THE PRICING OF SWAPTIONS AND CAPS UNDER THE GAUSSIAN MODEL OF INTEREST RATE

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The Pricing of Swaptions and Caps under the Gaussian Model of Interest Rate

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Abstract

In this paper, we use Jamshidian's [8] approach to express the swaption, often priced as an interest rate option, as a bond option. We derive a simple closed-form solution for the swaption pricing formula. Modifications so as to fit the current term structure of the interest rate level and volatility are introduced. In addition, the sensitivity of this option, and bond options in general, with respect to the volatility of the interest rate is analyzed. Furthermore, we outline the difference between swaptions and caps, floors and collars. Adapting Longstaff's [10] analysis to the Ornstein-Uhlenbeck process, we express these contracts as options on yields. This analysis outlines the misuse of the Black Scholes model that consists of simply plugging the forward rate in the payoff function as an approximation.

1 Introduction

Since the original proposition of a Gaussian interest rate model by Vasicek [13], the modelling of the term structure of interest rates has considerably evolved. The specification of the interest rate process has thus taken into account more and more features that ought to be translated in the behavior of the interest rate. Unfortunately, this is accompanied by considerable complications, reducing the analytical tractability of the models. Moreover, the numerous parameters to estimate lead to a considerably more difficult implementation, resulting in proxies instead of exact formulations.

Specifically, the pricing of bonds depends crucially on the specification of the interest rate process. In addition to the choice of the number of state variables (sources of uncertainty), the care about fitting the current term structure of interest rates and volatilities makes formulas difficult to handle, except if they do not give up too much on simplicity. Since there are many derivative securities that depend directly or indirectly on bonds, this aspect is of great importance for their pricing.

In order to get rid of this bond pricing problem, practitioners have found it convenient to use a trick: replace the bond option by a corresponding number of options on the interest rate [1]. Then, it is possible use Black-Scholes [2] formula (see Hull [6, p.382]). However, this method has two very serious drawbacks. First, it provides only a proxy for the price of the option, even if the interest rate model is correctly specified. Second, and more fundamentally, it counters the underlying philosophy of the Black-Scholes analysis: the option is priced using a portolio that perfectly replicates it, and that can be used to hedge the position. Here, it is not possible to do so on an interest rate, which is not a traded asset. This issue can be solved using Black's [1] formula for options on futures, but the fact that it yields a proxy cannot be avoided by simply plugging the forward rate instead of the current one. Moreover, it is difficult to justify the replacement of the true payoff function by an artificial one on other grounds than looking for simplicity. A better approach is to stick to the true payoff and try to find the analytical expression for the option value.

[1] For a specific example of this approach concerning swaptions using the forward swap rate as a state variable, see Neuberger [11].

Caps, floors and collars are not exactly swaption-like contracts, in that the holder of the option receives payment of the interest rate differential *after* the exercise decision, whereas exercising a swaption involves receiving a bond at the maturity date of the option. The implications of this distinction are to be discussed, and lead to a different valuation framework: caps (as well as floors and collars) should be considered as options on yields.

It is possible to value analytically bond options and options on yields when the interest rate process is simply defined. Jamshidian [8], in a note about bond options pricing, derives a closed form solution for a bond option under the assumption that the interest rate follows a Ornstein-Uhlenbeck process with one state variable described by Vasicek [13]. In his paper about options on yields, Longstaff [10] starts from the Cox, Ingersoll and Ross [4] model. His methodology can be adapted to the Vasicek model, leading to a tractable option pricing formula.

The Vasicek model incorporates mean reversion in the process, but allows for negative values of the interest rate and does not fit the current term structure of the interest rate nor the current volatilities. Hull and White [7] have solved the second issue, and noted that provided that the parameters of the process are adequately chosen, a nonzero probability of negative interest rates does not empirically lead to sensible differences with the closest alternative, i.e. Cox, Ingersoll and Ross [4] model, for which negative values are ruled out automatically.

In this paper, we use Jamshidian's [8] approach to express the swaption, often priced as an interest rate option, as a bond option. We derive a simple closed-form solution for the swaption pricing formula. Modifications so as to fit the current term structure of the interest rate level and volatility are introduced. In addition, the sensitivity of this option, and bond options in general, with respect to the volatility of the interest rate is analyzed. Furthermore, we outline the difference between swaptions and caps, floors and collars. Adapting Longstaff's [10] analysis to the Ornstein-Uhlenbeck process, we express these contracts as options on yields. This analysis outlines the misuse of the Black Scholes model that consists of simply plugging the forward rate in the payoff function as an approximation.

The paper is organized as follows. Section 2 presents the swaptions formula

expressed in Jamshidian's framework. Section 3 introduces modifications so as to consider explicitly the current term and volatility structure. Section 4 studies the sensitivity of the bond option with respect to the interest rate volatility. Section 5 compares the swaption to other types of options, like caps. In Section 6, these options are expressed as options on yields, with the corresponding put-call parity. Section 7 concludes the paper.

