_{\{e'R>0\}}(e'R)u'^{+}((x+te)'R)] = S^{+}E^{+}(e)$$

and

$$E[1_{\{e'R<0\}}(e'R)u'^{-}((x+te)'R)] = S^{-}E^{-}(e)$$

for all $t \geq 0$. Equivalently,

$$\pi(e'R > 0 \text{ and } u'^{+}((x+te)'R) \neq S^{+}) = 0$$

and

$$\pi(e'R < 0 \text{ and } u'^{-}((x+te)'R) \neq S^{-}) = 0$$

for all $t \geq 0$. These equations, in turn, are equivalent to

$$\pi(x'R > r \text{ and } e'R < 0) = 0 \text{ and } \pi(x'R < s \text{ and } e'R > 0) = 0.$$

□

Corollary 1 *If e is a direction of improvement of $\bar{V}$, then $E(e'R) \geq 0$, and $E(e'R) > 0$ unless u is risk neutral or $e = 0$.*

PROOF: It follows from 1 of Theorem 1 that $S^+E^+(e) + S^-E^-(e) \geq 0$. If $E^+(e) = 0$ then $E^-(e) = 0$, contradicting Assumption 4. So, $E^+(e) > 0$. If $S^+ = 0$ or $S^- = +\infty$, then $E^-(e) = 0$, so that $E(e'R) = E^+(e) > 0$. If $0 < S^+ < S^- < +\infty$, then u is not risk neutral, and

$$0 \leq S^+E^+(e) + S^-E^-(e) < S^-(E^+(e) + E^-(e)) = S^-E(e'R),$$

implying that $E(e'R) > 0$. Finally, if $0 < S^+ = S^- < +\infty$, then u is risk neutral, and

$$0 \leq S^+E^+(e) + S^-E^-(e) \leq S^-(E^+(e) + E^-(e)) = S^-E(e'R),$$

implying that $E(e'R) \geq 0$. □

For most von Neumann-Morgenstern utility functions, directions of recession with non-negative return for sure are the only directions of improvement. This is true of all utility functions except those that will be called asymptotically risk neutral. Say that the utility function or the investor is *asymptotically risk neutral* if $0 < S^+$ and $S^- < +\infty$.

Corollary 2 *Assume that u is not asymptotically risk neutral. A direction of recession e of X is a direction of improvement of V if and only if $e'R \geq 0$ for sure (in which case either $e = 0$ or e is a direction of global strict improvement).*

PROOF: Since e is a direction of recession of X, it is a direction of improvement of V if and only if it is a direction of improvement of $\bar{V}$, which is the case if and only if $S^+E^+(e) + S^-E^-(e) \geq 0$. Since $0 < S^-$ and either $S^+ = 0$ or $S^- = \infty$, it is clear that $S^+E^+(e) + S^-E^-(e) \geq 0$ if and only if $E^-(e) = 0$. If so, and if $e \neq 0$, then e is a direction of global strict improvement of $\bar{V}$. □

Corollary 2 corresponds to Lemma 5.2 of Page (1987) (except that Page assumes a bounded returns distribution).

It follows from Corollary 2 that if u is not asymptotically risk neutral, then

$$I = D \cap \mathrm{supp}(\pi)^+.$$

In this case, every non-zero direction of improvement of V is a direction of global strict improvement.

If e is a portfolio with $S^+E^+(e) + S^-E^-(e) = 0$ and $e \neq 0$, then e may or may not be a direction of global strict improvement of $\bar{V}$. Statement 3 of Theorem 1 is a necessary and sufficient condition. That condition is somewhat complicated, but some simpler consequences in special cases are developed below.

First, if $e \neq 0$ is a direction of recession of X such that $e'R \geq 0$ for sure, then e is a direction of global strict improvement.

A direction of improvement $e \neq 0$ can fail to be a direction of global strict improvement only if some indifference curve contains a half-line in the direction of e. This can only be the case if the numbers r and s in Statement 3 of Theorem 1 are finite. Say that the investor (or the von Neumann–Morgenstern utility function u) is *risk neutral at the extremes* if there exist finite numbers r and s such that u is affine on both of the intervals $(-\infty, r]$ and on $[s, \infty)$. We conclude from Theorem 1 that if u is not risk neutral at the extremes, then every non-zero direction of improvement of $\bar{V}$ is a direction of global strict improvement of $\bar{V}$.

Of course, if u is risk neutral at the extremes, then it is also asymptotically risk neutral.

In most cases, the distribution followed by R will be such that its closed support $\text{supp}(\pi)$ is convex. If so, a direction of improvement can fail to be a direction of global strict improvement only if the von Neumann-Morgenstern utility function u either is risk neutral or consists of two risk-neutral parts joined at a single kink. This is a consequence of the following proposition.

