THE RESIDUAL LIFE OF THE RENEWAL PROCESS: A SIMPLE ALGORITHM

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ABSTRACT

We develop a simple algorithm, that does not require convolutions, for computing the distribution of the residual life when the renewal process is discrete. We also analyze the algorithm for the particular case of lattice distributions.

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1 - Introduction

The residual life of a renewal process is defined as the time elapsed from some fixed time t until the following renewal. It is one of the random variables that describes the local behavior of a renewal process, the other being the age at time t, or the time since the last renewal. The residual life is widely used in modelling stochastic processes. In reliability theory it appears as the time until the next failure. In queueing theory, it is the time until a customer starts service in an M/G/1 queue (see Gross & Harris (1985)); and in inventory theory, it is the undershoot of the reorder point when (s, S) policies are used (see Tijms (1976)).

Early work on the residual life includes that of Karlin (1958) who defines the residual life, or excess random variable, for a renewal process and presents its Laplace transform. He also describes the age, or deficit random variable. Karlin then presents the value of the residual life for the case of the exponential distribution. He applies the residual life to the case of the (s,S) inventory policy, but he restricts the application to exponential demands. (In the case of exponential renewal processes, the residual life is also exponential.) Karlin also notes that the residual life and age of a renewal process are identical. Ross (1983) (pp. 63ff) discusses the residual life and age and notes their asymptotic behavior.

Although presented as a way of describing the local behavior of a stochastic process at a fixed time t, the expressions for the residual life, or for some of its parameters such as its expected value, generally assume that $t \to \infty$. In other words, asymptotic results are typically used. These results are clearly approximate for small t, but they are often adequate for the modelling purpose at hand. Unfortunately, this is not always so. In the context of the (s, S) inventory model, Baganha, Pyke & Ferrer (1994) show that there are cases for which the asymptotic approximation generates large errors. In this note we develop a simple recursive algorithm for computing the residual life of a renewal process when the renewal process is

discrete. The algorithm is quite fast, can be developed on a spreadsheet in minutes, and does not require convolutions. It does, however, require knowledge of the distribution of the renewal process whereas the asymptotic approximation requires knowledge of only its first three moments.

The remainder of this paper is organized as follows: In Section 2 we discuss the renewal theoretic basis of the residual life distribution and its approximation. In Section 3 we present our algorithm for computing the residual life distribution, and in Section 4 we present a summary.

2 - The Residual Life

In this section we briefly discuss the residual life of a renewal process. We refer the reader for a more complete development to Heyman & Sobel (1982) (Chapter 5), and Ross (1983) (Chapter 3). We use the standard notation of renewal theory from Heyman & Sobel (1982) , with the addition of some problem-specific variables. Let $\{N(t); t \geq 0\}$ be a renewal process and define:

X_i	=	time between the (i-1)st and the ith renewal.
$F(x)$	=	cumulative distribution function (*cdf*) of the random variable X_i.
$f(x)$	=	density function or probability function of the random variable X_i.
$F_n(x)$	=	n-fold convolution of $F(x)$ with itself.
$f_n(x)$	=	n-fold convolution of $f(x)$ with itself.
$M(j)$	=	$\sum_{n=1}^{\infty} F_n(j)$. = the renewal function, which can be thought of as the number of renewals at time j.
$m(j)$	=	$\sum_{n=1}^{\infty} f_n(j) \quad j \geq 0$. Often called the renewal density. $m(j) * \Delta j$ represents the probability that a renewal occurs during $(j, j + \Delta j)$.
μ	=	expected value of X_i
σ^2	=	variance of X_i.
$U(t)$	=	the residual life at time t.

$h_t(u)$ = P($U(t)$ = u) = density or probability function of $U(t)$

$H_t(u)$ = distribution function or *cdf* of $U(t)$

When X_i is continuous $H_t(u)$ is given by the following equation (see Heyman & Sobel (1982), pp 131-132)

$$H_t(u) = F(t+u) - \int_0^t [1 - F(t+u-x)]dM(x)$$

If X_i is discrete we have:

$$H_t(u) = F(t+u) - \sum_{x=1}^{t-1} [1 - F(t+u-x)]m(x) \tag{1}$$

and,

$$h_t(u) = f(t+u) - \sum_{x=1}^{t-1} f(t+u-x)m(x) \tag{2}$$

When the inter-renewal time is exponential (continuous case) or geometric (discrete case) then $H_t(u)$ is also exponential or geometric and its distribution is independent of t. In all other cases the time until the next renewal is dependent on t. When using renewal theory it is generally assumed that the process has reached steady state; therefore, the asymptotic distribution is often used.