2 Swaption Pricing

2.1 The Swaption Contract

The swaption is an option to enter a swap agreement. The underlying swap contract in which we are interested is the interest rate swap. In this framework, the first party agrees to pay to the other party a cash flow equal to the interest at a fixed rate on a given notional principal, while receiving from this other party a cash flow equal to the interest at a floating rate, say the London Interbank Offer Rate (LIBOR), on the same notional. This exchange operation is repeated for a number of periods.

The swap is typically a private agreement, and so traded over the counter. The contract execution is governed by the ISDA (International Swap Dealer's Association) rules. We are considering options on "plain vanilla" swaps, and especially the first exchange payment of the swap. The two parties receive both only one payment at a prespecified deadline. This assumption does not create any problem using Jamshidian's methodology, since multiple payments at specific deadline can be considered as coupon payments, that are explicitly taken into account by Jamshidian and do not add much complexity to the formulas adapted to swaptions. The extension to several payments is thus straightforward.

We are interested in pricing the swaption at time t; the maturity date of the option, i.e. the moment when the right to enter the swap must be exercised, is fixed (corresponding to a European option) and is denoted T, where $T > t$. The reciprocal payments consecutive to the swap agreement occur at time s, with $s > T > t$. Therefore, the option is alive from t to T whereas the swap runs from T to s.

The terms of the swaption contract are the following: the *principal* is the sum, denoted L, on which the swap holds. At T, one party commits to pay a fixed rate on L, denoted r_x, to the other party. In the meantime, the counterparty payment will be the LIBOR on the same amount L. That means that the swap is written on the same principal amount L at time T, leaving presumably different exchange payments at s.

In order to do the easiest link with Jamshidian's note, the holder of the option is in the subsequent analysis the party who receives a fixed payment and commits to pay the LIBOR. Thus, this swaption is a call option.

At maturity of the option, the holder of the swaption pays the LIBOR on L from T to s. He could at the same time borrow the sum L at the spot interest rate on the market, and close his position on the LIBOR market by lending L on the current LIBOR rate. Therefore, her positon is short on a floating-rate bond with a current value of L, maturing at s. This approach allows to say that the strike price of the swaption is L at time T, and that it need not be discounted.

In counterpart, since the swap holds on a single reciprocal payment ("zero-coupon swap"), the swaption holder receives $Le^{r_x(s-T)}$ at time s, which has to be discounted to T [2].

The payoff to the swaption at time T is then here $\max\{0, \tilde{P}Le^{r_x(s-T)} - L\}$, where $\tilde{P}$ is the price at time T of a pure discount bond paying off 1 at time s, denoted $P(r(T), T, s)$, where $r(T)$ is the value of the spot interest rate at time T; this price is a random variable at time t. When computed at time t, it must be equal then to:

$$E_\lambda[\max\{0, \tilde{P}Le^{r_x(s-T)} - L\}]$$

where λ denotes the (constant) market price of risk, and $E_\lambda(.)$ stands for the expected swaption payoff at time T under the risk-adjusted probability measure.

It appears that the option is exercised only when $P \geq e^{-r_x(s-T)}$ [3]. Expressing P as $E_\lambda \exp[-\int_T^s r(u)du] = e^{-Y(T)(s-T)}$, where $Y(T)$ is the bond yield at time

[2] This exercise rate is continuously compounded, whereas the quoted rate is generally a discrete rate. The transformation is straightforward.

[3] P is not random anymore at time T, and thus no tilde is needed.

T, we fall back on the interest-option-like decision rule for swaptions: exercise when $r_x \geq Y(T)$, which applies to a put option. However, as it will be seen later, expressing the swaption as an option on a yield would not be really incorrect, but at least imprecise. This is because the decision to enter the swap involves becoming long in a discount bond, which can be physically hedged: the bond option pricing formula provides the delta needed for this purpose. This is why the method that we develop is much more accurate than considering merely the option on a yield or, what is often practised, the option on the spot interest rate. The story will reveal to be different with the cap on LIBOR.

Note that, if we set $r^*(T)$ such that $P(r^*(T), T, s) = e^{-r_x(s-T)}$, we alternatively have that the option is exercised when $r^*(T) \geq r(T)$. But considering the swaption as an option on the spot interest rate is even less natural than as an option on a yield. It is important to notice that, when computed at time t, the risk-adjusted expected payoff of these two put options on the interest rate should be equal.

2.2 Jamshidian's Model Applied to Swaptions

The payoff function of the swaption is the one of a call option. The only difference between swaptions and bond options lies in constants appearing in this function. This swaption contract, when considered as an interest rate option, is indeed a put option.