Theorem 2 *Suppose* $\text{supp}(\pi)$ *is convex. Let x and $e \neq 0$ be portfolios. Suppose $S^+E^+(e)+S^-E^-(e) = 0$. Then $\bar{V}(x+te)$ is constant as a function of $t \geq 0$ if and only if either u is risk neutral or else there exists a finite number r such that u is affine on each of the intervals $(-\infty, r)$ and $(r, +\infty)$ and x has the form $x = y + \lambda e$, where y is riskless with total return r and $\lambda \geq 0$.*

PROOF: To show that the condition is necessary, assume that $\bar{V}(x + te)$ is a constant function of $t \geq 0$ and let $r = \sup\{a : u'^-(a) = S^-\}$ and $s = \inf\{a : u'^+(a) = S^+\}$. It follows from Statement 3 of Theorem 1 that

$$\pi(x'R > r \text{ and } e'R < 0) = 0 \text{ and } \pi(x'R < s \text{ and } e'R > 0) = 0.$$

Consequently, every point R in supp(π) satisfies either $e'R = 0$, or $e'R < 0$ and $x'R \leq r$, or $e'R > 0$ and $x'R \geq s$. Since e is not a direction of global strict improvement of $\bar{V}$, u is risk neutral at the extremes, and so $S^- < +\infty$ and $0 < S^+$. Assuming that u is not risk neutral, r and s are finite with $r \leq s$. Since $S^+E^+(e) + S^-E^-(e) = 0$, $E^+(e) > 0$ and $E^-(e) < 0$, so that $\pi(e'R < 0) > 0$ and $\pi(e'R > 0) > 0$. If R is a point in supp(π) with $e'R = 0$, then $x'R \geq s$, because otherwise there would be a point $\tilde{R}$ in supp(π), close to R, such that $x'\tilde{R} < s$ and $e'\tilde{R} > 0$. Similarly, R has to satisfy $x'R \leq r$, so that $x'R = r = s$. Consequently, if R is a point in aff(supp(π)) with $e'R = 0$, then $x'R = r = s$. Since there are such points R, it follows that $r = s$. If aff(supp(π)) contains zero, then it equals $\mathbb{R}^n$. In that case, $r = 0$ and x has the form $x = \lambda e$. Otherwise, if aff(supp(π)) does not contain zero, then there is a portfolio y such that $y'R = r$ for all R in aff(supp(π)). Then $(x - y)'R = 0$ for all R in aff(supp(π)) with $e'R = 0$, and it follows that $x - y = \lambda e$ for some λ. In both cases, since

$$\pi(x'R > r \text{ and } e'R < 0) = 0 \text{ and } \pi(x'R < s \text{ and } e'R > 0) = 0,$$

it follows that $\lambda \geq 0$. Conversely, to show that the condition is sufficient, consider first the case where u is risk neutral. Then $S^+ = S^-$, $E(e'R) = E^+(e) + E^-(e) = 0$, u has the form $u(c) = a + S^+c$, and

$$\begin{aligned}\bar{V}(x + te) &= a + S^+E((x + te)'R) \\ &= a + S^+(E(x'R) + tE(e'R)) \\ &= a + S^+E(x'R),\end{aligned}$$

which is independent of t. Alternatively, consider the case where $x = y + \lambda e$ for some $\lambda \geq 0$, $y'R = r$ for sure, and u is affine on each of the intervals $(-\infty, r)$ and $(r, +\infty)$. Then u has the form

$$u(c) = \begin{cases} a + S^+(c - r) & \text{for } c \geq r \\ a + S^-(c - r) & \text{for } c \leq r \end{cases},$$

so that

$$\begin{aligned}\bar{V}(x + te) &= \bar{V}(y + (\lambda + t)e) \\ &= a + E[1_{e'R \geq 0}S^+(\lambda + t)(e'R)] + E[1_{e'R < 0}S^-(\lambda + t)(e'R)]\end{aligned}$$

$$
\begin{aligned}
&= a + S^{+}(\lambda + t)E^{+}(e) + S^{-}(\lambda + t)E^{-}(e) \\
&= a + (\lambda + t)(S^{+}E^{+}(e) + S^{-}E^{-}(e)) \\
&= a,
\end{aligned}
$$

which is independent of t. □

When interpreting Theorem 2, it is useful to distinguish whether or not there exists a riskless asset (or portfolio).

First, suppose that there is no riskless asset or portfolio (other than zero). In this case the condition in Theorem 2 requires that either u is risk neutral, or else u is affine on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$ (and x is a non-negative multiple of e). If $\mathrm{supp}(\pi)$ is convex but u is not affine on both of the intervals $(-\infty, 0)$ and $(0, +\infty)$, then every direction of improvement of V is a direction of global strict improvement.

Next, suppose that the first asset is riskless with non-zero return R_f and that the closed support $\mathrm{supp}(\nu)$ of the marginal distribution of the other assets is convex. Theorem 2 implies that if there is no finite r such that u is affine on both of the intervals $(-\infty, r)$ and $(r, +\infty)$, then every non-zero direction of improvement of V is a direction of global strict improvement.

The following proposition summarizes some sufficient conditions under which no indifference curve contains a half-line.

