WHY INCLUDE WARRANTS IN NEW EQUITY ISSUES? A THEORY OF UNIT IPOS

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Abstract

We develop a theory of unit IPOs, in which the firm going public issues a package of equity with warrants. We model an equity market characterized by asymmetric information, where insiders have private information about the riskiness as well as the expected value of their firm's future cash flows. We demonstrate that, in equilibrium, high risk firms issue "units" of equity and warrants, and the package of equity and warrants is underpriced; lower risk firms, on the other hand, issue underpriced equity alone. An important feature of our model is that, in contrast to the existing literature, underpricing is used as a signal in equilibrium in the context of a *one-shot* equity offering. While the model is developed in the context of IPOs of equity, it is also applicable with minor modifications to the case of seasoned equity offerings packaged with warrants; further, the intuition behind the model generalizes readily to provide a new rationale for packaging call option like claims with other risky securities (e.g., convertible debt, debt with warrants) as well.

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1. Introduction

The use of packaged offerings in security issues has been the subject of considerable debate in corporate finance. One such instance of packaged security issues is the use of warrants in the initial public offerings of certain companies, popularly referred to as "unit offerings." In a unit offering, the firm going public issues a package of shares of equity and warrants (commonly referred to as a "unit") rather than equity alone.[1] Often, the companies issuing units are high-risk firms, and marketed by less reputable underwriters; further, a significantly large proportion of these firms tend to be concentrated in certain highly speculative industries.

Unit offerings confront financial economists with a puzzle: In a full information setting, issuing such securities with contingent claims attached to them provides no special advantage to the firm. The justification given by practitioners for issuing units is that they act as "sweeteners" to otherwise unattractive, high risk equity issues.[2] Of course, such casual explanations do not shed much light on how attaching warrants to equity creates value, and why firm insiders might prefer to make use of a unit IPO rather than simply setting a lower share price for the equity sold. Further, any theory of unit equity

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[1]For example, consider the case of Microprobe Corporation, a biomedical company which went public on September 29, 1993. The firm issued 2,500,000 units, each consisting of one share of common stock and one warrant, priced at $6.60 per unit (with the warrants having exercise price of $6.60 and a time to expiration of five years). The number of warrants and shares of equity in a unit vary considerably across IPOs: in his empirical study of unit IPOs, Schultz (1993) documents that in his sample, the ratio of shares to warrants varied from 1/3 to 4, with the median offering containing twice as many shares as warrants. He also documents that the median time to expiration of the warrants attached to equity as part of a unit IPO was 4.75 years; the median ratio of warrant exercise price to the offering price of each unit was 1.25. While investors can buy equity in these IPOs only as part of the unit prior to the IPO, the stock and warrants typically start trading separately soon after the IPO.

[2]For instance, a recent article in Business Week (May 13, 1991) advises investors:"...And keep away from 'unit offerings'--new stock packaged with warrants to buy more shares. Quality deals don't need such warrants as sweeteners." Such advice deepens the unit offering puzzle, since, far from "sweetening" the pot, issuing warrants along with the equity seems to convey to investors the information that these are indeed high risk equity offerings. However, the theory we develop in this paper will demonstrate that, despite revealing true firm riskiness, it is optimal for high risk firms to make use of unit initial public offerings.

offerings should also explain why only relatively high risk firms choose such offerings, with less risky firms issuing equity alone in their IPO.

In this paper, we develop a theory of unit offerings in a simple asymmetric information set-up, which sheds some light on the above issues.[3] We develop a two-type model in which type G firms (defined as those with a higher expected value of future cash flow) may have a greater, the same, or a lower variance of future cash flow (riskiness) compared to type B firms (which have a lower expected value of future cash flow). Firm insiders, who are risk-averse, have private information about the type of their own firm (they know both the mean and the riskiness of future cash flows), while outsiders cannot distinguish between the two types. In this setting, the firm approaches the equity market to raise capital in an initial public offering. When making the stock issue, insiders optimally choose the fraction of the equity to be sold, the share price, the number of warrants (if any) to be issued, and the exercise price of these warrants.

In order to understand the broad intuition leading to our results, it is useful to compare it to the seminal model of Leland and Pyle (1977). Leland and Pyle also model a one shot equity offering, where risk-averse firm insiders have private information about the type (mean of the firm's future cash flow distribution) of their firm. In this setting, insiders of the better type firm(s) signal true firm value to the equity market by retaining a larger fraction of their firm's equity than the poorer type firm. Such a signaling equilibrium exists, since the cost of retaining a larger fraction of the firm's equity (than would be optimal from purely risk-sharing considerations) is lower for the better type firm.

In contrast to the Leland and Pyle (1977) model, here we allow the two types of firms to differ

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[3]By unit offerings we mean only those IPOs where equity and warrants are sold to *all* investors as a package, and investors are allowed to buy equity in the IPO only as part of such a package. Such unit IPOs should not be confused with the issue of a limited number of warrants to the underwriters of many IPOs as part of their compensation ("underwriter warrants"). Underwriter warrants, which are used as part of the underwriters' compensation in IPOs with equity alone as well as in unit IPOs (perhaps in order to ensure that underwriters' incentives are aligned with those of the issuing firm), will not be our focus here. (See the empirical study by Barry, Muscarella and Vetsuypens (1991) for a discussion of underwriter warrants.)

in their riskiness as well as in the mean of their future cash flows. It is this difference in riskiness which allows for underpricing as well the issue of warrants to act as signals in our model. To see the intuition behind how underpricing can act as a signal in our setting, consider the case where the type G firm is significantly riskier than the type B firm. Now, starting from the single-signal equilibrium point (where the type G firm is constrained to use only the fraction of equity retained as a signal), allow the firm to underprice its equity (i.e., set the price below the full-information value) as well. It now becomes optimal (in terms of maximizing insiders' expected utility or equivalently, minimizing signaling costs) for the type G firm insiders to cut back on the fraction of equity they retain, simultaneously underpricing their firm's equity to deter mimicking by the type B firm. Since the marginal benefit to cutting back on equity retained while simultaneously underpricing is greater for the insiders of the (riskier) type G firm compared to that for type B firm insiders, the equilibrium thus involves the use of both underpricing and the fraction of equity retained as signals.

Now allow the type G firm to use a third signal, namely the issue of warrants along with equity in the IPO. To see the intuition behind how warrants may be used as part of the equilibrium signaling mix by the type G firm, it is useful to compare the cost of issuing a warrant for a type G firm and for a type B firm (assuming as before that the type G firm is riskier than the type B firm, and assuming, for concreteness, that the exercise price of each warrant is set at the mean of the type G firm's cash flow distribution). On the one hand, each warrant issued by a type G firm is more valuable (in terms of the true "dollar value" of the expected cash flow yielded to the holders of the warrant) than a warrant issued by a type B firm. On the other hand, given that firm insiders are risk averse (and noting that type G firm has a greater mean and riskiness than type B firms), the marginal utility (i.e., insiders' "private valuation") of each dollar of cash flow that they have to yield to outsiders (as part of the payoff when the warrants are in the money) will be lower for insiders of type G firms compared to those of type B firms. Warrants will be part of the equilibrium signaling mix (and the type G firm will therefore make

use of a unit IPO) only if the latter *private value effect* dominates the former *dollar value effect*, since in this case issuing warrants will be less costly for type G firm insiders than for type B insiders.

It is useful to consider how issuing warrants differs as a signaling device from underpricing equity. Both signaling devices impose dissipative costs (unavoidable in an asymmetric information setting) on the firm. However, warrants provide a way to incur some of these dissipative costs only selectively, in the higher realizations of the firm's future value. Since insiders are risk averse, this ability to dissipate value selectively in only the higher states becomes important, since it is precisely in these states that insiders' valuation of the firm's cash flows is the lowest (the actual states in which value is yielded to outsiders can be specified by suitably choosing the exercise price of the warrants issued). Alternatives to selling warrants, such as selling underpriced equity, do not have this ability to yield value only in specified states of the world; embedding call-option like claims in securities offerings thus helps to minimize the dissipative costs associated with asymmetric information, leaving firm insiders better off (relative to the case where only underpriced equity is issued) in situations where the firm's cash flow stream is highly risky.

We thus show that packages of underpriced equity and warrants are issued in equilibrium by firms making IPOs when type G firms have greater cash flow variability compared to type B firms. Further, such unit offerings are employed only when the cash flow variance of the type G firm exceeds a certain threshold level; if the type G firm's variance is below this threshold level, no warrants are issued (underpriced equity alone is issued in this case). This leads to the testable prediction that, in an ex ante indistinguishable pool of firms, firms which have made unit IPOs should be associated with greater ex post variance compared to those which have made IPOs with equity alone, a prediction borne out by the evidence (see Schultz (1993)). A second testable implication of our model is that for firms which have made unit IPOs, the fraction of firm value sold as warrants (i.e., the ratio of the number of shares that warrant holders will receive upon exercise to the total number of shares in the firm) will be increasing

in firm riskiness. We will discuss these and other testable predictions of our model in section 5.

Unit IPOs have been a relatively unexplored area in both the theoretical and the empirical literature. In the only empirical study on unit offerings that we are aware of, Schultz (1993) documented that a significant proportion of IPOs (167 of the 797 IPOs which took place during 1986-1988) were unit offerings. Motivated by Jensen's (1986) free-cash flow hypothesis, Schultz conjectures that perhaps these are multi-stage equity financing arrangements which minimize agency costs. The argument is that managers/insiders may squander any cash raised in excess of the firm's immediate investment requirements by investing in negative NPV projects, and warrants provide a means by which the firm may raise new financing later on, if required. One problem with such an explanation is that the possibility of squandering value arises in almost all firms, and if this were the sole driving factor, unit offerings should be the financing vehicle of choice for most firms (rather than being confined to high risk firms in speculative industries). This suggests that agency cost based arguments *alone* cannot explain the use of unit IPOs.

Our paper is related to the theoretical literature which explains the well-documented "underpricing" of initial public offerings.[4] A closely related paper is Grinblatt and Hwang (1989), which also uses a framework similar to that of Leland and Pyle (1977), along with the assumption (similar to our paper) of differences in riskiness across firm types; however, unlike our paper, they also assume that the firm makes a second, seasoned offering soon after the IPO (with some probability of the revelation of true firm type between the initial and seasoned offerings). Thus, while Grinblatt and Hwang (1989) also show that underpricing, along with the fraction of firm value retained by firm insiders, can serve

[4]The underpricing of IPOs has been documented by a number of papers (see Ritter (1991) for recent evidence, and Ibbotson, Sindelar and Ritter (1988) for a survey of the empirical literature). Theoretical models of IPO underpricing are presented in Allen and Faulhaber (1989), Benveniste and Spindt (1989), Chemmanur (1993), Grinblatt and Hwang (1989), Hughes and Thakor (1992), Mauer and Senbet (1992), Rock (1986), Spatt and Srivastava (1991), and Welch (1989). Unlike these papers, our primary objective here is to analyze the optimality of unit IPOs (though underpricing arises in equilibrium here as well).

to signal true firm value, ours is the first paper which demonstrates that underpricing may serve as a signal in the context of a *one-shot equity* offering (we thus demonstrate that a second, seasoned equity offering is not a necessary ingredient in order to generate underpricing as a signal in IPOs).[5] Further, neither Leland and Pyle (1977) nor Grinblatt and Hwang (1989) address the question of why firms issue warrants along with equity in IPOs, which is the main issue addressed in this paper.