We begin by modelling the interest rate process using a gaussian model similar to the one proposed by Vasicek:

$$r(t) = (1 - e^{-at})b + e^{-at}r_0 + \sigma e^{-at} \int_0^t e^{as} dW(s)$$

Where $W(s)$ is a standard one-dimensional Wiener process. This, in differential notation, becomes:

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \tag{1}$$

The term b is the long run mean of the interest rate. The factor a can be taken as the speed of adjustment of the process; in order to make sense

economically, it should be positive. The parameter σ is the instantaneous standard deviation of the process [4].

Setting a constant price of risk $\lambda = a(b - \bar{r})/\sigma$[5], we can define the risk-adjusted process which serves as a basis for the pricing of interest rate derivative securities:

$$dr(t) = a(\bar{r} - r(t))dt + \sigma dW_\lambda(t) \tag{2}$$

where W_λ is a standard Wiener process under the risk-adjusted probability measure.

Under these conditions, the price at time t of a pure discount bond paying off 1 at time T with the instantaneous interest rate $r(t)$ equal to r, denoted by $P(r, t, T)$, is written as [6]:

$$P(r, t, T) = A(t, T)e^{-B(t,T)r} \tag{3}$$

where, if $a \neq 0$,

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \tag{4}$$

$$A(t, T) = \exp[\frac{(B(t, T) - T + t)(a^2\bar{r} - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}] \tag{5}$$

This formula for the bond price is indeed equivalent to the following, as shown by Jamshidian:

$$P(r, t, T) = \exp[\frac{1}{2}k^2(t, T) - n(r, t, T)] \tag{6}$$

where

$$\begin{aligned} k^2(t, T) &= \frac{\sigma^2}{2a^3}(4e^{-a(T-t)} - e^{-2a(T-t)} + 2a(T - t) - 3) \\ &= var_\lambda(Y(t, T) \mid r(t)) \end{aligned} \tag{7}$$

[4]For expositional purposes, we consider here that the swap rate is the same riskless interest rate. In a forthcoming paper, we will consider that the swap rate is indeed driven by a two-factor model, as in Chen and Scott [3], still in the framework of gaussian processes.

[5]This differs from Jamshidian's specification of λ by the minus sign, but this is the equation that makes the most sense.

[6]see Vasicek [13] for the derivation of the formulas.

$$
\begin{aligned}
n(r,t,T) &= \bar{r}(T-t) + \frac{1}{a}(r-\bar{r})(1-e^{-a(T-t)}) \\
&= E_\lambda(Y(t,T) \mid r(t))
\end{aligned} \tag{8}
$$

$\bar{r}$ results from the equation $\bar{r} = b - \sigma\lambda/a$, whereas $E_\lambda(.)$ and $var_\lambda(.)$ denote respectively the expectation and the variance of the bond yield $Y(t,T)$ conditional on the current value of r under the risk-adjusted probability measure. The conditional variance is independent of r.

The bond option formula presented by Jamshidian is nothing more than Black formula. The fact that the random price at time T of a bond maturing at time s is lognormally distributed under the risk-adjusted probability measure allows to use it, with the appropriate expression for the variance of the log of the bond price:

$$
var_\lambda[\log \tilde{P}] = var_\lambda[\log P(r(T),T,s) \mid r,t] \equiv \sigma_P^2 \tag{9}
$$

$$
\sigma_P = v(t,T)\frac{(1-e^{-a(s-T)})}{a} \tag{10}
$$

$$
v^2(t,T) = var_\lambda[r(T) \mid r,t] = \frac{\sigma^2(1-e^{-2a(T-t)})}{2a} \tag{11}
$$

This expression for the variance allows to compute the value of the option. The strike price in the Black formula is K (strike price of the bond) in Jamshidian; here, it is L (principal of the swap contract).

The present value of the payment in case of exercise is $\tilde{P}$ for Jamshidian; here, it is simply multiplied by a constant: $Le^{r_x(s-T)}$. Thus, we can readily write the expression for the Black formula applied to swaptions:

$$
C(r,t,T,s,L) = Le^{r_x(s-T)}P(r,t,s)N(d_1) - LP(r,t,T)N(d_2) \tag{12}
$$

where

$$
d_1 = \frac{log[\frac{P(r,t,s)e^{r_x(s-T)}}{P(r,t,T)}]}{\sigma_P} + \frac{\sigma_P}{2}
$$

$$
d_2 = d_1 - \sigma_P
$$

An interesting comment that can be made about this formula is the 'neutrality' of the contract size in the formula. The parameter L does not appear in

the formulation of d_1, despite the fact that it is the strike price. Normalizing the contract size to the value $L = 1$ does not change any of the sensitivity characteristics of the swaptions; L is indeed just a scaling factor. This formulation corresponds to the underlying philosophy of the Black-Scholes approach, because the option pricing formula involves that the underlying is a traded asset. In this context, it makes perfect sense since the exercise of the swaption allows its holder to go to the market to unfold her position in the swap and to become long in a bond.