Proposition 1 *Under each of the following assumptions (seperately), any non-zero direction of improvement of X is a direction of global strict improvement of V:*

1. $S^{+}E^{+}(e) + S^{-}E^{-}(e) > 0$ *for every direction of recession* $e \neq 0$ *of* X.
2. *All directions of recession of X have non-negative return for sure.*
3. *u is not risk neutral at the extremes.*
4. *There is a riskless asset or portfolio with non-zero return,* $\mathrm{supp}(\pi)$ *is convex, and there is no finite r such that u is affine on both of the intervals* $(-\infty, r)$ *and* $(r, +\infty)$.
5. *There is no non-zero riskless asset or portfolio,* $\mathrm{supp}(\pi)$ *is convex, u is not affine on both of the intervals* $(-\infty, 0)$ *and* $(0, +\infty)$.

4 Satiation

Proposition 2 *The set of satiation portfolios is non-empty and bounded if and only if there is no non-zero direction of improvement.*

PROOF: Pick a portfolio x in X. The set

$$\{y \in X : V(y) \geq V(x)\}$$

is non-empty, closed and convex. It is bounded if and only if it has no non-zero direction of recession, and if and only if the set of satiation portfolios is non-empty and bounded. A a direction of recession of this set is the same thing as a direction of improvement. □

For satiation portfolios to exist, it is not necessary that there be no non-zero direction of improvement, because there may be an unbounded set of satiation portfolios. However, if all non-zero directions of improvement of V are directions of global strict improvement, and if there exists a portfolio x in X with $V(x) > -\infty$, then the set of satiation portfolios is bounded if it is non-empty, and the condition in Proposition 2 is indeed a necessary condition for existence of a satiation portfolio. Recall that Proposition 1 lists sufficient conditions ensuring that all non-zero directions of improvement of V are directions of global strict improvement.

In Model 1, there is a satiation portfolio if and only if X is bounded. In the typical case where $X = [0, \infty)^n$, there is no satiation portfolio.

In the rest of this section, we consider only Model 2.

Proposition 3 *If u is not asymptotically risk neutral, then there is a satiation portfolio in X if and only if all non-zero directions of recession of X have positive probability of negative return.*

PROOF: Follows from Corollary 2. □

The condition in Proposition 3, that no non-zero direction of recession of X have non-negative return for sure, can formally be expressed as

$$D \bigcap \mathrm{supp}(\pi)^+ = \{0\}.$$

For example, if $X = \mathbb{R}^n$, then $D = \mathbb{R}^n$, and the condition says that $\text{supp}(\pi)^+ = \{0\}$, or equivalently, that $\text{conv}(\text{supp}(\pi))$ contains the origin in its interior.

So, in Model 2, if there is a direction of recession of X which has non-negative return for sure, then there is no satiation portfolio. On the other hand, if all non-zero directions of recession of X have positive probability of negative return, then we need to distinguish two cases. First, if u is not asymptotically risk neutral, then there is a satiation portfolio. Second, if u is asymptotically risk neutral, then there may or may not be a satiation portfolio, depending on the limiting behavior of u and on the ratio between the positive and negative incomplete means of directions of recession of X.

Specifically, if there is no direction of recession $e \neq 0$ of X such that

$$S^-/S^+ \geq -E^+(e)/E^-(e),$$

then there is a satiation portfolio. If there is a direction of recession $e \neq 0$ of X such that

$$S^-/S^+ > -E^+(e)/E^-(e),$$

then there is no satiation portfolio. If there is no direction of recession $e \neq 0$ of X which satisfies the strict inequality, but there is one which satisfies the corresponding equality,

$$S^-/S^+ = -E^+(e)/E^-(e),$$

then we need to refer to 3 of Theorem 1. However, the typical situation will be where u is not risk neutral at the extremes, and there is no satiation portfolio.

Example 1 Model 2, normal distributions. Suppose $X = \mathbb{R}^n$ and R is normally distributed with mean $\bar{R}$ and a positive definite covariance matrix Ω. Then all non-zero directions of recession of X have positive probability of negative return.

Set $m(\alpha) = E[(\alpha + Y)1_{(\alpha+Y)>0}]$, where Y is some standard normal variate. Then $m(\alpha)$ is the non-negative incomplete mean of a normal distribution with mean α and unit variance.

If u is asymptotically risk neutral (but not risk neutral), let $\bar{\alpha}$ denote the unique solution to[1]

$$\frac{S^+}{S^-} = 1 - \frac{\alpha}{m(\alpha)}.$$

Nielsen (1987) observes that $\bar{\alpha}$ is the limiting slope of the indifference curves of the preference relation for standard deviation and mean of return, and that there exists a global satiation portfolio if and only if

$$(\bar{R}'\Omega\bar{R})^{1/2} < \bar{\alpha}.$$

See also Nielsen (1993). □

5 Optimal Portfolios and Arbitrage-Free Prices

Proposition 4 *Let $p \neq 0$ be a price system and w a wealth level. Assume that there is some x in X such that $p'x \leq w$. The set of optimal portfolios is non-empty and bounded if and only if $p'e > 0$ for all directions of improvement $e \neq 0$, and if and only if p belongs to the interior of I^+.*