This is the first paper to analyze the optimality of packaging warrants along with *equity* offerings, thus making an important contribution to the security-design literature as well. Further, while our model is developed in the context of new issues of equity, it is also applicable (with minor modifications) to seasoned issues of equity packaged with warrants; therefore, our paper has implications for such equity issues as well.[6] Finally, our intuition generalizes readily to provide a new rationale for packaging call-option like claims with other risky securities (e.g., debt with warrants, convertible debt) as well. A well-known explanation for issuing debt with warrants or convertible debt is the "hidden-action" argument provided by Green (1983), who argues that such claims are a way of controlling firm insiders' incentives to overinvest in riskier projects which arises from the presence of debt in the firm's capital structure (see also Barnea, Haugen and Senbet (1985)). Of course, such arguments would not provide a rationale for issuing warrants along with *equity*: the risk-shifting incentive driving Green's and other related analyses

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[5]Two other papers which have the extent of underpricing as a signal are Welch (1989) and Allen and Faulhaber (1989). However, in these papers, insiders are risk neutral, and the riskiness of the firm's cash flows is irrelevant. The driving force which generates underpricing as a signal in these models is a second equity offering, with some probability of an exogenous release of additional information about true firm value between the IPO and the seasoned offering. As such, the intuition driving these papers is not directly related to the one in this paper.

[6]Of course, the impact of asymmetric information, though significant, can be expected to be much less severe in the seasoned equity market than in the market for IPOs; therefore, the various phenomena we model here will also to be much less pronounced in the seasoned equity market. However, Jayaraman, Shastri and Tandon (1991) document that a significant number of seasoned firms make offerings of equity packaged with warrants. Also, Smith (1977) (among several others) documents that the offering price of seasoned equity is set below the market value of the equity; he finds a statistically significant abnormal return from the offer price to the closing price of the offer date (though the magnitudes involved are smaller compared to the underpricing of IPOs).

will clearly not be present when there is no debt involved.[7]

The rest of the paper is organized as follows. Section 2 presents the essential features of the model. Section 3 characterizes the nature of the equilibrium that will obtain under different settings. Section 4 develops a detailed analysis of the equilibrium involving unit IPOs. Section 5 concludes, after summarizing the testable implications of our model. The proofs of all propositions are in the appendix.

2. The Model

The model has two dates (time 0 and time 1). At time 0, entrepreneurs, who are risk-averse, take their private firms "public" by selling equity in these firms in an equity market dominated by risk-neutral outsiders.[8] Each firm has a positive NPV project which can be implemented by investing an amount K at this date. For simplicity, we normalize the total number of shares in each firm to 1; entrepreneurs offer a fraction $(1 - \alpha)$ of the firm's equity to the public in the IPO, retaining the remaining fraction α of equity.

Firms are of two types: type G ("good") and type B ("bad") firms. Type G and type B firms differ in both the mean and the variance of the distribution of future cash flows. At time 0, the equity market is characterized by asymmetric information: While outside investors cannot distinguish between the two types of firms, entrepreneurs ("firm insiders") know the types of their own firms. However, at time 0 even entrepreneurs are uncertain about their firm's time 1 value. We assume a simple three-point probability distribution for each firm's time 1 value: For the type G firm, the value at time 1 will be one of $\mu^G + \delta/p_H$ (with a probability p_H), μ^G (with probability p_M), or $\mu^G - \delta/p_L$ (with probability p_L). We will

[7]Other rationales for issuing convertible debt, also based on asymmetric information but unrelated to ours, are provided by Brennan and Kraus (1987) and Constantinides and Grundy (1989), who argue that (under certain restrictive conditions) issuing convertible debt may allow the firm to reveal its type costlessly. See also Stein (1993), who assumes the existence of significant deadweight costs of financial distress to drive a signaling equilibrium involving the use of convertible debt.

[8]While the assumption of risk neutral outsiders serves to simplify our computations significantly, it does not drive any of our results. The intuition behind our results goes through even if outsiders are risk-averse.

often refer to these three possible realizations of time 1 firm value as high, medium and low "states." For type B firms, the time 1 value will be one of $\mu^B + \varepsilon/p_H$, μ^B, or $\mu^B - \varepsilon/p_L$ (with the same state probabilities p_H, p_M and p_L). Clearly, δ is a measure of the variance of the time 1 value for a type G firm (since the variance will be larger for a larger δ); similarly ε is a measure of the variance of the time 1 value of the type B firm. We will therefore refer to δ and ε as the "risk parameters" of the two types of firms from now on. We will assume throughout that the type G firm has a greater mean time 1 value than the type B firm (i.e., $\mu^G > \mu^B$).

Since, at time 0, insiders observe their own firm type, they know both the true mean (μ^G for type G firms or μ^B for type B firms) and the risk parameter (δ for type G or ε for type B) of their firm; however, they do not know which firm value state will occur at time 1. Outsiders, on the other hand, do not know the type of a given firm approaching the equity market; based on publicly available information, they form a prior probability assessment of the firm making the IPO being a type G firm. They may revise these prior beliefs after observing firm insiders' actions (i.e., the package of securities issued in the IPO, the price of these securities, and the fraction of the firm's equity α retained by insiders). The state probabilities p_H, p_M and p_L, as well as other model parameters, are common knowledge.

At time 1, the firm's true value is realized, and is observed by both insiders and outsiders. Thus, all asymmetric information between insiders and outsiders about the firm's future cash flows is resolved at this date.[9]

Insiders may choose to sell either equity alone, or a combination of equity with a certain number

[9]We *do not* assume that all uncertainty about the firm's future cash flows is removed at time 1, but only that this uncertainty is diminished between time 0 and time 1. Thus, the firm's time 1 market value will be the expectation of its cash flows beyond this date conditional on the common information set of firm insiders and outsiders. Notice that the presence of residual uncertainty about the firm's future cash flows at time 1 does not alter the insiders' problem (even though insiders are assumed to be risk-averse), since firm insiders are free to liquidate their equity positions in the firm (at the full information value) at time 1. We can think of the period between time 0 and time 1 as the firm's "seasoning" period; in practice, the impact of private information is greatest before a firm goes public, and is greatly diminished after it has become seasoned.

γ of warrants in the IPO. We assume that any warrants sold expire at time 1, after the resolution of information asymmetry. Investors holding warrants will exercise them if (and only if) these warrants are in the money at time 1, so that the number of shares in the firm will then be $(1 + \gamma)$. Therefore, the fraction $\gamma/(1 + \gamma)$, which we will denote by θ, represents the fraction of the firm value surrendered to warrant holders upon exercise. We denote the exercise price of the warrants by k. To minimize computational complexity, we will restrict the insiders' choice of warrant exercise price to values greater than μ^G.[10] Now, for warrants to play any signaling role in our model, it is important that even the type B firm's warrants are in the money for *some* realization of the firm's time 1 value (otherwise, knowing that their warrants will never be in the money, the type B firm can always issue warrants whenever the type G firm does so, thus enabling it to mimic the type G firm costlessly); we will therefore impose the parametric restriction that $\mu^B + \varepsilon > \mu^G$, thus ensuring that the type G firm can choose an exercise price greater than or equal to μ^G for which the type B firm's warrants are also in the money with a positive probability.[11] Thus, if warrants are issued in equilibrium, type G firm insiders will always choose an exercise price in the range $\mu^G \leq k < \mu^B + \varepsilon$. As a result, any warrants issued will be in the money if the time 1 firm value is high, and out of the money otherwise, regardless of whether the firm issuing the warrants is a type G or a type B firm (though the amount of the payoff will of course be higher for a warrant issued by a type G firm than that issued by a type B firm).

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[10]This restriction on the exercise price of the warrants issued in the IPO is made with the objective of keeping the insiders' optimization problem tractable, since it ensures that the warrants are in the money only in the high firm value state. However, it is straightforward, in principle, to generalize the model to relax this restriction on the range of possible exercise prices. Such a generalization would involve solving the insiders' optimization problem in various steps, picking that value of k which is optimal for each possible range of k values, and then choosing that exercise price from among these constrained optimal k values that one which maximizes the insiders' equilibrium utility. It is, however, obvious that such a generalization will complicate our model further, without generating commensurate additional insights into the trade-offs driving the issuance of warrants along with equity in unit IPOs.

[11]Intuitively, warrants can play an important informational role in stock issues even when this parametric restriction is not satisfied, since the only requirement is that there be some overlap in the possible time 1 values of type G and type B firms. Thus, the parametric restriction we impose here is stronger than required for the intuition behind our model to go through; it is needed only because of the earlier restriction (made simply to minimize computational complexity) on the range of exercise prices that type G insiders may choose (to values above μ^G).

We assume, without loss of generality, that the risk-free rate is zero; outsiders therefore value the package of securities issued by the firm at the expected value (conditional on their equilibrium beliefs) of the stream of future cash flows that accrues to these securities. This is therefore the highest price that any rational investor is willing to pay for the securities issued; a package of securities offered at a higher price will be rejected by the capital market.[12]

2.1 The Insiders' Objective

At time 0, firm insiders choose the fraction $(1 - \alpha)$ of equity to be sold, the number of warrants γ (or equivalently, the fraction of the firm's value to be surrendered to warrants in the event of exercise, θ), the exercise price k of each warrant, and the price of the package of securities issued, V. The objective of insiders in making these choices is to maximize the expected value of their end-of-period (i.e., time 1) utility, which in turn, depends on their time 1 wealth. We will assume that insiders invest their net wealth after the IPO (i.e., their initial wealth before the IPO, w_0, plus the proceeds from the stock issue V, minus the project investment K) in the risk-free asset.[13] Thus, in our model, the insiders' only alternative to investing in the equity of their own firm is to invest in the risk free asset.[14] We assume that firm insiders' utility functions are characterized by constant absolute risk-aversion,

[12]We assume that no firm would choose to set the price of its securities in the IPO at such a level that the offering is rejected. This is consistent with any non-zero frictional cost of issuing securities.

[13]As is standard in this literature (see, e.g., Leland and Pyle (1977) or Grinblatt and Hwang (1989)), the investment in the risk-free asset can be positive or negative. Also, setting K = 0 addresses the case where the purpose of the IPO is only to diversify the insiders' portfolio (and not to raise any capital for the firm).