2.3 Extension to Multiple Payments at Fixed Periods

Of course, the swap agreement is not limited to one reciprocal payment of interest at s. In practice, several such exchanges take place at different points in time. Fortunately, we note that all bond prices are decreasing functions of r. Therefore, the option on a portfolio of bonds is equivalent to a portfolio of options on discount bonds, whose pricing has been defined in the case of a single payment in a swap.

If we express the swaption as an option on a portfolio of n fixed payments, which are generally on a same principal, each payment j occurring at time s_j, we have that the call price C_n must respect the following:

$$C_n = P(r,t,T)E_\lambda[\max\{0, L\sum_{j=1}^{n} \tilde{P}_j e^{r_x(s_j-T)} - nL\}] \tag{13}$$

where $\tilde{P}_j = \tilde{P}(r(T), T, s_j)$. With this expression, we see that the option will be exercised at time T when $\sum_{j=1}^{n} P_j e^{r_x(s_j-T)} \geq n$. Defining, as before, $r^*(T)$ as the rate such that $\sum_{j=1}^{n} P(r^*(T), T, s_j)e^{r_x(s_j-T)} = n$, we have that the swaption is exercised when $r^*(T) \geq r(T)$. Moreover, the payoff at time T of the swaption is:

$$\sum_{j=1}^{n} Le^{r_x(s_j-T)} \max[0, P(r(T), T, s_j) - X_j] \tag{14}$$

where $X_j \equiv P(r^*(T), T, s_j)$. Equation (14) is the sum of payoffs at time T of individual bond options, which can be computed by Jamshidian's formulas.

We see that the value of the swaption is thus a weighted sum of the values of the individual bond options.

3 Fitting the Initial Term Structure

The Vasicek model, because of its lack of flexibility, is often criticized because it might yield values of the interest rate that do not correspond empirically to reality, namely to the actual interest rate structure. Hull and White [7] have found that the introduction of an additional term in the drift of the differential equation (1) could solve this empirical issue. Moreover, the same kind of controversy holds for the volatility parameter, which is constant in Vasicek's model: they allow it to be a deterministic function of t. They thus rewrite:

$$dr(t) = (\theta(t) + a(b - r(t))dt + \sigma(t)dW(t) \tag{15}$$

Of course, λ and $\bar{r}$ become functions of t too. The result of this change in specification is very interesting: equations (3), (4) and (5) are virtually unchanged. The only difference comes from the new function $\phi(t) = a(t)b + \theta(t) - \lambda(t)\sigma(t)$ that replaces $a^2\bar{r}$ by $a\phi(t)$ in equation (5). Consequently, the bond price formula proposed by Jamshidian in equation (6) differs only slightly:

$$P(r,t,T) = \exp[\frac{1}{2}k^2(t,T) - n^*(r,t,T)] \tag{16}$$

where $n^*(r,t,T)$ is the same as in equation (9) with now $\bar{r}(t) = \phi(t)/a(t)$.

The interest of this virtually unchanged formulation is that $a(t)$, $\phi(t)$, $A(t,T)$ and $B(t,T)$ can be expressed in terms of $A(0,T)$, $B(0,T)$ and $\sigma(t)$, which entirely depend on the initial interest rate and volatility structures. Therefore, the same model as before can be applied but now taking explicitly into account the initial data in the calibration of the functions. Hull and White show the exact relationship between those parameters and the initial structures, using bond prices at time 0, which we will not repeat here.

Concerning the bond option, and therefore the swaption, the only parameter that needs to be determined is the volatility term σ_P. Hull and White show

that a very general formulation is:

$$\sigma_P^2 = [B(0,s) - B(0,T)]^2 \int_t^T [\frac{\sigma(u)}{\partial B(0,u)/\partial u}]^2 du \tag{17}$$

The function B, defined in (4), being only a function of t and T, it is insensitive to the functional form of r and $\partial B(0,u)/\partial u = e^{-au}$. Therefore, we can reexpress (17) :

$$\sigma_P^2 = [\frac{e^{-aT}}{a}(1 - e^{-a(s-T)})]^2 \int_t^T \sigma(u)^2 e^{2au} du \tag{18}$$

From this expression, it turns out that σ_P is unchanged *provided that a and σ are constant.* However, the formula for the swaption holds with σ_P defined as in equation (18) for any sigma being a deterministic function of time: the calibration for the volatility structure is more flexible here than in Hull and White framework, who work with a constant σ. The only requirement left is that a, the speed of adjustment of the interest rate, must remain constant, which is not too disturbing since the functional form of a would be extremely difficult to discover from reality.

4 Determining the Vega

As shown in this section, the process of the interest rate proves to be very useful in determining the vega of the swaption in this context.

We are here interested in determining the vega, denoted Λ, of the swaption, i.e. its sensitivity to a volatility parameter computed by $\frac{\partial C}{\partial \sigma}$. The question is: what volatility parameter should be used?