PROOF: The set

$$\{y \in X : V(y) \geq V(x) \text{ and } p'y \leq w\}$$

is non-empty, closed and convex. It is bounded if and only if it has no non-zero direction of recession, and if and only if the set of optimal portfolios is non-empty and bounded. A portfolio $e \neq 0$ is a direction of recession of this set if and only if e is a direction of improvement and $p'e \leq 0$. So, the set is bounded if and only if $p'e > 0$ for all $e \neq 0$ in I. Since I is a closed cone, the latter is the case if and only if p belongs to the interior of the negative polar cone I^+. □

Proposition 4 states a necessary and sufficient condition for the set of optimal portfolios to be non-empty and bounded. It is not a necessary condition for optimal portfolios to exist, because there may be an unbounded

[1] The equation in statement 3 of the proposition in Nielsen (1987) should also read like this (but contains a misprint).

set of optimal portfolios. However, if all non-zero directions of improvement of V are directions of global strict improvement, and if there exists a portfolio x with $p'x \leq w$ and $V(x) > -\infty$, then the set of optimal portfolios is bounded if it is non-empty, and the condition in Proposition 4 is indeed a necessary condition for existence of an optimal portfolio. Recall that Proposition 1 lists sufficient conditions ensuring that all non-zero directions of improvement of V are directions of global strict improvement.

Proposition 5 *Suppose every non-zero direction of improvement is a direction of global strict improvement (this will be the case in Model 1). Suppose $p \neq 0$. Suppose there is some x in X with $p'x \leq w$ and $V(x) > -\infty$. The set of optimal portfolios is bounded if it is non-empty.*

PROOF: The set of optimal portfolios is closed and convex. It cannot have a non-zero direction of recession, since every non-zero direction of recession of X is a direction of global strict improvement. Hence, the set of optimal portfolios is bounded if it is non-empty. □

In Model 1, there exists an optimal portfolio if and only if the price system is positive (in all coordinates).

The rest of this section investigates the relation between absence arbitrage and existence of optimal portfolios.

A price system $p \neq 0$ will be called *arbitrage-free* for the investor if $p'e > 0$ for every direction of recession $e \neq 0$ of X such that $e'R \geq 0$ for sure according to π. The interpretation is this: there is no portfolio which the investor can add to his initial portfolio at a non-positive cost, increasing his payoff with positive probability and decreasing it with zero probability, staying in the choice set X no matter what the initial portfolio is.

Proposition 6 *A price system $p \neq 0$ is arbitrage-free for the investor if and only if it belongs to the interior of*

$$(D \cap \operatorname{supp}(\pi)^+)^+.$$

PROOF: The price system is arbitrage-free if and only if $p'e > 0$ for all $e \neq 0$ in $D \cap \operatorname{supp}(\pi)^+$. The latter is a closed cone, and $p'e > 0$ for all $e \neq 0$ in a closed cone if and only if p belongs to the interior of the negative polar cone. □

If all directions of recession of X have non-negative return for sure, then

$$D \cap \text{supp}(\pi)^+ = D = I.$$

This will be the case, for example, in Model 1. In this case, a price system $p \neq 0$ is arbitrage-free if and only if it belongs to the interior of the cone $D^+ = I^+$, and if and only if there exists an optimal portfolio (whenever there exists x in X such that $p'x \leq w$). In Model 1, a price system is arbitrage-free if and only if it is positive (in all coordinates).

Example 2 If no non-zero portfolio has non-negative return for sure, then $\text{supp}(\pi)^+ = \{0\}$ and

$$(D \cap \text{supp}(\pi)^+)^+ = \mathbb{R}^n,$$

so that every price system $p \neq 0$ is arbitrage-free. In particular, this will be the case if R follows a normal distribution. □

Example 3 Riskless asset. Suppose the first asset is riskless with total return $R_f \neq 0$ per share, and suppose the portfolio $(1, 0)$ is a direction of recession of X. Let ν be the distribution of returns to the remaining $n-1$ assets. If $\text{conv}(\text{supp}(\nu)) = \mathbb{R}^{n-1}$, then

$$\text{supp}(\pi)^+ = \{(t, 0) \in \mathbb{R} \times \mathbb{R}^{n-1} : t \geq 0\} = D \cap \text{supp}(\pi)^+,$$

so that every price system $(1, p)$ is arbitrage-free. □

Proposition 7 *Let $p \neq 0$ be a price system and w a wealth level. If there exists an optimal portfolio x given p and w, with $V(x) > -\infty$, then p is arbitrage-free.*

PROOF: Let $e \neq 0$ be a direction of recession such that $e'R \geq 0$ for sure. Then e is a direction of global strict improvement of V. If $p'e \leq 0$, then $p'(x+e) \leq w$ but $V(x+e) > V(x)$, contradicting the optimality of x. □

In the final proposition of this section, we consider only Model 2.