In the discrete case, the only case we will consider in this note, the limit of H_t only exists if the distribution is non-lattice. It is given by (Heyman & Sobel, p. 131)

$$\lim_{t\to\infty} H_t(u) = \frac{1}{\mu} \sum_{u=0}^{t-1} [1 - F(u)] \tag{3}$$

Thus, the expected value and variance of $U(t)$ as $t \to \infty$ are given by (Silver & Peterson (1985) p. 346 or Hill (1988))

$$\lim_{t\to\infty} \mu_u = \frac{\sigma^2 + \mu^2}{2\mu} - \frac{1}{2} \tag{4}$$

$$\lim_{t\to\infty} \sigma_u^2 = \frac{E(X^3)}{3\mu} - \left[\frac{\sigma^2 + \mu^2}{2\mu}\right]^2 - \frac{1}{12} \tag{5}$$

In Baganha et al. (1994) we tested the robustness of these approximations, in the context of inventory theory, indicating conditions of good and poor performance. In cases of bad

performance, one should use the exact distribution of the residual life. But normally this is computationally burdensome because of the convolutions necessary to obtain $m(x)$ and then to find $h_t(u)$. In the next section we present an algorithm to determine $h_t(u)$ without the use of convolutions.

3 - Computation of the Residual Life

Recall from elementary renewal theory that

$$m(j) = \sum_{i=1}^{\infty} f_i(j) \text{ for } j \geq 0$$

Using standard results of renewal theory, we have

$$m(j) = f(j) + \sum_{i=0}^{j} f(j-i)m(i)$$

Clearly, $m(0) = f(0) + f(0)m(0)$

and

$$m(0) = \frac{f(0)}{1-f(0)}. \tag{6}$$

So,

$$m(j) = f(j) + \sum_{i=0}^{j-1} f(j-i)m(i) + f(0)m(j)$$

From equation (2) we get

$$m(j) = h_j(0) + f(0)m(j)$$

Thus,

$$m(j) = \frac{h_j(0)}{1-f(0)}. \tag{7}$$

Also by equation (2), we get

$$h_{t+1}(u-1) = f(t+u) + \sum_{j=0}^{t} m(j) f(t+u-j)$$
$$= f(t+u) + \sum_{j=0}^{t-1} m(j) f(t+u-j) + m(t) f(u)$$
$$= h_t(u) + m(t) f(u)$$

Substituting (7) implies

$$h_{t+1}(u-1) = h_t(u) + \frac{h_t(0) f(u)}{1 - f(0)}. \tag{8}$$

When $t = 1$, using equations (2) and (6) we obtain

$$h_1(u) = f(u+1) + m(0) f(u+1)$$

or

$$h_1(u) = \frac{f(u+1)}{1 - f(0)} \tag{9}$$

Using equations (8) and (9) we have the following algorithm, where u_{max} is the largest value of the residual life we are willing to consider, and T is the time since the process started. Thus, T is the desired value of t.

ALGORITHM

1. Let $t = 1$. Compute for all $u \le u_{max}$, :

   $$h_1(u) = \frac{f(u+1)}{1 - f(0)}$$

   Go to Step 3.

2. Compute for all $u \le u_{max}$:

   $$h_{t+1}(u-1) = h_t(u) + \frac{h_t(0) f(u)}{1 - f(0)}.$$

3. Let $t = t + 1$. If $t \le T$ go to Step 2; otherwise Stop.

This algorithm can be easily implemented on a spreadsheet, requiring no more than $O(T*u_{max})$ basic operations. The choice of u_{max} is clear when the support of the renewal process is bounded. When $f(x)$ has an unbounded support the algorithm provides the exact distribution just for the range studied. In this case, given an ε as close to zero as desired, it is always possible to get a u_{max} such that

$$1 - H_T(u_{\max}) \le \varepsilon$$

Using the above results the expected value of the residual life may also be computed by means of a recursive equation. It is easy to show that:

$$\begin{aligned}
E[U(t+1)] &= \sum_{u=0}^{\infty} u h_{t+1}(u) \\
&= \sum_{u=0}^{\infty} u \left[h_t(u+1) + \frac{h_t(0) f(u+1)}{1-f(0)} \right] \\
&= \sum_{u=0}^{\infty} u h_t(u+1) + \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} u f(u+1) \\
&= \sum_{u=0}^{\infty} (u+1) h_t(u+1) + \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} (u+1) f(u+1) \\
&\quad - \sum_{u=0}^{\infty} h_t(u+1) - \frac{h_t(0)}{1-f(0)} \sum_{u=0}^{\infty} f(u+1) \\
&= E[U(t)] + \frac{h_t(0)}{1-f(0)} \mu - 1.
\end{aligned}$$

Notice that a condition for the convergence of the mean residual life is the following Corollary of the Key Renewal Theorem (see Ross (1983), p 65)

$$\lim_{t \to \infty} \frac{h_t(0)}{1-f(0)} = \frac{1}{\mu}$$

Now the Key Renewal Theorem requires that the distribution f not be lattice. A distribution is lattice if the random variable X, with density f, is such that there exists $k \ge 0$ such that $\sum_{n=0}^{\infty} P(X = nk) = 1$, where k is the periodicity of f. Strictly speaking, any discrete distribution is lattice. We generalize this application of the Key Renewal Theorem showing that only if $k > 1$

will the above algorithm to compute the residual life density not converge. To show this we begin by stating two lemmas without proof.