Two parameters are considered: one is the σ_P computed using Jamshidian's formula (10) and the one of the volatility parameter σ in the Vasicek model (on which σ_P is contingent). Although the application of σ_P looks more direct, the value of σ is in our view fundamental, for three reasons. First, the parameter σ stands for the instantaneous volatility of the interest rate, hence of the true state variable in the model: it is the effect of a change of this standard deviation that is ultimately sought for. Second, if we consider

swaps in practice, it is clear that several payments take place. For a general effect on the swaption, one has to consider volatilities of coupons one by one, whereas the overall effect of σ can be known. Finally, σ is the value that is indeed empirically calibrated (see previous section) using data on the bond market. The value of σ_P, for any bond, involves knowledge of a, which also results from the calibration of the model. Therefore, a variation in σ could be more accurately detected than a variation in σ_P.

4.1 Sensitivity With Respect to σ_P

The determination of Λ with the use of σ_P is straightforward:

$$\Lambda_P = \frac{\partial C}{\partial \sigma_P} = e^{r_x(s-T)} P(r,t,s) N'(d_1)^7 \tag{19}$$

This expression for Λ_P suggests that it is always positive. This is consistent with the conjecture that the option value is positively related to the volatility of the underlying. However, it does not imply that the same sign must be expected for the sensitivity of the swaption (and hence the bond option) with respect to the instantaneous volatility of the interest rate process σ.

4.2 Sensitivity With Respect to σ

The expression of the sensitivity to the volatility parameter of the interest rate process is more complex. However, thanks to the formulas proposed by Jamshidian [8] , this vega has a closed-form solution.

The generic function that we have to compute is:

$$\Lambda = \frac{dC}{d\sigma} = \frac{\partial C}{\partial \sigma_P}\frac{\partial \sigma_P}{\partial \sigma} + \frac{\partial C}{\partial P(r,t,T)}\frac{\partial P(r,t,T)}{\partial \sigma} + \frac{\partial C}{\partial P(r,t,s)}\frac{\partial P(r,t,s)}{\partial \sigma} \tag{20}$$

But it has been shown that:

$$\frac{\partial C}{\partial \sigma_P} = e^{r_x(s-T)} P(r,t,s) N'(d_1) \tag{21}$$

[7]From now on, L will be assumed to be unity.

$$\frac{\partial C}{\partial P(r,t,T)} = -N(d_2) \tag{22}$$

$$\frac{\partial C}{\partial P(r,t,s)} = e^{r_x(s-T)}N(d_1) \tag{23}$$

Moreover, the functional forms of all three intermediary variables is known:

$$\sigma_P = \frac{\sigma(1-e^{-2a(T-t)})^{1/2}(1-e^{-a(s-T)})}{\sqrt{2}\,a^{3/2}}$$

$$P(r,t,T) = \exp[\frac{1}{2}k^2(t,T) - n(r,t,T)]$$

$$P(r,t,s) = \exp[\frac{1}{2}k^2(t,s) - n(r,t,s)]$$

which derivatives with respect to σ are:

$$\frac{\partial \sigma_P}{\partial \sigma} = \frac{(1-e^{-2a(T-t)})^{1/2}(1-e^{-a(s-T)})}{\sqrt{2}\,a^{3/2}} \tag{24}$$

$$\begin{aligned}\frac{\partial P(r,t,T)}{\partial \sigma} = \; & P(r,t,T) \; [\frac{\sigma(4e^{-a(T-t)} - e^{-2a(T-t)} + 2a(T-t) - 3)}{2a^3} \\ & + \frac{\lambda(T-t)}{a} \\ & - \frac{\lambda(1-e^{-a(T-t)})}{a^2}]\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial P(r,t,s)}{\partial \sigma} = \; & P(r,t,s) \; [\frac{\sigma(4e^{-a(s-t)} - e^{-2a(s-t)} + 2a(s-t) - 3)}{2a^3} \\ & + \frac{\lambda(s-t)}{a} \\ & - \frac{\lambda(1-e^{-a(s-t)})}{a^2}]\end{aligned} \tag{26}$$

Expressions (25) and (26) respect Nielsen's [12, p.122] version of $n(r,t,s)$, where $\bar{r} = b - \sigma\lambda/a$, contradicting Jamshidian.

And therefore, plugging (21), (22), (23), (24), (25) and (26) into (20), we can express Λ:

$$\begin{aligned}
\Lambda &= e^{r_x(s-T)}P(r,t,s)N'(d_1)\frac{(1-e^{-2a(T-t)})^{1/2}(1-e^{-a(s-T)})}{\sqrt{2}\,a^{3/2}} \\
&- N(d_2)P(r,t,T)[\frac{\sigma(4e^{-a(T-t)}-e^{-2a(T-t)}+2a(T-t)-3)}{2a^3} \\
&\quad +\frac{\lambda(T-t)}{a}-\frac{\lambda(1-e^{-a(T-t)})}{a^2}] \\
&+ e^{r_x(s-T)}N(d_1)P(r,t,s)[\frac{\sigma(4e^{-a(s-t)}-e^{-2a(s-t)}+2a(s-t)-3)}{2a^3} \\
&\quad +\frac{\lambda(s-t)}{a}-\frac{\lambda(1-e^{-a(s-t)})}{a^2}] \qquad (27)
\end{aligned}$$

Clearly, we can now express vega as $\Lambda(r, r_x, r_0, a, b, \lambda, \sigma, t, T, s)$, which is the expression we were looking for.