Proposition 8 *Assume that u is not asymptotically risk neutral. Let $p \neq 0$ be a price system and w a wealth level. An optimal portfolio exists given p and w if and only if p is arbitrage-free, and if and only if p belongs to the interior of I^+.*

PROOF: It follows from Proposition 6 and Corollary 2 that p is arbitrage-free if and only if it belongs to the interior of I^+. Also by Corollary 2, any non-zero direction of improvement of V is a direction of strict global improvement. By Proposition 5, the set of optimal portfolios is bounded if it is non-empty. Hence, by Proposition 4, an optimal portfolio exists if and only if p belongs to the interior of I^+. □

So, in Model 2, if all directions of recession of X have non-negative return for sure, or if u is not asymptotically risk neutral, then there exists an optimal portfolio if and only if the price system is arbitrage-free. If there is a direction of recession of X which has positive probability of negative return, and if u is asymptotically risk neutral, then it is possible that there is no optimal portfolio even if the price system is arbitrage-free. This depends on the limiting behavior of u and on the ratio between the positive and negative incomplete means of directions of recession of X.

Example 4 (Continuation of Example 1). Model 2, $X = \mathbb{R}^n$, R is normally distributed with mean $\bar{R}$ and a positive definite covariance matrix Ω, u is asymptotically risk neutral. As noted in Example 2, every price system $p \neq 0$ is arbitrage-free. But whether an optimal portfolio exists depends on the utility function.

Set $A = \bar{R}'\Omega^{-1}p$, $B = \bar{R}'\Omega^{-1}\bar{R}$, $C = p'\Omega^{-1}p$, and $D = BC - A^2$. Assume that p and $\bar{R}$ are linearly independent. The efficient portfolios with cost w trace out the upper branch of a hyperbola in (σ, μ)-space:

$$\mu = \bar{\mu} + \rho(\sigma^2 - \bar{\sigma}^2)^{1/2}, \quad \sigma \geq \bar{\sigma},$$

where $\mu = Aw/C$, $\bar{\sigma} = |w|C^{-1/2}$, and $\rho = (D/C)^{1/2}$. Note that ρ is the slope of the asymptote to the hyperbola. Nielsen (1987) observes that there exists an optimal portfolio if and only if $\rho < \bar{\alpha}$. As in Example 2, $\bar{\alpha}$ is the limiting slope of the indifference curves of the preference relation for standard deviation and mean of return, and it is computed on the basis of the limiting slopes of the graph of the von Neumann-Morgenstern utility function u. □

6 Equilibrium

After introducing the necessary concepts and notation, this section states two conditions which are jointly sufficient for the existence of a general equilibrum (or quasi-equilibrium). The following sections analyze what these conditions mean specifically in the context of expected utility.

There are m investors, $i = 1, \ldots, m$. Each investor i has a choice set X^i, a von Neumann–Morgenstern utility function u_i and considers the n–vector R of total returns to the assets to be distributed in $\mathbb{R}^n$ according to a probability distribution π_i. If the first asset is riskless (in the judgment of investor i), then its total return per share is denoted R^i_f, and the (marginal) distribution of the other assets is denoted ν_i. The choice set X^i, the von Neumann–Morgenstern utility function u_i and the probability distribution π_i satisfy the assumptions of Model 1 or Model 2 and Assumptions 1 through 4. The extended utility function $\bar{V}_i$ for portfolios is defined on $\bar{X}^i$ by

$$\bar{V}_i(x) = E_i u_i(x'R),$$

where the expectations symbol "E_i" refers to the mathematical expectation with respect to the probability distribution π_i. Investor i's utility function V_i for portfolios is the restriction of $\bar{V}_i$ to X^i. The set of directions of recession of X^i is denoted D^i, and the set of directions of improvement of V_i is denoted I_i.

In Model 2, the incomplete means of the total return $e'R$ to a portfolio e according to investor i's probability distribution π_i are denoted $E_i^+(e)$ and $E_i^-(e)$, and the asymptotic slopes of u_i are denoted S_i^+ and S_i^-.

An *allocation* is an m-tuple $(x^i) = (x^1 \ldots x^m)$ consisting of a portfolio x^i in X^i for each investor. Each investor i has an *initial portfolio* ω^i, which is assumed to belong to X^i.

The initial portfolios constitute the *initial allocation* (ω^i). An allocation (x^i) is *attainable* (without free disposal) if $\sum_i x^i = \sum_i \omega^i$. Obviously, the initial allocation is attainable.

A *general equilibrium* is a is a price vector $p \neq 0$ and an attainable allocation (x^i) such that for each i, $p'x^i \leq p'\omega^i$ and if y^i belongs to X^i and $V_i(y^i) > V_i(x^i)$ then $p'y^i > p'\omega^i$.

A *general quasi-equilibrium* is a price vector $p \neq 0$ and an attainable allocation (x^i) such that for each i, $p'x^i \leq p'\omega^i$ and if y^i belongs to X^i and $V_i(y^i) > V_i(x^i)$ then $p'y^i \geq p'\omega^i$.