[14]Our results go through even if we assume (along the lines of Leland and Pyle (1977)) that insiders may choose to allocate their net wealth after the IPO between the market portfolio and the risk-free asset (in addition to holding equity in their own firm). We choose not to add this unnecessary complication to the model, and focus instead on the effect of issuing warrants in the IPO.

$$U(\tilde{w}_1) = -e^{-A\tilde{w}_1}, \tag{1}$$

where A is the co-efficient of absolute risk aversion, and $\tilde{w}_1$ is their (uncertain) time 1 wealth level.[15]

The time 1 wealth level of firm insiders clearly depends on the realization of the firm's time 1 value, since it determines the value of the equity they retain in the firm after the IPO. Let w_H^G, w_M^G, and w_L^G respectively denote the three possible realizations of the type G firm insiders' time 1 wealth level $\tilde{w}_1$ (corresponding to the high, medium and low firm value states). These are given by:

$$w_H^G \equiv w_0 + \alpha\frac{\mu^G+\delta/p_H+\gamma k}{1+\gamma} + V - K = w_0 + \alpha(\mu^G+\delta/p_H) - \alpha\theta(\mu^G+\delta/p_H-k) + V - K, \tag{2}$$

$$w_M^G \equiv w_0 + \alpha\mu^G + V - K, \tag{3}$$

$$w_L^G \equiv w_0 + \alpha(\mu^G-\delta/p_L) + V - K. \tag{4}$$

In each case above, the insiders' time 1 wealth in a particular state is given by the sum of the value of their equity holdings in the firm in that state and the value of their investment in the risk free asset; notice that the expression for w_H^G incorporates the effect of warrant conversion (which occurs in the high state) on the firm's equity value. Similarly, the three possible realizations of the time 1 wealth level of type B firm insiders, denoted by w_H^B, w_M^B, and w_L^B, respectively, are given by:

$$w_H^B \equiv w_0 + \alpha(\mu^B+\varepsilon/p_H) - \alpha\theta(\mu^B+\varepsilon/p_H-k) + V - K, \tag{5}$$

[15]The assumption that insiders have constant absolute risk-aversion is made to simplify the analysis in general, and in particular, to facilitate the development of comparative static results. However, the central results of this paper (propositions 1 through 4) go through for any risk-averse utility function. Leland and Pyle (1977) and Grinblatt and Hwang (1989) also assume that firm insiders have constant absolute risk aversion. These papers combine this assumption with the additional assumption that stock returns are normally distributed, so as to further simplify their analysis; the combination of these two assumptions implies that the analysis of the insiders' problem can proceed in a mean-variance framework. In our model, since we allow the firm to issue warrants along with equity in a setting of *asymmetric information*, assuming normally distributed stock returns complicates the analysis considerably rather than simplifying it. We therefore do not make the latter assumption, and consequently, our analysis does not proceed in a mean-variance framework.

$$w_M^B \equiv w_0 + \alpha\mu^B + V - K, \tag{6}$$

$$w_L^B \equiv w_0 + \alpha(\mu^B - \varepsilon/p_L) + V - K. \tag{7}$$

The objective (i.e., the time 1 expected utility) of insiders of type G and type B firms, denoted by $\bar{U}^G$ and $\bar{U}^B$ respectively, are now given by:

$$\begin{aligned}\bar{U}^G(\alpha,\theta,V,k) \equiv\ & p_H U\left[w_0 + \alpha(\mu^G + \frac{\delta}{p_H})(1-\theta) + \alpha\theta k + V - K\right] \\ & + p_M U\left[w_0 + \alpha\mu^G + V - K\right] + p_L U\left[w_0 + \alpha(\mu^G - \frac{\delta}{p_L}) + V - K\right],\end{aligned} \tag{8}$$

$$\begin{aligned}\bar{U}^B(\alpha,\theta,V,k) \equiv\ & p_H U\left[w_0 + \alpha(\mu^B + \frac{\varepsilon}{p_H})(1-\theta) + \alpha\theta k + V - K\right] \\ & + p_M U\left[w_0 + \alpha\mu^B + V - K\right] + p_L U\left[w_0 + \alpha(\mu^B - \frac{\varepsilon}{p_L}) + V - K\right].\end{aligned} \tag{9}$$

2.2 The full information outcome

Before characterizing the solution to the insiders' problem under asymmetric information, it is useful to note the solution under full information (i.e., in the case where outsiders observe firm type at time 0). Denote by V^G the proceeds of the type G firm's IPO under full information, if insiders sell a fraction $(1 - \alpha)$ of equity (retaining a fraction α), as well as warrants with claim to a fraction θ of firm value upon exercise. V^G is simply the expected value of the stream of cash flows that will accrue to the holders of the package of equity and warrants in the type G firm at time 1, and is given by:

$$V^G(\alpha,\theta,k) \equiv (1-\alpha)\mu^G + \alpha p_H \theta(\mu^G + \delta/p_H - k)\ . \tag{10}$$

The full-information value of any package of securities sold by the type B firm, denoted by V^B, is given by an expression similar to (10), with μ^G replaced by μ^B, and δ replaced by ε. Lemma 1 characterizes

the full information outcome.

Lemma 1. Under full information, $\alpha^* = 0$, $\theta^* = 0$, optimizes the objectives of both types of firms. Further, the type G firm prices its equity at μ^G, and the type B firm prices its equity at μ^B.

In the full-information setting, insiders of the type G firm can receive the true value of any security they sell in the IPO. Therefore, being risk-averse, they choose to divest their equity holdings in the firm completely, since the time 1 value of the firm is uncertain. Warrants are therefore irrelevant in this setting. The intuition behind the behavior of type B firm insiders is similar. From the full-information analysis, it is clear that there is no reason for issuing warrants in the absence of asymmetric information: it is the interaction between asymmetric information and risk-aversion on the part of insiders that provides a rationale for issuing warrants in our model.

3. Equilibrium Under Asymmetric Information

We now proceed to the analysis of the insiders' problem under asymmetric information. Under asymmetric information, insiders of type G firms may find it optimal to distinguish themselves from type B firms, which have a lower intrinsic value; type B firms, on the other hand, have an incentive to mimic the more valuable type G firms, unless it is too costly for them to do so. We will focus here only on separating equilibria, where the type G firm structures the IPO in such a way that any attempt by the type B firm to mimic it imposes such a high cost on the type B firm that it is deterred from doing so, and instead sells its securities in the IPO at their true full information value. Thus, equilibrium strategies and beliefs in our model are defined as those which constitute a separating sequential equilibrium (see Kreps and Wilson (1982)), and which are such that the dissipative costs of separation incurred are the least (in other words, the equilibrium concept used is that of a Pareto dominant or efficient separating sequential

equilibrium).[16] Such an equilibrium emerges as the solution to the following *non-mimicry problem* faced by type G firm insiders:

$$\text{Max}_{\alpha,\theta,V,k} \quad \bar{U}^G(\alpha,\theta,V,k) \tag{11}$$

subject to

$$\bar{U}^B(\alpha,\theta,V,k) \quad \leq \quad U(\mu^B) \tag{12}$$

$$V \leq \quad V^G(\alpha,\theta,k) \tag{13}$$

$$0 \leq \alpha \leq 1, \quad 0 \leq \theta \leq 1, \quad V \geq 0, \quad k \geq \quad \mu^G. \tag{14}$$

The solution to the problem (11) to (14) gives the type G firm's equilibrium choice of the fraction of equity to be retained, α, the fraction of firm value that would accrue to warrants upon exercise, θ, the exercise price of each warrant, k, and the price of the package of securities offered, V. The constraint (12) is the non-mimicry constraint, which ensures that the value of the type B firm's objective if they mimic the type G firm is less than or equal to that obtained by choosing the full information equilibrium values of α, θ and V. Constraint (13) reflects the fact that the maximum price that outsiders will be willing to pay for the package of securities of a firm that has been revealed as a type G firm is V^G, its full information value (given by (10)). We will refer to this constraint as the competitive rationality constraint. Finally, the constraints (14) reflect the range of allowable values of α, θ, V and k. The equilibrium that meets our requirements therefore involves the type B firm picking its full information optimum values of α, θ, V and k (characterized in lemma 1), while the type G firm does just enough

[16]Using such an equilibrium concept ensures that investors' beliefs in response to out-of-equilibrium moves by firm insiders are explicitly specified, and allows us to rule out equilibria with excessive, inefficient amounts of signaling by the type G firm. It also rules out equilibria where the type B firm, even though revealed as type B, deviates from its full-information optimal price for the package of securities sold in the IPO. The reasonableness of such an equilibrium criterion in the context of signaling games has been established by a rather large literature, and will not be discussed here. See Milgrom and Roberts (1986) and Engers (1987) for discussions of this issue.

signaling to distinguish itself from the type B firm. Further, given this equilibrium behavior by issuers, the equilibrium beliefs of outside investors are such that they infer true firm type with probability 1. Finally, if investors observe any firm choosing out-of-equilibrium values for α, V, or θ, they infer with probability 1 that it is a type B firm, and value its securities accordingly (it is easy to verify that such beliefs by outsiders when confronted with out-of-equilibrium actions by issuing firms can support the equilibria that we will describe in propositions 1 to 4 as sequential equilibria).

We now proceed to characterize the nature of the equilibrium under various alternative settings. For ease of exposition, we will assume in sections 3.1 and 3.2 that the firm is exogenously constrained not to issue warrants, and will discuss how the firm chooses between only the two signals, α (the fraction retained by insiders), and V (the price of the equity sold). (We will re-introduce the possibility of the firm issuing warrants as well in section 3.3, and then go on to characterize the firm's efficient choice from the three signals available to it.)

3.1 The No-Underpricing Equilibrium

In this section, we will study the case where the firm is allowed to sell only equity, and the variability of future cash flows of the type G firm is the same as that for the type B firm ($\delta = \varepsilon$).

Proposition 1 (Equilibrium with one signal). *If $\delta = \varepsilon$, the separating equilibrium involves the type G firm retaining a positive fraction of the firm's equity ($\alpha^* > 0$), and will not involve any equity underpricing (i.e., $V^* = V^G$).*[17]

The equilibrium that emerges here is thus akin to that characterized by Leland and Pyle (1977), where the fraction of equity retained is the only signal (however, the above proposition demonstrates that

[17]For convenience of comparison with Leland and Pyle (1977), we have presented this proposition in a setting where equity is the only security issued by the firm. However, it can shown that even when the firm is allowed to issue a combination of equity and warrants, there will be no underpricing of this package of securities if $\delta = \varepsilon$. Further, this more general result holds also for the case where $\delta < \varepsilon$, provided that $\varepsilon \geq \delta \geq \varepsilon - p_H(\mu^G - \mu^B)$. Interested readers are referred to Chemmanur and Fulghieri (1995) for a proof.

if the riskiness of the two firm types is the same, the firm will not underprice its equity even if allowed to do so. in contrast to Leland and Pyle (1977), where underpricing is exogenously ruled out).