It is not obvious that Λ is positive, but we will show that the assumptions of the model lead to the positivity of the coefficient.

4.3 Sign of Λ

In order to find out the sign of Λ, it is useful to interpret $\frac{\partial P(r,t,T)}{\partial \sigma}$. Indeed, using (25), (8) and (9), the formula can be rewritten:

$$\frac{\partial P(r,t,T)}{\partial \sigma} = \frac{P(r,t,T)}{\sigma}[k^2(t,T) - n(b,t,T) + b(T-t)] \qquad (28)$$

In this expression, a new function has appeared: $n(b,t,T)$, which can be interpreted, similarly to (9), as $E_\lambda(Y(t,T) \mid r(t) = b)$. The term $n(b,t,T)$ is thus the risk-adjusted expected yield of a bond if the current interest rate were b, the long run mean of the process. Using the law of iterated expectations, it can be viewed as the unconditional risk-adjusted expected yield of the bond.

The terms between brackets are independent of r. The two first terms, $k^2(t,T)$ and $n(b,t,T)$ represent the log of the bond price $P(b,t,T)$ in the

Vasicek model with $\frac{1}{2}k^2(t,T)$ added to it. The third term adds the mean interest rate computed from t to T.

We are thus able to write the elasticity of the bond price with respect to the volatility parameter, denoted η.

$$\eta(t,T) = \frac{\partial P(r,t,T)}{\partial \sigma}\frac{\sigma}{P(r,t,T)} = k^2(t,T) - n(b,t,T) + b(T-t) \tag{29}$$

This important function does not depend on the current value of r. Therefore, the choice of the maturity of the bond alone defines its elasticity with respect to the diffusion term of the interest rate process. The f.o.c. gives:

$$\frac{\partial \eta(t,T)}{\partial (T-t)} = \frac{\sigma^2}{a^2}(1 - 2e^{-a(T-t)} + e^{-2a(T-t)}) + (b-\bar{r})(1 - e^{-a(T-t)}) \tag{30}$$

At $T = t$, this function has value 0, and the second order derivative is:

$$\frac{\partial^2 \eta(t,T)}{\partial (T-t)^2} = \frac{2\sigma^2}{a}(e^{-a(T-t)} - e^{-2a(T-t)}) + a(b-\bar{r})e^{-a(T-t)}$$

This second derivative is positive provided that $a > 0$ and $b > \bar{r}$, which is the case if λ is positive. This means that the elasticity is an increasing, convex function of T.

Now consider the following transformation of the vega formula (27):

$$\begin{aligned}
\Lambda &= \Lambda_P \frac{\partial \sigma_P}{\partial \sigma} \\
&- N(d_2)P(r,t,T)\frac{1}{\sigma}(k^2(t,T) - n(b,t,T) + b(T-t)) \\
&+ e^{r_x(s-T)}N(d_1)P(r,t,s)\frac{1}{\sigma}(k^2(t,s) - n(b,t,s) + b(s-t))
\end{aligned} \tag{31}$$

which can be rewritten, using equation (12) as:

$$\begin{aligned}
\Lambda &= \Lambda_P \frac{\partial \sigma_P}{\partial \sigma} \\
&+ C(r,t,T,s)\frac{\eta(t,s)}{\sigma} \\
&+ N(d_2)P(r,t,T)\frac{\eta(t,s) - \eta(t,T)}{\sigma}
\end{aligned} \tag{32}$$

Since the call price is always positive, and the first term of this equation too, we see that a set of sufficient condition for Λ to be positive is: $\eta(t,s) \geq 0$ and $\eta(t,s) \geq \eta(t,T)$, which is obviously true since the elasticity is 0 when $T = t$, is increasing in T and $s \geq T$. We can thus conclude that vega is positive. The negative effect of the interest rate volatility on the discount factor is thus more than offset by a set of three positive effects displayed in the last equation. The first represents the direct effect that σ has on the bond volatility. The second one is the effect due to the volatility value of the option. The third one is the "striking effect": exercising the option gives the right to receive fixed, and thus not be exposed to volatility risk from T to s.

5 Caps versus Swaptions

The interest rate swaption is not at all constrained to be the option to enter a swap agreement exchanging fixed for floating; the reverse can obviously occur. In this case, the holder has a put option on the bond, which can be valued using the put-call parity:

$$\Pi(r,t,T,s,L) = LP(r,t,T)N(-d_2) - Le^{r_x(s-T)}P(r,t,s)N(-d_1) \qquad (33)$$

Considering this option as an option on the interest rate leads to expressing it as a call option.