A general equilibrium is always a general quasi-equilibrium. If $(p,(x^i))$ is a general quasi-equilibrium, and if $p'x^i > \inf p'X^i$ for all i, then $(p,(x^i))$ is a general equilibrium. Note that if $p \neq 0$ and x^i is in the interior of X^i, then $p'x^i > \inf p'X^i$.

An allocation (x^i) is a *Pareto optimum* (or a *Pareto optimal allocation*) if it is attainable and there is no attainable allocation (y^i) such that $V_i(y^i) \geq V_i(x^i)$ for all i and $V_i(y^i) > V_i(x^i)$ for some i.

Both the possibility of satiation and the possible unboundedness of choice sets may lead to non-existence of general equilibrium. In an economy with abstract preferences, Nielsen (1989) identified two conditions which are jointly sufficient for the existence of equilibrium. We shall restate these conditions and analyze them in the specific context of expected utility.

Condition 1 *Positive semi-independence of directions of improvement: If for each i, e^i is a direction of improvement for investor i, and if $\sum_i e^i = 0$, then $e^i = 0$ for all i.*

Condition 1 says that there is no feasible system of net trades which can be repeated indefinitely without eventually making somebody worse off. This condition always holds in Model 1. In Model 2, it can be restated as follows: If for each i, e^i is a direction of recession of X^i such that $S_i^+E_i^+(e^i) + S_i^-E_i^-(e^i) = 0$, and if $\sum_i e^i = 0$, then $e^i = 0$ for all i.

Recall that an allocation (x^i) is attainable if $\sum_i x^i = \sum_i \omega^i$. A portfolio x is *attainable* for investor i if it is part of some attainable allocation, *i.e.*, if there is an attainable allocation $(x^1, \ldots, x^i, \ldots, x^m)$ such that $x = x^i$. An allocation (x^i) is *individually rational* if it is attainable and Pareto dominates the initial allocation, which means that $V_i(x^i) \geq V_i(\omega^i)$ for all i. Let A denote the set of individually rational allocations, *i.e.*,

$$A = \{(x^i) : \sum_i x^i = \sum_i \omega^i,\ x^i \in X^i,\ V_i(x^i) \geq V_i(\omega^i) \text{ for all } i\}.$$

Say a portfolio x is *Pareto attainable* for investor i if it is part of some allocation in A. Let A^i be the set of portfolios that are Pareto attainable for i, *i.e.*,

$$A^i = \{x : x = x^i \text{ for some } (x^1, \ldots, x^i, \ldots, x^m) \text{ in A}\}.$$

Condition 1 implies that the set of individually rational allocations A has no non-zero direction of recession, and therefore it is compact. It follows that the set of Pareto attainable portfolios A^i for each investor is compact.

Condition 2 *Non-satiation at Pareto attainable portfolios: If x is a portfolio in A^i then there exists a portfolio y in X^i such that $V_i(y) > V_i(x)$.*

Theorem 3 *Existence of equilibrium. Under Conditions 1 and 2 there exists a general quasi-equilibrium which Pareto dominates the initial allocation.*

Theorem 3 (existence of equilibrium) can now be translated to the present setting as follows: In Model 1, assume that no X^i is bounded. In Model 2, assume Conditions 1 (as restated above) and 2. In either model, there exists a general quasi-equilibrium which Pareto dominates the initial allocation.

7 Positive Semi-Independence

In order to understand better why Condition 1 is needed in Theorem 3, it is useful to relate the condition to the existence of viable price systems and the existence of Pareto optima.

A system of prices is said to be viable if there exist optimal portfolios for all investors when they choose subject to wealth constraints given by those prices. Formally, $p \neq 0$ is a *viable* price system if for each investor i there exists a portfolio x^i in X^i such that $p'x^i \leq p'\omega^i$ and if y^i belongs to X^i and $V_i(y^i) > V_i(x^i)$ then $p'y^i > p'x^i$.

If a price system $p \neq 0$ is viable, then it is arbitrage-free for all investors. This follows from Proposition 7.

There seems to be no condition in terms of directions of improvement which is generally necessary and sufficient for a price system to be viable. For this reason, we shall use a slightly stronger concept. A price system will be called *boundedly viable* if each investor's set of optimal portfolios is non-empty and bounded. Formally, $p \neq 0$ is a *boundedly viable* price system if for each investor i the set of portfolios x^i in X^i such that $p'x^i \leq p'\omega^i$ and if y^i belongs to X^i and $V_i(y^i) > V_i(x^i)$ then $p'y^i > p'x^i$, is non-empty and bounded.