Figure 1 depicts the equilibrium choice of the firm between the two signals, α and V. In this figure. the straight line $\mu^G 1$ represents the full-information value V^G of the equity issued by the type G firm as a function of the fraction of equity retained by insiders, α. This line is downward sloping since, as the firm sells a larger fraction of the equity to outsiders (i.e., as α falls), the proceeds from the sale of equity increases. The indifference curves of type G as well as the type B firm insiders are also downward sloping, since, as insiders sell a larger fraction of the firm's equity to outsiders, the total amount received from the sale, V, also has to be larger in order to maintain a given expected utility level. For α close to 1, the insiders' marginal benefit from diversification is very high, so that the price they need to receive for each additional percentage of equity divested in order to maintain a given expected utility level is low (i.e., the magnitude of $\partial V/\partial\alpha$ is small); however, as α falls, the marginal benefit from diversifying also falls, so that the price they need to receive (so as to maintain a given expected utility level) for each additional percentage of equity divested increases (i.e., the magnitude of $\partial V/\partial\alpha$ increases as α falls). Thus, the insiders' indifference curves are flattest for α close to 1, and become steeper for smaller values of α. Notice also that, for any given α, the slope of the insider's indifference curve depends on the riskiness of their firm's future cash flows. Thus, if $\delta = \varepsilon$ (as in figure 1), the type B insider's indifference curves are always flatter than that of the type G: since type G firms have a greater mean future cash flow, insiders of such firms will need to receive a greater price (than type B insiders) for any given fraction of equity sold in the IPO in order to maintain a given utility level. If, on the other hand, δ is sufficiently larger than ε (as in figure 2), the type G insiders' indifference map will be flatter than that of type B insiders; here the ordering is reversed since the type G firm is riskier than the type B firm, and consequently, the benefits from diversifying are greater for type G insiders than for type B insiders (thus dominating the effect of the type G firm having a greater mean future cash flow than the

type B firm).

As a starting point, assume that the type G firm is exogenously constrained to use only the fraction of equity retained, α, as a signal: i.e., the firm is constrained to set the price of equity sold, V, equal to the full information value, V^G (no underpricing). In this case, the equilibrium will be at the point P' in figure 1, which is the intersection of the full-information line $\mu^G 1$ and the type B insiders' indifference curve at the utility level $U(\mu^B)$, which they would receive if they divested the firm's entire equity at its true value μ^B. This point P' satisfies the competitive rationality constraint (13) (as an *equality*, since there is no underpricing here) as well as the incentive compatibility constraint (12) and the constraints (14). Thus, in this single-signal equilibrium, type G insiders retain a fraction of equity just large enough to dissuade the type B insiders from mimicking them (type B insiders have no incentive to mimic the type G firm if they hold the fraction of the equity represented by P', since they can receive the same expected utility simply by divesting their firm's entire equity at its full information value μ^B). From now onwards, we will use $\hat{\alpha}_0$ to denote this "single-signal" equilibrium fraction of the equity retained by type G insiders. Such a separating equilibrium involving the signal α alone exists because the marginal cost of a maintaining a greater α is greater for type B firm insiders than type G insiders (recall that, since $\mu^G > \mu^B$, any amount of equity retained by type B insiders is held in an intrinsically less valuable firm compared to that held by type G insiders).

Now, allow the type G firm to underprice its equity as well. The equilibrium would not shift inward from the point P' even now. To see why, notice that, even when underpricing is allowed, the area below the thick curve, $0\mu^B P'1$ represents the α, V combinations satisfying the incentive compatibility, competitive rationality, and other constraints (notice that the competitive rationality (13) constraint now needs to be satisfied only as a weak inequality). Given this feasible region, and given that the type G firm's indifference map is steeper than that of the type B firm, it is not possible to find another type G indifference curve which yields insiders a greater utility level compared to the expected utility level

associated with the one passing through P' while simultaneously deterring the type B firm from mimicking. Thus, $\alpha^* = \hat{\alpha}_0$, and the type G firm does not choose to underprice its equity in equilibrium.

3.2 One-shot IPOs with Equity Underpricing

We will now study the equilibrium outcome in the case where the riskiness of the type G firm is significantly greater than that of the type B firm (in this section also, we will continue to maintain the assumption that the firm is exogenously constrained not to issue warrants).

Proposition 2 (Equilibrium with two signals). *If the riskiness of the type G firm is sufficiently greater than that of the type B firm, the equilibrium involves underpricing the firm's equity ($V^* < V^G$). Further, in this case the equilibrium fraction of equity retained by insiders will be positive, but lower than the fraction that would be retained if insiders were constrained to use the fraction of equity alone as a signal (i.e., $\alpha^* < \hat{\alpha}_0$).*

The intuition behind this proposition can be seen from figure 2. In this case, the feasible region consisting of the set of α, V values satisfying the incentive compatibility condition (12), the competitive rationality constraint (13), as well as the other constraints is given by the area below the thick curve $0\mu^B P^* P' 1$. To arrive at the equilibrium in this case, we can think of the type G firm as starting at the single-signal equilibrium point P' (the intersection of the full-information value line $\mu^G 1$ and the type B indifference curve corresponding to the expected utility level $U(\mu^B)$) and cutting back on the equity retained, underpricing at the same time to prevent the type B firm from mimicking. Since the type G firm is significantly riskier in this case than the type B, resulting in the type G insiders' indifference map being flatter than that of type B insiders (so that the marginal benefit of cutting back on the fraction of equity retained, at the expense of a lower equity price, is greater for type G insiders than for type B insiders), it is now possible to find a point in the feasible region which yields type G firm insiders a greater expected utility level compared to that associated with the point P'. The equilibrium point therefore moves

to this point of maximum expected utility, P*, where the type G indifference map is tangent to the type B indifference curve corresponding to the utility level $U(\mu^B)$. The equilibrium fraction of equity retained thus falls to α^*, and the firm underprices its equity in equilibrium.

It is useful to place proposition 2 in the context of previous models which obtain underpricing as a signal (Allen and Faulhaber (1989), Grinblatt and Hwang (1989), and Welch (1989)). In these models, the IPO occurs at time 0, full resolution of information asymmetry occurs at the final date (time 2 in the above models), and a second, seasoned offering by the firm (or its insiders) after the IPO at an intermediate date (time 1 in those models), with a positive probability of additional information about true firm type arriving between the two equity issues. In contrast to these models, Proposition 2 demonstrates that underpricing may arise as part of the efficient signaling mix even in the context of a one-shot equity offering (thus, a seasoned equity offering is not a necessary ingredient required to generate the use of underpricing as a signal).[18]

3.3 Unit IPOs and IPOs Without Warrants

In this section, we will re-introduce the possibility of the firm issuing warrants: i.e., firm insiders may use any combination of the three signals, α, V and θ, to distinguish the type G from the type B firm. In this case the type G firm's problem is to choose that triplet $\{\alpha^*, V^*, \theta^*\}$ which maximizes the expected utility of firm insiders, while ensuring that the type B firm does not mimic (i.e., incentive compatibility is maintained).

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[18]To make a closer comparison between proposition 2 and the results of the above models, it is useful for a moment deviate from the main theme of this paper and rule out the possibility of issuing warrants. If we do so, we can re-interpret the time line of our model in terms similar to those of the above models, with time 0 in our paper coinciding with time 0 in those models, and the final date (time 1) in our paper coinciding with the final date (time 2) in those models (i.e., we can think of time 1 in our model as being far into the future, so that firm insiders hold a fraction $(1 - \alpha)$ of the equity in the firm indefinitely). The essential difference, then, between our time line and those of the above models is that we have no intermediate date (at which revelation of additional information about firm type, and a subsequent seasoned offering is assumed to take place in the above models). It thus becomes evident that, if insiders are risk-averse, a separating equilibrium with underpricing as a signal may arise in a one-shot equity offering as well.

Proposition 3. (Unit IPOs with underpricing and a positive fraction of equity retained) *If the risk parameter δ of the type G firm exceeds a certain value $\hat{\delta}$ (where $\hat{\delta} > \varepsilon$, and is defined after (A7)), then the equilibrium strategy of the type G firm involves the following:*[19]

(a) The firm issues equity with warrants (i.e., $\theta^* > 0$);

(b) the strike price of the warrants, k, is set equal to μ^G;

(c) the package of equity and warrants is underpriced (i.e., $V^* < V^G$).

(d) firm insiders retain a positive fraction of the firm's equity ($1 > \alpha^* > 0$).

Such a separating equilibrium will always exist, provided that $\mu^G - \mu^B > (\delta - \varepsilon)/(1 - p_H)$.[20]

In order to study how issuing warrants along with equity allows the type G firm to signal firm value more efficiently, it is useful to study how issuing warrants differs as a signaling device from underpricing equity. We saw before that underpricing may be part of the optimal combination of signals because giving away value by underpricing allows the type G firm to cut back on α, the fraction they retain in the firm (while deterring the type B firm from mimicking). Now, underpricing *equity* amounts to dissipating value in *all* states. In contrast, issuing warrants allows firm insiders to yield a part of the firm's time 1 value to outsiders *in the high state alone*. This ability to yield firm value only in the high state becomes attractive when the insiders' assessment of firm riskiness is very large. This is because insiders' private valuation of each dollar of firm value is lowest in the high state, and consequently, packaging warrants along with equity in the IPO allows high risk type G firms to satisfy the incentive compatibility condition more efficiently.

There are two broad factors which determine whether warrants are part of the equilibrium

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[19]The condition that $\delta > \hat{\delta}$ is a sufficient (but not necessary) condition: If this condition is met, the separating equilibrium always involves the issue of warrants along with equity in the IPO; however, such an equilibrium involving warrants *may* exist even when this condition is not met.

[20]This is a sufficient (but *not necessary*) condition, required only to ensure that the solution to the type G insiders' optimization problem leads to their setting $\alpha^* > 0$; such a separating equilibrium *may* exist even when this condition is not satisfied.

package of securities issued in the IPO. First, for $\delta > \varepsilon$, the marginal utility of income in the high state is lower for type G firm insiders than for type B firm insiders; however, the marginal utility of income in the low state is higher for type G insiders than for type B insiders. In other words, the *private valuation* of every dollar of cash flow that they have to yield to outsiders in the high state is lower for type G insiders than for type B insiders, while the inequality is reversed for the low state. This factor, taken alone, would indicate that warrants, which allows the type G firm to sell cash flows in the high state alone, should always be part of the efficient signaling package; because of the difference in private valuations, packaging warrants along with equity in the IPO would allow the type G firm to cut back on one or both of the other two signals (namely, the fraction of firm value retained by insiders, and the underpricing of the package of securities sold) required to deter mimicking by the type B firm (i.e., to maintain incentive compatibility).

There is, however, a second factor which determines whether warrants will be a part of the equilibrium package of securities issued in the IPO. The type G firm's expected time 1 value is greater than that of the type B firm ($\mu^G > \mu^B$). Further, since $\delta > \varepsilon$, type G cash flows are also riskier than type B cash flows. This means that the true magnitude of time 1 cash flows that each warrant is entitled to (if the high state is realized) is greater for type G warrants than for type B warrants. This second factor, taken alone, would indicate that warrants should not be part of the efficient signaling package, since the *dollar value* of cash flows that they have to yield to each warrant is always higher for the type G firm than for the type B firm. Thus, warrants are issued in equilibrium by type G firms if, and only if, the first factor (the "private value effect") dominates the second ("dollar value effect"), which is indeed the case if the riskiness δ of the type G firm is higher than that of the type B firm ε, and exceeds $\hat{\delta}$, a certain threshold value.[21] Further, the number of warrants, θ^*, is also determined from the tension between

[21]Unit IPOs are puzzling to those who try to analyze them based on the intuition that the warrants of riskier, higher value firms are more valuable than those of less risky, less valuable firms: Such a casual intuition might suggest (in opposition to the stylized facts about unit IPOs) that it is the *less risky* firm that would issue warrants

these two opposing effects.