Lemma 1. *If f is lattice, with positive probability only at values that are multiples of k, then for all $j \neq nk$, $n \geq 1$ and integer, $m(j) = \sum_{i=0}^{\infty} f_i(j) = 0$.*

For example, a random variable that is of the two-mass-point type has positive probability only for two values. Denote these kn_1 and kn_2. The convolution will yield positive probability at values that are linear combinations of the type $\lambda_1 n_1 k$ and $\lambda_2 n_2 k$, where λ_1 and λ_2 are integers. These linear combinations are also multiples of k.

Lemma 2. *If f is lattice, with positive probability only at values that are multiples of k, then for all $t \neq nk$, $n \geq 1$ and integer, $h_t(0) = 0$.*

Based on these lemmas it is easy to show that:

Proposition 1. *If f is lattice with positive probability only at values that are multiples of k, then for all $t + u \neq nk$, $n \geq 1$ and integer, $h_t(u) = 0$.*

Proof: From equation (9) and the definition of lattice it is shown for $t = 1$. Now, assuming that it is true for $t = m$, from equation (8)

$$h_{m+1}(u-1) = h_m(u) + \frac{h_m(0) f(u)}{1 - f(0)}$$

Because the first term of the right hand side is equal to zero, by hypothesis, $h_{m+1}(u-1) \neq 0$ only if

$$h_m(0) \neq 0$$

and

$$f(u) \neq 0$$

But these conditions imply that t and u are both multiples of k and thus $t + k$ will also be a multiple of k, which contradicts that hypothesis. **Q.E.D.**

Besides reducing the number of calculations, the above results shed some light on the question of using the approximation to the residual life presented in equation (3). If f is lattice

with a period $k > 1$, $h_t(u)$ will never converge to the approximation. When $k = 1$ the result in Proposition 2 holds. First, we recall Blackwell's theorem (see Ross (1983), p 63-64): If f is lattice with period k, $\lim_{t\to\infty} P[\text{ renewal occurs at } tk] = \frac{k}{\mu}$.

PROPOSITION 2: *If f is lattice with period $k = 1$, then* $\lim_{t\to\infty} h_t(u) - h_{t-1}(u) = 0$. In other words, for any u, $h_t(u)$ converges as $t \to \infty$.

Proof: Recall from equation (8)

$$h_{t+1}(u-1) = h_t(u) + \frac{h_t(0)f(u)}{1-f(0)}$$

implying that

$$h_t(u-1) = h_{t-1}(u) + \frac{h_{t-1}(0)f(u)}{1-f(0)}.$$

Thus,

$$h_{t+1}(u-1) - h_t(u-1) = h_t(u) - h_{t-1}(u) + \frac{f(u)}{1-f(0)}\left(h_t(0) - h_{t-1}(0)\right)$$

By (7), and Blackwell's theorem

$$\begin{aligned}\lim_{t\to\infty}\left[h_{t+1}(u-1) - h_t(u-1)\right] &= \lim_{t\to\infty}\left[h_t(u) - h_{t-1}(u) + \left(m(t) - m(t-1)\right)f(u)\right] \\ &= \lim_{t\to\infty}\left[h_t(u) - h_{t-1}(u)\right]\end{aligned} \tag{10}$$

The latter equality follows from the fact that $m(t)$ is the probability that a renewal occurs at time t, and from Blackwell's theorem: If $k = 1$, $m(t) = 1/\mu$ as $t \to \infty$. Thus $m(t-1) = 1/\mu$ as $t \to \infty$, and $m(t) - m(t-1) = 0$ as $t \to \infty$.

Because f is lattice, u can only take on integer values. In particular, let $u = 1$. Then,

$$\lim_{t\to\infty}\left[h_{t+1}(0) - h_t(0)\right] = \lim_{t\to\infty}\left[h_t(1) - h_{t-1}(1)\right]$$

Since $h_t(0) = m(t)(1 - f(0))$, we also have that

$$\lim_{t\to\infty}\left[h_{t+1}(0)-h_t(0)\right]=\lim_{t\to\infty}\left[\left(m(t+1)-m(t)\right)\left(1-f(0)\right)\right]=0$$

Hence, $\lim_{t\to\infty}\left[h_t(1)-h_{t-1}(1)\right]=0$.

Using (10) recursively, for $u = 1, 2, ..., u_{max}$, we find that, for any integer u,

$$\lim_{t\to\infty}\left[h_t(u)-h_{t-1}(u)\right]=0$$

Q.E.D.

Proposition 2 implies that if f is lattice with period k=1, the residual life distribution converges.

4 - Summary

In this paper we have developed a simple algorithm for computing the distribution of the residual life of discrete renewal processes. We also have provided a recursive formula to determine the expected value of the residual life as a function of cumulative time. We show that if the distribution of the time between renewals is lattice with period greater than 1, then the distribution of the residual life will not converge asymptotically. In these cases, the generally used approximation should be substituted by the algorithm that we have offered.

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