Proposition 9 *A price system $p \neq 0$ is boundedly viable if and only if it belongs to the interior of $\bigcap_i I_i^+$.*

PROOF: The price system p is boundedly viable if and only if none of the sets

$$\{y^i : y^i \in X^i \text{ and } V_i(y^i) \geq V_i(\omega^i) \text{ and } p'y^i \leq p'\omega^i\}$$

has a non-zero direction of recession. This is so if and only if $p'e^i > 0$ for all $e^i \neq 0$ in I_i, all i, or equivalently, $p'e > 0$ for all $e \neq 0$ in $\sum_i I_i$. The latter is true if and only if p belongs to the interior of $\bigcap_i I_i^+$, since $\sum_i I_i$ is a closed cone with negative polar cone $\bigcap_i I_i^+$. □

An allocation (x^i) is a *bounded Pareto optimum* if it is a Pareto optimum and the set

$$\{(y^i) : \sum_i y^i = \sum_i \omega^i \text{ and } y^i \in X^i \text{ and } V_i(y^i) \geq V_i(x^i) \text{ for all } i\ \}$$

is bounded. This is the set of attainable allocations (y^i) that are equivalent to (x^i) in the sense that each investor i is indifferent between x^i and y^i.

If no indifference curve of any investor contains a half-line, then a price system $p \neq 0$ is boundedly viable if and only if it is viable (because every investor's set of optima is compact if it is non-empty), and an allocation (x^i) is a bounded Pareto optimum if and only if it is a Pareto optimum (because the set of attainable allocations (y^i) that are equivalent to (x^i) has no non-zero directions of recession).

If $(p, (x^i))$ is a general equilibrium, then p is a viable price system and (x^i) is a Pareto optimum. So, the existence of a viable price system and a Pareto optimum is a necessary condition for the existence of a general equilibrium. Theorem 3 relies on the slightly stronger Condition 1, which, by Proposition 10, implies the existence of a boundedly viable price system and the existence of a bounded Pareto optimum.

Proposition 10 *Statements 1–5 are equivalent.*

1. *Condition 1 holds.*
2. *$\bigcap_i I_i^+$ has non-empty interior.*
3. *The set of individually rational allocations is bounded.*

4. A boundedly viable price system exists.

5. There exists a bounded Pareto optimum.

PROOF: The set $\sum_i I_i$ is a closed convex cone. Its negative negative polar cone is $-\bigcap_i I_i^+$. Condition 1 holds if and only if the cone $\sum_i I_i$ contains no line. It follows from Rockafellar (1970, Theorem 14.6) that the cone $\sum_i I_i$ contains no line if and only if the negative polar cone, $-\bigcap_i I_i^+$, has dimension n, which is the case if and only if $\bigcap_i I_i^+$ has non-empty interior. This shows the equivalence of Statements 1 and 2. Condition 1 is equivalent to 3, because each holds if and only if the set of individually rational allocations A has no non-zero direction of recession. The equivalence of Statements 2 and 4 follows from Proposition 9. If the set A of individually rational allocations is bounded, then there exists a bounded Pareto optimum which Pareto dominates the initial allocation, because the continuous function $\sum_i V_i(x^i)$ has a maximum on A. To show that the existence of a bounded Pareto optimum implies Condition 1, let (x^i) be a bounded Pareto optimum and suppose e^i is in I_i for each i and $\sum_i e^i = 0$. Then $V_i(x^i + te^i) = V_i(x^i)$ for all i and all $t \geq 0$, which implies that $e^i = 0$ for all i because (x^i) is a bounded Pareto optimum. □

The equivalence of 1 and 4 in Proposition 10 was shown in Nielsen (1989) in an abstract setting.

If every direction of improvement is a direction of global strict improvement, for example if no u_i is risk neutral at the extremes, then 4 of Proposition 10 is equivalent to the existence of a viable price system, and 5 in Proposition 10 is equivalent to the existence of a Pareto optimum. In this case, Condition 1 is obviously a necessary condition for existence of equilibrium. Hart (1974) noted that the condition is necessary when no u_i is risk neutral at the extremes.

Say that *unlimited Pareto arbitrage* is possible if there exists a family (e^i) consisting of a direction of recession e^i of X^i for each i, not all zero, such that $e^i R \geq 0$ for sure according to π_i and $\sum_i e^i = 0$.

If Condition 1 holds, then unlimited Pareto arbitrage is not possible. In particular, unlimited Pareto arbitrage is impossible in Model 1.

If all investors' probability distributions π_i are identical, then unlimited Pareto arbitrage is not possible. If, in addition, no u_i is asymptotically risk neutral, then it follows from Proposition 11 below that Condition 1 holds.

In the rest of this section, we consider only Model 2.

Proposition 11 *Assume that no u_i is asymptotically risk neutral.*

1. *A price system $p \neq 0$ is arbitrage-free for all investors if and only if it is viable, and if and only if it belongs to the interior of $\bigcap_i I_i^+$.*
2. *Condition 1 holds if and only if unlimited Pareto arbitrage is impossible, and if and only if there exists a price system $p \neq 0$ which is arbitrage-free for all investors.*

PROOF: By Corollary 2, any non-zero direction of improvement of V_i is a direction of global strict improvement. Consequently, a price system is viable if and only if it is boundedly viable. Statement 1 follows from Propositions 8 and 10. It follows from Statement 1 and Proposition 10 that Condition 1 holds if and only if there exists a price system $p \neq 0$ which is arbitrage-free for all investors. Since all directions of improvement have non-negative return for sure, the condition holds if and only if unlimited Pareto arbitrage is impossible. □

Statement 2 of Proposition 11 was shown by Hart (1974) (under the assumption of a bounded and non-negative returns distribution).