In equilibrium, the type G firm sets the exercise price of any warrants issued equal to μ^G. The intuition here is that any warrants issued will be most effective as a separating device if it can be ensured that the type B firm has to give away as much value as possible for each warrant it issues as part of any attempt to mimic the type G firm. Setting $k = \mu^G$ accomplishes this, since doing so maximizes the "private value effect" discussed above, while the "dollar value effect" is not sensitive to the exercise price of the warrants issued (it is easy to verify that the *difference* in the cash flow yield between type G and type B warrants is independent of k).

We have seen so far that issuing warrants allow the type G firm to signal true value more efficiently when the type G firm is significantly riskier compared to the type B firm. This naturally raises the question: What are the conditions under which the type G firm chooses *not* to include warrants in the equilibrium package of securities issued (thus using only α and V as signals), even when it is free to do so? (This is clearly a question different from that addressed by proposition 2, which studies the conditions under which the type G firm uses underpricing as a signal in addition to the fraction of equity retained, in a situation where issuing warrants was exogenously ruled out.) We provide the answer to this question in proposition 4.

Proposition 4. (Underpriced IPOs without warrants) *If $\mu^G - \mu^B > C$ (defined after (A11)), and if the risk parameter δ of the type G firm falls within a certain interval (δ_1 , δ_2) lying between $\hat{\delta}$ and ε (δ_1 and δ_2 are defined after (A11) and (A8) respectively) then the equilibrium strategy of the type G firm involves the following:*

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in a world where outsiders are uncertain about firm riskiness (since, if the market price they obtain from selling warrants is the same, insiders of truly less risky firms would be selling an intrinsically less valuable security than those of more risky firms). However, our model demonstrates that such an argument is based on only half the story, since the marginal value of each dollar yielded to warrants by the risk-averse insiders of higher risk, higher value firms is also lower than that yielded by the insiders of lower risk, lower value firms.

(a) The firm issues equity alone in the IPO (i.e., no warrants are issued, $\theta^* = 0$);

(b) the equity is underpriced (i.e., $V^* < V^G$); and

(c) insiders retain a positive fraction of the firm's equity ($\alpha^* > 0$). Further, the equilibrium fraction retained, α^*, is always smaller here compared to the case where α alone is used as a signal ($\alpha^* < \hat{\alpha}_0$).

The intuition behind this proposition is as follows. For given values of μ^G and μ^B, if the type G firm's riskiness is greater than that of the type B firm but falls in an interval less than the threshold value $\hat{\delta}$ (discussed under proposition 3), the dollar value effect discussed above (i.e., the fact that the magnitude of the cash flows promised to type G warrants is always greater than that promised to type B warrants) dominates the private value effect (the fact that type G insiders have a lower valuation of each dollar of cash flow in the high state than type B insiders), so that the efficient signaling mix of the type G firm does not involve issuing warrants. However, the signalling mix does involve the firm selling underpriced equity in addition to insiders retaining a fraction of the equity α^* in the firm, and consequently, the equilibrium fraction of equity retained, α^*, will be less than the single-signal equilibrium fraction $\hat{\alpha}_0$ (for the same reasons that we discussed in detail under proposition 2).

4. Analysis of Unit IPOs

We will now develop the implications of our model for IPOs in general and unit IPOs in particular. In further discussion, we will use u^* to denote the proportion (*percentage*) of equilibrium underpricing of the package of securities issued (whether equity alone or equity with warrants). Clearly, u^* is given by the *amount* of underpricing divided by the full information value of the securities issued in the IPO:

$$u^* \equiv \frac{V^G - V^*}{V^G} = 1 - \frac{V^*}{(1-\alpha^*)\mu^G + \alpha^* p_H \theta^* (\mu^G + \delta/p_H - k^*)}. \tag{15}$$

Propositions 5, 6 and 7 below will present the comparative static results on unit IPOs (i.e., the

comparative statics of the equilibrium characterized in proposition 3, where the type G firm issues an underpriced package of equity and warrants). We will then briefly discuss the comparative static results of the equilibrium characterized in proposition 4, where the firm sells underpriced equity alone (even though free to issue warrants as well).

Proposition 5. (Comparative statics on θ^*) *Holding α constant, the equilibrium amount of warrants issued θ^* is: (a) positively related to firm riskiness δ, provided δ is greater than a certain value δ_3 and $\alpha^* > \underline{\alpha}$ (both values are defined after (A12));*

(b) negatively related to the difference in intrinsic values between the type G and the type B firm, $(\mu^G - \mu^B)$.

Proposition 6. (Comparative statics on α^*) *Holding θ constant, the equilibrium fraction of equity α^* retained by insiders is: (a) negatively related to firm riskiness δ; and*

(b) positively related to the difference in intrinsic values between the type G and the type B firms $(\mu^G - \mu^B)$, provided that $\mu^G - \mu^B > 1/p_H$.

Proposition 7. (Comparative statics on underpricing) *The equilibrium amount of underpricing $(V^G - V^*)$ is: (a) positively related to firm riskiness δ, holding θ constant; and*

(b) negatively related to the difference in intrinsic values between the type G and the type B firm, $(\mu^G - \mu^B)$, holding θ constant, and provided that $\mu^G - \mu^B > 1/p_H$.

In addition, for u^* less than a certain value $\bar{u}$, the equilibrium percentage of underpricing is:

(c) positively related to firm riskiness δ, holding θ constant;

(d) positively related to the amount of warrants issued θ^*, holding α constant;

(e) negatively related to the fraction of equity α^* retained by insiders, holding θ constant.

The intuition behind the above comparative statics on the equilibrium where the type G firm uses unit IPOs is as follows. As firm riskiness δ increases, it becomes relatively more expensive (in terms of

expected utility) for type G firm insiders to retain equity in the firm, so that they cut back on α, at the same time increasing the extent of underpricing and warrants in the equilibrium signaling mix (thus ensuring that incentive compatibility is maintained). Similarly, when $(\mu^G - \mu^B)$ is increased, selling underpriced equity or issuing warrants becomes more expensive for the type G firm relative to the other signal α, so that the firm cuts back on warrants and underpricing while retaining more of the equity in the firm (thus maintaining incentive compatibility).

The comparative static results for IPOs without warrants (i.e., on the equilibrium characterized in proposition 4) are broadly similar to that presented above. In the case of all equity IPOs also, the extent of underpricing is positive related to firm riskiness, and negatively related to the fraction of equity retained by firm insiders; further, the fraction of equity retained by insiders is declining in firm riskiness.[22] The intuition driving these results is also quite similar to that underlying the comparative static results discussed for unit IPOs: as firm riskiness δ increases, employing the signal α becomes relatively more expensive, thus leading to a reduction in α and an increase in underpricing in the equilibrium signaling mix.

5. Empirical Implications and Conclusion

This paper has analyzed the optimality of unit IPOs, in which the firm going public issues a package of equity with warrants. We showed that, in an equity market characterized by asymmetric information, where insiders have private information about the riskiness as well as the expected value of their firm's future cash flows, high risk firms package their equity with warrants, and the package of equity and warrants is underpriced. Lower risk firms, on the other hand, issue underpriced equity alone.

Our model provides several testable predictions. First, it predicts that in a group of firms which are indistinguishable prior to the IPO, the subset of firms employing unit IPOs will be associated with

[22]Details of these results and their derivations are available in Chemmanur and Fulghieri (1995).

greater variability of future cash flows compared to those which employ share IPOs.[23] Strong empirical support for this prediction is provided by several empirical results reported in Schultz (1993)). First, he provides evidence regarding the regarding the probability of a firm surviving one, two, or three years *after* the IPO. Schultz (1993) defines failure as being de-listed from the NASDAQ for various reasons: 88.9% of the firms in his sample that had IPOs of equity alone were still listed three years after the IPO, while only 58.8% of the companies that had unit IPOs were listed. Further, when he estimated the probability of failure using a logistic model, the likelihood of failing was significantly higher for firms that used unit offerings. Finally, Schultz (1993) documents that unit IPOs are associated with smaller firms, tend to be concentrated in speculative industries, have shorter operating histories, and have lower values of sales divided by IPO proceeds and firm assets divided by IPO proceeds.

A second prediction of our model is that, in unit IPOs, the proportion of firm value sold as warrants (i.e., the ratio of the number of shares that warrant holders will receive upon exercise to the total number of shares in the firm) is increasing in firm riskiness, keeping the fraction of equity retained by firm insiders constant. We believe that there is as yet no empirical work testing this hypothesis, so that this prediction can perhaps serve as a test of our model. A third prediction of our model is that in unit IPOs, as well as in IPOs without warrants, the percentage of underpricing is increasing in firm riskiness. Evidence consistent with this implication is provided by Schultz (1992) for unit IPOs; Beatty and Ritter (1986) and Ritter (1991) provide such evidence for IPOs without warrants. A fourth prediction is that, in unit IPOs, the fraction of equity retained by firm insiders is decreasing in firm riskiness, keeping the proportion of firm value sold as warrants constant; our model implies a negative relationship between firm riskiness and the fraction of equity retained by insiders for IPOs with equity alone as well.

The predictions of our model are consistent with the widely observed fact that unit IPOs tend to

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[23]Applying our model to seasoned equity issues yields a similar implication: firms characterized by greater uncertainty about future cash flows use packaged offerings of equity and warrants, while less risky firms issue equity alone.

be marketed by less prestigious underwriters (see, e.g., Schulz, 1993). Assuming that the economic role of an investment bank underwriting an IPO is that of an information-producing intermediary (i.e., investment banks can, by incurring a certain monetary or other cost unearth information about firms otherwise unavailable to outside investors), we know from the existing theoretical and empirical literature on investment bank reputation that more prestigious underwriters will prefer to underwrite the IPOs of less risky firms.[24] Therefore, given that the model here demonstrates that it is the intrinsically more risky firms which choose unit IPOs in equilibrium, it is not surprising to find that such unit offerings are associated with less reputable underwriters.

A secondary contribution made by the paper lies in demonstrating (in contrast to the existing literature) that underpricing may emerge as an equilibrium strategy even if the firm going public does not plan a seasoned equity offering (regardless of whether the IPO involves warrants or not). When the firm making the IPO is very risky, underpricing allows firm insiders to cut back on the fraction of equity that they have to retain in the firm after the IPO, thus increasing insiders' equilibrium welfare relative to the case where the firm uses the fraction of equity alone as a signal.

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[24]Chemmanur and Fulghieri (1994) develop a model of reputation acquisition by investment banks, who act as information producing intermediaries in an equity market characterized by asymmetric information. Their model demonstrates why more reputable investment banks tend to set stricter standards in terms of the kind of firms for which they underwrite an equity issue. Further, their model demonstrates that the variance in the true value of the equity marketed by an investment bank is decreasing in its reputation. (See also Carter and Manaster (1990), who document this relationship between underwriter reputation and firm riskiness.)

References

Allen, F. and G. R. Faulhaber, 1989, Signalling by underpricing in the IPO market, *Journal of Financial Economics* 23, 303-323.