For a cap, the story is completely different. The cap allows its holder to pay never more than a predetermined level of the interest rate on a bond, even if the LIBOR rate goes higher than this threshold. The floor on LIBOR is the symmetric of the cap, in that it allows a lender to never get less than a given level of the LIBOR. The collar is a portfolio of the two options.

The main difference, in essence, between a cap on LIBOR and a swaption on the same interest rate lies in the timing of the physical settlement: for a swaption, as we have seen in the previous sections, the maturity date coincides with the possibility of unfolding the operation on the swap market. The holder of the swaption has thus payoff at time T, the striking date of the option.

At time T, the cap is exercised if the LIBOR rate for loans running from T to s is higher (or lower for a floor) than the exercise rate. Hence, this is a call

option that is directly written on an interest rate. More precisely, exercising a cap gives the right to a yield on a bond running from T to s. The physical settlement is not a bond, but a cash payment at maturity of the bond. Since the yield is not a traded asset, it is not possible to go to the bond market at time T.

The distinction is very important when one refers to the meaning of the Black-Scholes approach. The whole development of the Black-Scholes analysis rests on an arbitrage argument, assuming that the option can be perfectly replicated by traded assets. In the context of caps, there is no asset called "yield" that is traded on the market. However, there is a well-known analytical relationship between bond prices, spot interest rate and bond yield. We intend to use it in order to price options on yields. The interesting feature of this comes from the fact that computing a hedge ratio in terms of yields allows to determine the hedge ratio in terms of bonds.

The methodology which Hull [6] suggests to apply to caps on LIBOR is, as he remarks, very approximative. This comes from the fact that the payoff of the cap is discounted to the decision date by using the forward rate at the exercise date, which does not take into account the fact that the payoff appears later than this date.

6 Caps and Floors as Options on Yields

The case of options on yields has been studied by Longstaff [10], but only under the assumption that the interest rate follows a CIR process. The theoretical analysis carried out in his paper can be adapted, with results considerably simplified, to the case of the Vasicek model. The usefulness of this exercise is to become able to express the cap and the floor as options on yields in a proper manner.

As Longstaff does, it is useful to begin by considering the call option on the yield (i.e. the cap on LIBOR), and then use a proper put-call parity relationship in order to determine the formula for a put option (i.e. the floor). The analysis is carried out under the original Vasicek process, and can be easily extended to the modified model by Hull and White [7].

As it has been mentioned in section 2, the exercise rule for a bond option can be stated as follows (for a put on the bond): exercise when $P \leq e^{-r_x(s-T)}$. Since P can be reexpressed in terms of yield as $P = e^{-Y(T,s)(s-T)}$ where $Y(T,s)$ is the instantaneous yield at time T of a zero-coupon bond maturing at s, we have that the option is exercised if $Y(T,s) \geq r_x$. This is the payoff of a call option on the yield. This is the starting point of the subsequent analysis.

Under the Ornstein-Uhlenbeck process, the bond pricing formula can be written as in (3). However, in terms of bond yield, we can also define P as:

$$P(r, t_1, t_2) = e^{-Y(t_1,t_2)(t_2-t_1)} \tag{34}$$

Combining (3) and (34), defining τ as $t_2 - t_1$, we have that

$$Y(\tau) = \alpha(\tau) + \beta(\tau)r(t_1) \tag{35}$$

with

$$\beta(\tau) \equiv \frac{B(\tau)}{\tau} = \frac{1-e^{-a\tau}}{a\tau} \tag{36}$$

$$\alpha(\tau) \equiv \frac{1}{\tau}\log\frac{1}{A(\tau)} = -\frac{1}{\tau}[\frac{(B(\tau)-\tau)(a^2\bar{r}-\sigma^2/2)}{a^2} - \frac{\sigma^2 B(\tau)^2}{4a}] \tag{37}$$

The call on the yield, which matures at time T, has a payoff function of $\max[0, Y(\tau) - r_x]$. In order to price this derivative security, we need the following theorem, which is directly adapted from the separation theorem proposed by Longstaff [10], that allows to express the value of the claim as the risk-adjusted expected payoff at time T discounted to t:

Theorem 1 *Let $U(Y(\tau))$ denote the payoff function for a contingent claim on $Y(\tau)$ maturing at T. Let $P(r,t,T)$ denote the value under a Ornstein-Uhlenbeck process for the instantaneous interest rate of a pure discount bond maturing at T. Then the value of the claim can be written as*

$$P(r,t,T)E_\lambda[U(Y(\tau))] \tag{38}$$

where the expectation is taken with respect to $Y(\tau)$ which is distributed as

$$\alpha(\tau) + \beta(\tau)N(m-q, v^2) \tag{39}$$

and where $N(m-q, v^2)$ is a normal variate, with

$$\begin{aligned} m &= E_\lambda[r(T) \mid r(t)] = e^{-a(T-t)}r(t) + (1-e^{-a(T-t)})\bar{r} \\ q &= cov_\lambda[r(T), Y(T-t) \mid r(t)] = \frac{\sigma^2}{a^2}(1-e^{-a(T-t)}) - \frac{\sigma^2}{2a^2}(1-e^{-2a(T-t)}) \\ v^2 &= Var_\lambda[r(T) \mid r(t)] = \frac{\sigma^2}{2a}(1-e^{-2a(T-t)}) \end{aligned}$$