Example 5 Suppose that each investor i judges that the first asset is riskless with total return $R_f^i > 0$ per share and that the distribution ν_i of total returns to the remaining assets is such that $\operatorname{conv}(\operatorname{supp}(\nu_i)) = \mathbb{R}^{n-1}$. The latter will be the case, for example, if the total returns to the last $n-1$ assets follow a regular joint normal distribution. According to Example 3, every price system $(1, p)$ is arbitrage-free. Assume that no u_i is asymptotically risk neutral. Then Condition 1 holds, by Proposition 11. If the portfolio $(1,0)$ is a direction of recession of X^i for each i, then there is no satiation portfolio, so that Condition 2 holds. Consequently, there exists a general quasi-equilibrium. If $X^i = \mathbb{R}^n$ for all i, then the general quasi-equilibrium is a general equilibrium. A special case is the classical mean-variance CAPM with normal distributions and with a riskless asset. A general equilibrium exists even if there is disagreement about the parameters (means, variances and covariances) of the returns distributions, provided that the investors are not asymptotically risk neutral. This result is the expected-utility version of Nielsen (1990b, Proposition 2, (2)). □

Note that satiation is not an issue in a model with a riskless asset. By contrast, in CAPM without a riskless asset, satiation can lead to non-existence of equilibrium.

Example 6 Suppose that for each investor i, no non-zero portfolio has non-negative return for sure. This will be the case, for example, if the total returns follow a regular joint normal distribution. According to Example 2, every price system $p \neq 0$ is arbitrage-free. If no u_i is asymptotically risk neutral, then Condition 1 holds, by Proposition 11. □

Proposition 12 *If no u_i is risk-neutral and all investors agree on the vector $E_i R = \bar{R}$ of expected returns, then Condition 1 holds.*

PROOF: It follows from Corollary 1 that all non-zero directions of improvement have positive expected return. If $\bar{R} = 0$ then any $p \neq 0$ is boundedly viable. If $\bar{R} \neq 0$, then $p = \bar{R}$ is boundedly viable. Since a boundedly viable price system exists, it follows from Proposition 10, that Condition 1 holds. □

Proposition 12 was shown by Hart (1974) (under the assumption of a bounded and non-negative returns distribution).

A consequence of Proposition 12 is that equilibrium exists in the classical CAPM with normal distributions and with a riskless asset, even if some investors are asymptotically risk neutral, provided that no investor is risk neutral and all investors agree on the expected returns. This result is the expected-utility version of Nielsen (1990b, Proposition 2, (1)).

8 Non-Satiation

If u_i is not asymptotically risk neutral, and if no non-zero portfolio has non-negative return for sure, then satiation somewhere in the choice set is bound to occur (by Proposition 3). Hence, ruling out satiation everywhere in the choice set is too strong an assumption in this model. Fortunately, Condition 2 (non-satiation at Pareto attainable portfolios) only requires non-satiation at Pareto attainable portfolios.

Obviously Condition 2 holds if there is a direction of global strict improvement. In that case, there is non-satiation not only at Pareto attainable portfolios but at all portfolios in the choice set. Some situations where a direction of global strict improvement exists are described in Proposition 1. Let

$$\omega = \sum_i \omega^i$$

be the market portfolio.

Proposition 13 *Assume that all investors agree on the vector $\bar{R}$ of expected returns. Assume that for all i, $V_i(\omega^i) \geq u_i(0)$, and if s^i is a satiation portfolio for i, then $s^i\bar{R} > \omega'\bar{R}$. Then Condition 2 holds.*

PROOF: Suppose (x^i) is a Pareto attainable allocation. Then $V_i(x^i) \geq V_i(\omega^i) \geq u_i(0)$ for all i, so $x^i\bar{R} \geq 0$. Since $\sum_i x^i\bar{R} = \omega'\bar{R}$, it follows that $x^i\bar{R} \leq \omega'\bar{R}$ for all i. So, a satiation portfolio for i cannot be Pareto attainable. □

The first requirement in Proposition 13 is that everybody is at least as happy with his initial endowment as with no endowment at all. The assumption about s^i says that a satiation point has to contain so many shares of the various assets that its expected total return is larger that the expected total return on the market portfolio.

Here is an example of an existence result that combines some of the results above:

Proposition 14 *Suppose no u_i is risk-neutral and there is agreement on the vector $\bar{R}$ of expected returns. Suppose that for all i, ω^i belongs to the interior of X^i, $V_i(\omega^i) \geq u_i(0)$, and if s^i is a satiation portfolio for i, then $s^i\bar{R} > \omega'\bar{R}$. Then there is a general equilibrium.*

PROOF: Condition 1 holds by Proposition 12, and Condition 2 holds by Proposition 13. So, there exists a general quasi-equilibrium. It is a general equilibrium because ω^i belongs to the interior of X^i for all i. □

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