Barnea, A., R. Haugen and L. W. Senbet, 1985, *Agency Problems and Financial Contracting* (Prentice Hall, Inc., Englewood Cliffs, NJ.)

Barry, C. B., C. J. Muscarella, and M. R. Vetsuypens, 1991, Underwriter warrants, underwriter compensation, and the costs of going public, *Journal of Financial Economics* 29, 113-135.

Beatty, R., and J. Ritter, 1986, Investment banking, reputation, and the underpricing of initial public offerings, *Journal of Financial Economics* 15, 213-232.

Benveniste, L. M., and P. Spindt, 1989, How investment bankers determine the offer price and allocation of new issues, *Journal of Financial Economics* 24, 343-361.

Brennan, M. and A. Kraus, Efficient financing under asymmetric information, *Journal of Finance* 42, 1225-1243.

Carter, R., and S. Manaster, 1990, Initial public offerings and underwriter reputation, *Journal of Finance* 45, 1045-1067.

Chemmanur, T., 1993, The pricing of initial public offerings: A dynamic model with information production, *Journal of Finance* 48, 285-304.

Chemmanur, T., and P. Fulghieri, 1994, Investment bank reputation, information production, and financial intermediation, *Journal of Finance* 49, 57-79.

Chemmanur, T., and P. Fulghieri, 1995, Why Include Warrants in New Equity Issues? A Theory of Unit IPOs, Paine-Webber Working Paper, Graduate School of Business, Columbia University.

Constantinides, G. and B. Grundy, 1989, Optimal investment with stock repurchase and financing as signals, *Review of Financial Studies* 2, 445-465.

Engers, M., 1987, Signalling with many signals, *Econometrica* 55, 663-674.

Green, R., 1982, Investment incentives, debt, and warrants, *Journal of Financial Economics* 13, 115-136.

Grinblatt, M. and C. Y. Hwang, 1989, Signalling and the pricing of unseasoned new issues, *Journal of Finance* 44, 393-420.

Hughes, P. J., and A.V. Thakor, 1992, Litigation risk, intermediation, and the underpricing of initial public offerings, *Review of Financial Studies* 5, 709-742.

Ibbotson, R., J. L. Sindelar, and J. Ritter, 1988, Initial public offerings, *Journal of Applied Corporate Finance* (Summer), 37-45.

Jayaraman, N., K. Shastri, and K. Tandon, 1991, The shareholder wealth effects of the inclusion of warrants in new security issues, Working paper, Georgia Institute of Technology.

Jensen, M.J., 1986, The agency cost of free cash flow, corporate finance and takeovers, *American Economic Review*, papers and proceedings, 76, 323-329.

Kreps, D. M., and R. Wilson, Sequential equilibrium, 1982, *Econometrica* 50, 863-894.

Leland, H. E., and D. H. Pyle, 1977, Informational asymmetries, financial structure and financial intermediation, *Journal of Finance* 32, 371-387.

Mauer, D. C., and L. W. Senbet, 1992, The effect of the secondary market on the pricing of initial public offerings: Theory and evidence, *Journal of Financial and Quantitative Analysis* 27, 55-79.

Milgrom, P. and J. Roberts, 1986, Price and advertising signals of product quality, *Journal of Political Economy* 94, 797-824.

Ritter, J., 1991, The long-run performance of initial public offerings, *Journal of Finance* 46, 3-27.

Rock, K., 1986, Why new issues are underpriced, *Journal of Financial Economics* 15, 187-212.

Schultz, P., 1993, Unit initial public offerings: A form of staged financing, *Journal of Financial Economics* 34, 199-229.

Smith, C. W., 1977, Alternative methods for raising capital: Rights versus underwritten offerings, *Journal of Financial Economics* 5, 273-307.

Spatt, C., and S. Srivastava, 1991, Pre-play communication, participation restrictions, and efficiency in initial public offerings, *Review of Financial Studies* 4, 709-726.

Stein, J. C., 1992, Convertible bonds as back-door equity financing, *Journal of Financial Economics* 32, 3-21.

Welch, I., 1989, Seasoned offerings, imitation costs and the underpricing of initial public offerings, *Journal of Finance* 44, 421-449.

APPENDIX

Consider the non-mimicry problem (11) and the corresponding Lagrangean expression:

$$\mathcal{L} = \bar{U}^G + \lambda_1\left[U(\mu^B) - \bar{U}^B\right] + \lambda_2\left[V^G - V\right] + \lambda_3\left[1-\alpha\right] + \lambda_4\left[1-\theta\right].$$

The Kuhn-Tucker conditions for that problem are:

$$\mathcal{L}_\alpha = \bar{U}^G_\alpha - \lambda_1 \bar{U}^B_\alpha + \lambda_2 V^G_\alpha - \lambda_3 \leq 0,$$

$$\mathcal{L}_\theta = \bar{U}^G_\theta - \lambda_1 \bar{U}^B_\theta + \lambda_2 V^G_\theta - \lambda_4 \leq 0,$$

$$\mathcal{L}_V = \bar{U}^G_V - \lambda_1 \bar{U}^B_V - \lambda_2 \leq 0,$$

$$\mathcal{L}_q = \bar{U}^G_k - \lambda_1 \bar{U}^B_k + \lambda_2 V^G_k \leq 0,$$

$$\alpha\mathcal{L}_\alpha + \theta\mathcal{L}_\theta + V\mathcal{L}_V + q\mathcal{L}_q = 0,$$

$$\lambda_1\left[U(\mu^B) - \bar{U}_B\right] + \lambda_2\left[V^G - V\right] + \lambda_3\left[1-\alpha\right] + \lambda_4\left[1-\theta\right] = 0,$$

$$\bar{U}^B \leq U(\mu^B), \quad V \leq V^G, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \theta \leq 1,$$

$$V \geq 0, \quad q \geq 0, \quad \lambda_i \geq 0, \; i = 1,2,3,4.$$

For all $(\theta,k) \geq 0$, define $\alpha_0(\theta,k) > 0$ as the (unique) solution to $\bar{U}^B[\alpha_0, \theta, V^G(\alpha_0, \theta, k), k] = U(\mu^B)$. Let $a(\theta,k) \equiv \min\{\alpha_0(\theta,k);1\}$ and $\hat{\alpha}_0 \equiv \alpha_0(0,k) < 1$. Note that, by definition, a triplet $\{\theta,k,\alpha_0(\theta,k)\}$ has the property that *both* the incentive-compatibility constraint (12) and the individual rationality constraint (13) hold as equalities. In the proofs below, we will use the fact that, if $\alpha < a(\theta,k)$, constraint (12) is binding while constraint (13) is not. The reverse is true for $\alpha > a(\theta,k)$. Hence, choosing $\alpha > a(\theta,k)$ is dominated by setting $\alpha = a(\theta,k)$, since the latter is incentive-compatible, and a type G may sell a greater fraction of the firm at V^G. Hence, all solutions (α,θ,k,V) must satisfy $\alpha \leq a(\theta,k)$.

Proof of Proposition 1. Consider the reduced version of the non-mimicry problem, where $\theta \equiv k \equiv 0$, and $\delta = \varepsilon$. A solution (α^*, V^*) with $\alpha^* > 0$ is an optimal strategy for a type G firm in a separating equilibrium. Note first that at an optimum, $\lambda_1 > 0$; if, on the contrary, $\lambda_1 = 0$, the unique solution to the non-mimicry problem is the full information optimum, which is not incentive-compatible. Hence, $\lambda_1 > 0$ and the incentive-compatibility constraint (12) must be binding. Since $\hat{\alpha}_0$ is incentive-compatible and

dominates $\alpha = 1$, a type G prefers to sell at least a fraction $(1 - \hat{\alpha}_0)$ of the firm at $V = (1 - \hat{\alpha}_0)\mu^G$, which implies that $V^* > 0$ and $\alpha^* < 1$. Hence, at an optimum $\mathcal{L}_V = \lambda_3 = 0$. From $\mathcal{L}_V = 0$, we have $\lambda_1 = [\bar{U}_V^G - \lambda_2]/\bar{U}_V^B$: substituting this value into $\mathcal{L}_\alpha$, we obtain:

$$\frac{\bar{U}_\alpha^G}{\bar{U}_V^G} - \frac{\bar{U}_\alpha^B}{\bar{U}_V^B} \quad \left(\equiv |MRS^G(V,\alpha)| - |MRS^B(V,\alpha)| \right) \leq -\frac{\lambda_2}{\bar{U}_V^G \bar{U}_V^B}\left[\bar{U}_\alpha^B + \bar{U}_V^B V_\alpha^G\right], \tag{A4}$$

where $|MRS^T(V, \alpha)| = \mu^T - (\rho_L^T - 1)\,\varepsilon/(p_H + p_M\rho_M^T + p_L\rho_L^T)$ is the marginal rate of substitution between V and α for T = G,B and $\rho_S^T \equiv U'(w_S^T)/U'(w_H^T) > 1$, for S = M,L. Uniqueness of $\alpha_0(\theta,k)$ implies that type B utility decreases with α along the V^G line (see Figure 2). Hence $\bar{U}_\alpha^B + \bar{U}_V^B V_\alpha^G \leq 0$, and the RHS of (A4) is non-negative. Suppose now that $\alpha^* < \hat{\alpha}_0$ so that $V^* < V^G$ and $\lambda_2 = 0$. $\delta = \varepsilon$ implies that $\bar{U}_\alpha^G/\bar{U}_V^G - \bar{U}_\alpha^B/\bar{U}_V^B = \mu^G - \mu^B > 0$, violating (A4). Hence $\alpha^* = \hat{\alpha}_0$ and $V^* = V^G$. QED

Proof of Proposition 2. Following a procedure similar to the one used in the proof of Proposition 1, we obtain again that $V^* > 0$, $\lambda_1 > 0$, and $\mathcal{L}_V = \lambda_3 = 0$. If $\hat{\alpha}_0$ is an optimum, (A4) must be satisfied as an equality and the LHS must be non-negative. Since it may be verified that $|MRS^G(V,\alpha)|$ is strictly decreasing in δ, there is $\underline{\delta} > \varepsilon$ such that for $\delta > \underline{\delta}$, (A4) is violated at $\hat{\alpha}_0$. This implies that $\alpha^* < \hat{\alpha}_0$ and $V^* < V^G$ and $\lambda_2 = 0$. Finally, at $\alpha^* = 0$ we have that $\rho_L^G = \rho_L^B = 1$ and $\bar{U}_\alpha^G/\bar{U}_V^G - \bar{U}_\alpha^B/\bar{U}_V^B = \mu^G - \mu^B > 0$, violating (A4). Hence $\alpha^* > 0$. QED

Proof of Proposition 3. Following a procedure similar to the one used in the proof of Proposition 1, we obtain again that $V^* > 0$, $\lambda_1 > 0$, and $\mathcal{L}_V = 0$. The proof proceeds now in four steps: First, we show that $k^* = \mu^G$. Second, we establish that $V^* < V^G$ and $\alpha^* < 1$. Next, we show that $\theta^* > 0$. Finally, we prove that $\alpha^* > 0$ for $\mu^G - \mu^B > (\delta - \varepsilon)/(1 - p_H)$.

Step 1: $k^* = \mu^G$. If at an optimum $\theta^* > 0$, $\theta\mathcal{L}_\theta = 0$ implies that $\mathcal{L}_\theta = 0$. By direct calculation, this implies that $\mathcal{L}_q < 0$. Hence, $q = k - \mu^G = 0$, and $k^* = \mu^G$.

Step 2: $V^* < V^G$ and $\alpha^* < 1$. We now show that $\alpha^* < a(\theta^*, k^*)$. $\alpha^* = a(\theta^*, k^*) > 0$ requires that $\mathcal{L}_\alpha = 0$. From $\mathcal{L}_V = 0$, substituting $\lambda_1 = [\bar{U}_V^G - \lambda_2]/\bar{U}_V^B$ into $\mathcal{L}_\alpha = 0$, we obtain:

$$\frac{\bar{U}_\alpha^G}{\bar{U}_V^G} - \frac{\bar{U}_\alpha^B}{\bar{U}_V^B} \quad \left(\equiv |MRS^G(V,\alpha)| - |MRS^B(V,\alpha)| \right) = -\frac{\lambda_2}{\bar{U}_V^G \bar{U}_V^B}\left[\bar{U}_\alpha^B + \bar{U}_V^B V_\alpha^G\right] + \frac{\lambda_3}{\bar{U}_V^G}. \tag{A5}$$

From $\bar{U}_\alpha^B + \bar{U}_V^B V_\alpha^G \leq 0$ and $\lambda_3 \geq 0$, the RHS of (A5) is again non-negative. We now show that there is a $\delta_a \geq \varepsilon$ such that for $\delta > \delta_a$ the LHS of (A5) is negative at $a(\theta,k^*)$, for all θ. Note that $|MRS^B(V,\alpha)|$ does not depend on δ and μ^G. As in the proof of Proposition 2, $|MRS^G(V,\alpha)|$ is strictly decreasing in δ. For any given θ, there is a $d_a(\theta, k^*) \geq \varepsilon$ such that for $\delta = d_a(\theta, k^*)$ it is $|MRS^G(V,\alpha)| = |MRS^B(V,\alpha)|$ at $a(\theta, k^*)$, and $|MRS^G(V,\alpha)| < |MRS^B(V,\alpha)|$ for all $\delta > d_a(\theta, k^*)$. Let $\delta_a = \max_\theta d_a(\theta, k^*)$. Hence, for $\delta > \delta_a$, (A5) is violated at $a(\theta, k^*)$ for all θ, and $\alpha^* < a(\theta^*, k^*)$, $V^* < V^G$ and $\lambda_2 = 0$.

Step 3: $\theta^* > 0$. We now show that there is a $\bar{\delta}$ such that $\theta^* > 0$ for $\delta > \bar{\delta}$. From $\mathcal{L}_\theta \leq 0$, and substituting for $\lambda_1 = [\bar{U}_V^G - \lambda_2]/\bar{U}_V^B$, the first order condition for θ is:

$$\frac{\bar{U}_\theta^G}{\bar{U}_V^G} - \frac{\bar{U}_\theta^B}{\bar{U}_V^B} \quad \left(\equiv MRS^B(V,\theta) - MRS^G(V,\theta) \right) \leq -\frac{\lambda_2}{\bar{U}_V^G \bar{U}_V^B}\left[\bar{U}_\theta^B + \bar{U}_V^B V_\theta^G\right] + \frac{\lambda_4}{\bar{U}_V^G}, \tag{A6}$$

where $MRS^T(V,\theta) = p_H\alpha(\mu^T + \tau^T/p_H - k)/(p_H + p_M\rho_M^T + p_L\rho_L^T)$ is the MRS between V and θ for T = G, B, and $\tau^G = \delta$ and $\tau^B = \varepsilon$. In step 2, we showed that $V^* < V^G$ and $\lambda_2 = 0$. A solution at $\theta = 0$ requires that $\lambda_4 = 0$ so that the LHS of (A6) must be non-positive. Let now $\phi \equiv (p_H + p_M\rho_M^G + p_L\rho_L^G)/(p_H + p_M\rho_M^B + p_L\rho_L^B)$ and $\psi \equiv (\mu^G + \delta/p_H - k)/(\mu^B + \varepsilon/p_H - k)$; (A6) may be restated as

$$MRS^B(V,\theta) - MRS^G(V,\theta) = \frac{\alpha p_H\left(\mu^B + \varepsilon/p_L - k\right)}{p_H + p_L\rho_L^G + p_L\rho_L^G}\left[\phi - \psi\right] \leq 0. \tag{A7}$$

Note that for $\delta = \varepsilon$ we have that at $\theta = 0$ it is $\rho_M^G = \rho_M^B$ and $\rho_L^G = \rho_L^B$. Hence $\phi = 1 < \psi$, and $MRS^G(V, \theta) > MRS^B(V, \theta)$. Note now that ψ is increasing and *linear* in δ, while ϕ is increasing and *convex* in δ. Hence, for any α there is a $\delta > \varepsilon$ for which $\phi = \psi$ at $\theta = 0$. Denote this value of δ by $d(\alpha)$. Letting $\bar{\delta} \equiv \max_\alpha d(\alpha)$, we have that at $\theta = 0$ it is $MRS^B(V, \theta) > MRS^G(V, \theta)$ for all $\delta > \bar{\delta}$, violating (A7). Letting $\hat{\delta} \equiv \max\{\bar{\delta}, \delta_a\}$, we have that $\theta^* > 0$ and $V < V^*$ for $\delta > \hat{\delta}$.

Step 4: $\alpha^* > 0$. $\alpha^* = 0$ requires $\lambda_2 = 0$, $\mathcal{L}_\alpha \leq 0$ and $\rho_M^G = \rho_L^G = \rho_M^B = \rho_L^B = 1$. Also, $k^* =$

μ^G and $\mu^G - \mu^B > (\delta - \varepsilon)/(1 - p_H)$ give $|MRS^G(V, \alpha)| = \mu^G - \theta\delta > \mu^B - \theta p_H(\mu^B + \varepsilon/p_H - \mu^G) = |MRS^B(V, \alpha)|$, violating (A4). Hence $\alpha^* > 0$, concluding the proof. QED

Proof of Proposition 4: We will show that there is a pair $\{\delta_1, \delta_2\}$ such that, if $\delta_2 > \delta_1$, then we have that for $\delta_1 < \delta < \delta_2$ the solution for θ occurs only at a corner, that is $\theta^* = 0$ and $V^* < V^G$. From the proof of Proposition 3 we know that there is a δ_a such that for $\delta > \delta_a$ it is $\alpha^* < a(\theta^*)$, $\lambda_2 = 0$ and $V^* < V^G$. Set $\delta \geq \delta_a$. (A6) implies that an interior solution for θ must satisfy:

$$MRS^B(V,\theta) - MRS^G(V,\theta) = \frac{\alpha p_H\left(\mu^B + \varepsilon/p_H - k\right)}{p_H + p_L\rho_L^G + p_L\rho_L^G}\left[\phi - \psi\right] = \frac{\lambda_4}{\tilde{U}_V^G} \geq 0 . \tag{A8}$$

There, we have also shown that at $\theta = 0$, $\delta = \varepsilon$ implies that $\phi = 1 < \psi$ and $MRS^G(V,\theta) > MRS^B(V,\theta)$. Define now $\delta_2 \equiv \min_\alpha d(\alpha)$, with $\varepsilon < \delta_2 < \bar{\delta}$. Hence, for $\varepsilon < \delta < \delta_2$ we have that at $\theta = 0$ it is $MRS^G(V,\theta) > MRS^B(V,\theta)$ so that (A8) is violated. The proof is concluded by showing that if at $\theta = 0$ it is $MRS^G(V,\theta) > MRS^B(V,\theta)$, then there cannot be another $0 < \theta < 1$ for which $MRS^G(V,\theta) = MRS^B(V,\theta)$. Note that $k^* = \mu^G$ implies that $MRS^G(V,\theta) > MRS^B(V,\theta)$ at $\theta = 1$, if:

$$Q(m) \equiv \frac{p_H + p_M e^{\alpha A m} + p_L e^{\alpha A(m + \varepsilon/p_L)}}{p_H + p_M + p_L e^{\alpha A \delta/p_L}} - \frac{\varepsilon/p_H - m}{\delta/p_H} > 0 , \tag{A9}$$

where $m \equiv \mu^G - \mu^B$. Let c be such that $Q(c) = 0$, with $c < (\delta - \varepsilon)/p_L$. Since $Q'(m) > 0$, we have that if $\mu^G - \mu^B > c$ then $Q(m) > 0$ and $MRS^G(V,\theta) > MRS^B(V,\theta)$ also at $\theta = 1$. Differentiating $MRS^G(V,\theta)$ and $MRS^B(V,\theta)$ with respect to θ, we obtain (after some algebra) that:

$$\frac{\partial MRS^T(V,\theta)}{\partial\theta} = \frac{\left[MRS^T(V,\theta)\right]^2}{p_H} A\left[p_M\rho_M^T + p_L\rho_L^T\right] > 0 , \tag{A10}$$

for T = G,B. Consider now a θ for which $MRS^G(V,\theta) = MRS^B(V,\theta)$. In this case

$$\frac{\partial MRS^G(V,\theta)}{\partial\theta} - \frac{\partial MRS^B(V,\theta)}{\partial\theta} = \frac{\left[MRS^T(V,\theta)\right]^2}{p_H} A\,\Delta(\theta) , \tag{A11}$$

where $\Delta(\theta) \equiv p_M(\rho_M^G(\theta) - \rho_M^B(\theta)) + p_L(\rho_L^G(\theta) - \rho_L^B(\theta))$. Also $\partial\rho_M^G/\partial\theta - \partial\rho_M^B/\partial\theta < 0$ if $J(\delta) \equiv \delta(p_L - m)/\varepsilon p_H$ -

$e^{\alpha A m} > 0$. Let δ_b be such that $J(\delta_b) = 0$. Then $J'(\delta) > 0$ implies that $J(\delta) > 0$ for $\delta > \delta_b$ so that $\partial\rho_M^G/\partial\theta - \partial\rho_M^B/\partial\theta < 0$. A similar argument shows that for $\delta > \delta_b$ also $\partial\rho_L^G/\partial\theta - \partial\rho_L^B/\partial\theta < 0$. Hence $\Delta'(\theta) < 0$ and $MRS^G(V,\theta)$ and $MRS^B(V,\theta)$ may intersect at most once. This, and the fact that at $\theta = 0.1$ it is $MRS^G(V,\theta) > MRS^B(V,\theta)$, gives that $MRS^G(V,\theta) > MRS^B(V,\theta)$ for all $\theta \in (0,1)$. Hence $\theta^* = 0$. Setting $\delta_1 \equiv \max\{\delta_2;\delta_3\}$ and $C \equiv \max\{c;(\delta - \varepsilon)/(1 - p_H)\}$ concludes the proof. QED

Proof of Proposition 5: (a): In an optimum with underpriced units, the optimal choice for θ is given by equating the LHS of (A6) to zero. It may immediately be verified that

$$\frac{\partial MRS^G(V,\theta)}{\partial\delta} < \alpha\frac{p_H + p_M\rho_M^G + p_L\rho_L^G\left(1 - \delta\alpha A/p_L\right)}{\left(p_H + p_M\rho_M^G + p_L\rho_L^G\right)^2}. \tag{A12}$$

Let $\underline{\alpha} \equiv \min_\theta \alpha_0(\theta, \mu^G) - \kappa$, for some $\kappa > 0$, and let δ^c satisfies: $p_H + p_M\rho_M^G(\delta^c) + p_L\rho_L^G(\delta^c)[1 - \delta^c\underline{\alpha}A] = 0$. Setting $\delta_3 \equiv \max\{\hat{\delta}, \delta^c\}$, we have that for $\alpha^* \geq \underline{\alpha}$ and $\delta > \delta_3$ the RHS of (A12) is non-positive and $MRS^G(V,\theta)$ is a decreasing function of δ. By implicit function differentiation of (A6), this and the second order conditions imply (a). (b): Similarly, from (A6) it may be verified that $\partial MRS^B(V,\theta)/\partial\mu^B > 0$. By implicit function differentiation of (A6), this, and the second order conditions, give (b). QED

Proof of Proposition 6: As in the proof of Proposition 5, part (a) is implied by the fact that in the proof of Proposition 3, step 2, we have shown that $\partial|MRS^G(V,\alpha)|/\partial\delta < 0$. Similarly, part (b) derives from the fact that, by direct calculation, it is $\partial|MRS^B(V,\alpha)|/\partial\mu^B > 0$ for $\mu^G - \mu^B > 1/p_H$. QED

Proof of Proposition 7: (a): In an optimum with warrants and underpricing, V^*, and the optimal amount of underpricing, $(V^G - V^*)$ are determined by the incentive-compatibility constraint (12). Since it may be shown that the $MRS^B(V,\alpha)$ is increasing in α, V^* is convex in α^*. Also, $\mu^G - \mu^B > (\delta - \varepsilon)/(1 - p_H)$ and $k = \mu^G$ imply that, at $\alpha = 0$, $|V_\alpha^G| > |MRS^B(V,\alpha)|$. Hence $(V^G - V^*)$ is a decreasing function of α^*. This and (6a) imply part (a), and that there is a $\bar{u}$ such that for $u^* < \bar{u}$ part (c) obtains. Parts (a) of this Proposition together with (6b) imply part (b). Finally, part (c) with $MRS^B(V,\theta) > 0$ and $MRS^B(V,\alpha) < 0$ together imply (d) and (e), respectively. QED

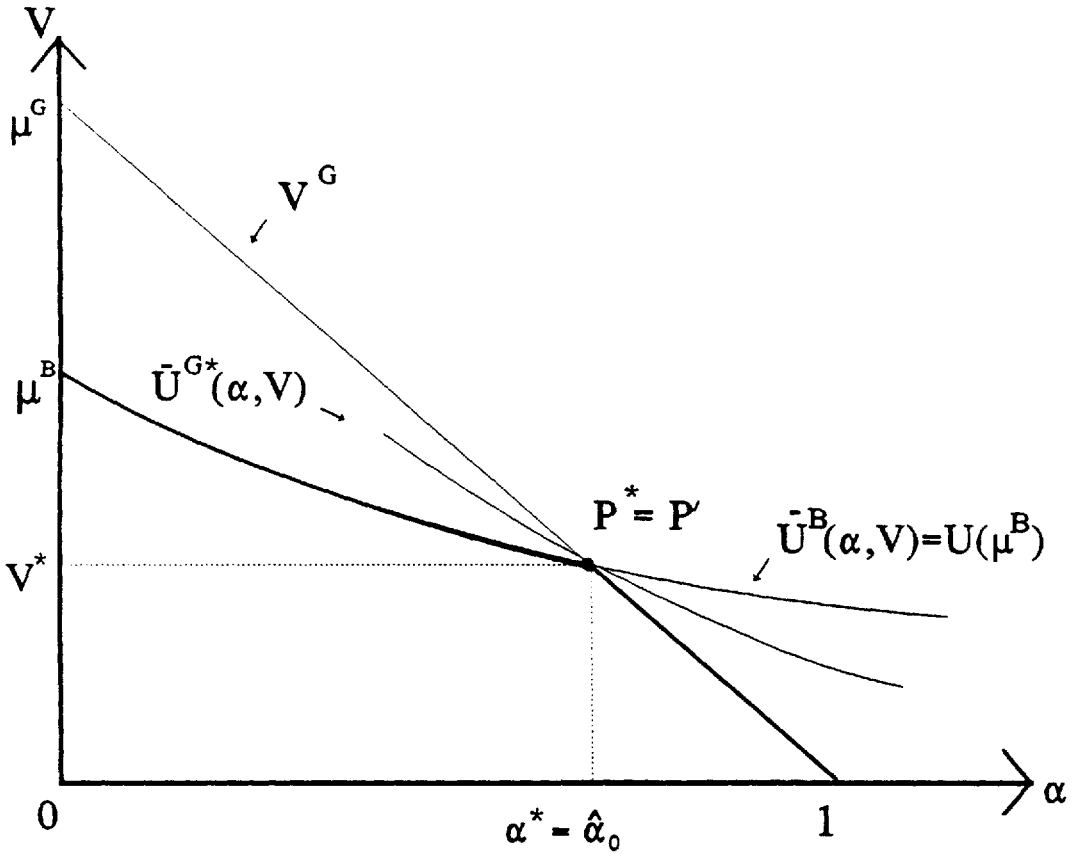

Figure 1. This figure illustrates the equilibrium when the two types of firms have the same riskiness (so that the indifference map of type G insiders is steeper than that of type B insiders) and the firm is exogenously constrained to issue equity alone (no warrants). The feasible region in this case is the area below the thick curve $0\mu^B P'1$, enclosed by the full-information value line $\mu^G 1$ (representing the competitve rationality constraint), the type B indifference curve corresponding to the expected utility level $U(\mu^B)$ (representing the incentive-compatibility constraint), and the two axes (representing the non-negativity constraints). The equilibrium point P* in this case is the same as the single-signal equilibrium point P' (i.e., the equilibrium is the same as the one which would have obtained if the firm were exogenously constrained not to set the price of its equity below its full-information value). Since the type G insiders' indifference map is steeper than that of type B insiders, the marginal benefit of cutting back on the fraction of equity retained (at the expense of a lower equity price) is greater for type B insiders than for type G insiders; it is, therefore, not possible to find a point in the feasible region which yields type G firm insiders a greater utility level compared to that associated with the point P' (thus ensuring that the equilibrium remains at P'even when underpricing is allowed).

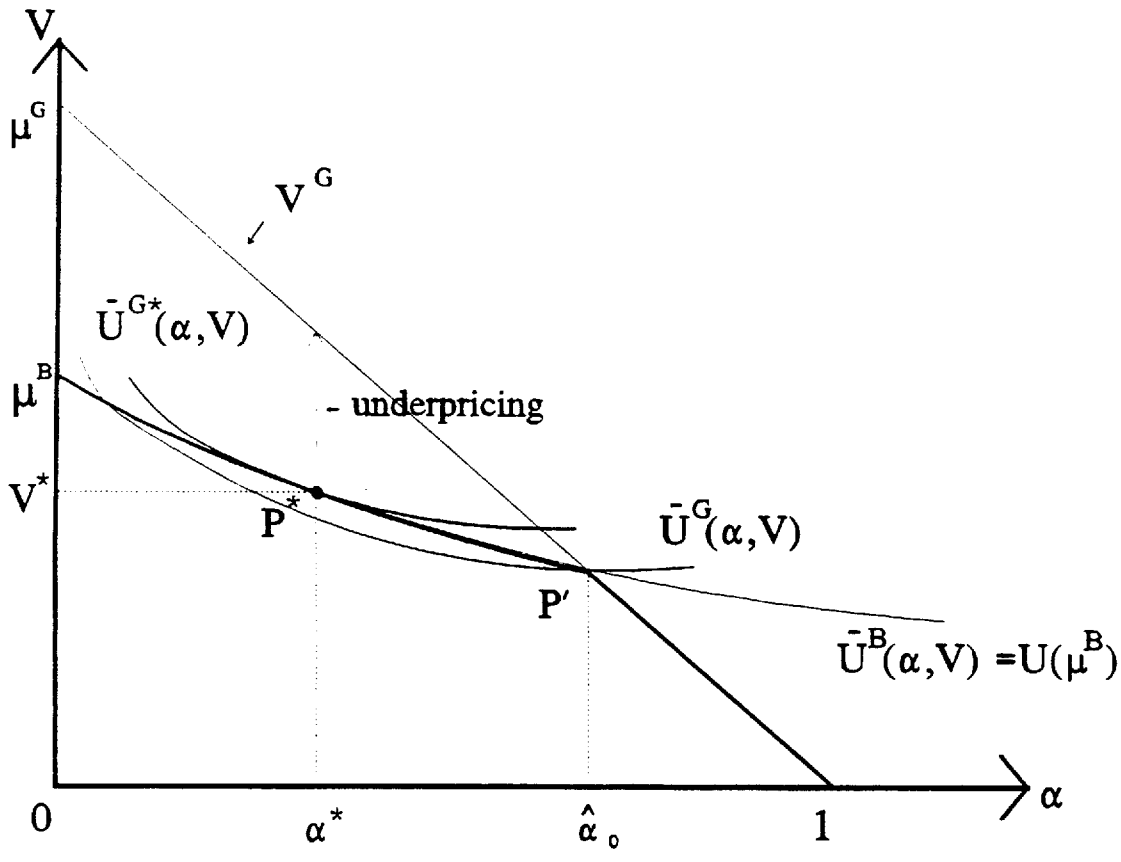

Figure 2. This figure illustrates the case where the type G firm is significantly riskier than the type B firm (so that type G insiders' indifference map is flatter than that of type B insiders), and the firm is exogenously constrained to issue equity alone (no warrants). To start with, let the firm be allowed to use only the fraction of equity retained, α, as a signal (no underpricing). Since the firm is constrained in this case to sell its equity at the full information value, the single-signal equilibrium point is at P', the intersection of the full-information value line $\mu^G 1$ and the type B indifference curve corresponding to the expected utility level $U(\mu^B)$. If the type G firm is now allowed to underprice its equity as well, then the feasible region becomes the area enclosed by the thick curve $0\mu^B P^* P' 1$ (i.e., the entire area enclosed by the full-information value line $\mu^G 1$, the type B indifference curve corresponding to the expected utility level $U(\mu^B)$, and the two axes). Since the type G insiders' indifference map is flatter in this case than that of type B insiders (implying that the marginal benefit of cutting back on the fraction of equity retained, at the expense of a lower equity price, is greater for type G insiders than for type B insiders), it is possible to find a point in the feasible region which yields type G firm insiders a greater expected utility level compared to that associated with the point P'. The equilibrium point therefore moves to this point of maximum expected utility, P*, where the type G indifference map is tangent to the type B indifference curve corresponding to the utility level $U(\mu^B)$. The equilibrium fraction of equity retained thus falls to α^*, and the firm underprices its equity in equilibrium.