Proof: In order to price any derivative security $U(r(t), T)$ maturing at time T whose payoff is $F(Y(\tau)) = F(\alpha(\tau) + \beta(\tau)r(t))$, where $\tau = s - T$, we notice that this security must satisfy the following usual partial differential equation (PDE):

$$-U_T + a(\bar{r} - r(t))U_r + \frac{1}{2}\sigma^2 U_{rr} - rU = 0 \tag{40}$$

where the subscripts represent partial derivatives with respect to the indicated arguments. If we rewrite $U(r(t), T) = P(r, t, T)G(r, T)$, we observe that $U_T = P_T G + PG_T$, $U_r = P_r G + PG_r$ and $U_{rr} = P_{rr}G + 2P_r G_r + PG_{rr}$. We can thus rewrite equation (40) as:

$$\begin{aligned} 0 &= G\left[\ -P_T + a(\bar{r} - r(t))P_r + \tfrac{1}{2}\sigma^2 P_{rr} - rP\ \right] \\ &+ P\left[\ -G_T + [\sigma^2 \tfrac{P_r}{P} + a(\bar{r} - r(t))]G_r + \tfrac{1}{2}\sigma^2 G_{rr}\ \right] \end{aligned}$$

Since P satistifies the same PDE as U, and noting that $P_r/P = -B(t, T)$, we have the following condition for G:

$$-G_T + [-B(t,T)\sigma^2 + a(\bar{r} - r(t))]G_r + \frac{1}{2}\sigma^2 G_{rr} = 0 \tag{41}$$

From Friedman's [5] theorem 5.2,

$$U(r(t), T) = P(r, t, T)E_\lambda[F(Y(\tau)] \tag{42}$$

where the expectation is taken with respect to the following process:

$$dr(t) = \left[\ a\left(\ \bar{r} - \tfrac{\sigma^2 B(t,T)}{a}\ \right) - ar(t)\ \right]dt + \sigma dW(t) \tag{43}$$

The form of equation (43) is exactly the same one as in the original risk-adjusted process (2), except that now $\bar{\bar{r}} = \bar{r} - \frac{\sigma^2 B(t,T)}{a}$. This differential

equation suggests that $r(T)$ is normally distributed conditionally on $r(t)$. Thus, we can immediately compute $E[r(T) \mid r(t)]$ and $var[r(T) \mid r(t)]$ under this process:

$$\begin{aligned} E[r(T) \mid r(t)] &= e^{-a(T-t)}r(t) + (1 - e^{-a(T-t)})(\bar{r} - \frac{\sigma^2}{a^2}) \\ &\quad + \frac{\sigma^2(1 - e^{-2a(T-t)})}{2a^2} \\ &\equiv m - q \\ var[r(T) \mid r(t)] &= \frac{\sigma^2}{2a}(1 - e^{-2a(T-t)}) \\ &\equiv v^2 \end{aligned}$$

Defining $Y(\tau) = \alpha(\tau) + \beta(\tau)r(T)$ completes the proof. □

This theorem has to be compared to the usual formulas for the forward yield in the Vasicek model: the instantaneous forward rate at time t, $f(r, t, T)$ is itself normally distributed with mean $m - q$ and variance v^2 (see Jamshidian [8]). This is a result corresponding to Longstaff's [10] formula computed for the CIR process.

Thanks to theorem 1, we now have the tool to construct the option pricing formula: since $Y(\tau) \rightsquigarrow N(\alpha(\tau) + \beta(\tau)(m - q), \beta(\tau)^2 v^2)$, we have:

$$C_Y = P(r, t, T)[E_\lambda[Y(\tau)1_{Y(\tau)\geq r_x}] - r_x \Pr(Y(\tau) \geq r_x)] \tag{44}$$

The expectation in the first term of the bracket is indeed the expectation of a truncated normally distributed random variable (see Judge et al. [9]):

$$E[Y(\tau)1_{Y(\tau)\geq r_x}] = \alpha(\tau)N(-d) + \beta(\tau)(m - 2q)N(-d) + \beta(\tau)vN'(d) \tag{45}$$

where $d = \frac{r_x - (\alpha(\tau) + \beta(\tau)(m-q))}{\beta(\tau)v}$

The probability in the second term is straightforward:

$$\Pr(Y(\tau) \geq r_x) = N(-d) \tag{46}$$

Combining (6) and (46) into (44) gives the formula for a call on the yield under a gaussian model of the interest rate:

$$\begin{aligned} C_Y &= P(r, t, T)[\alpha(\tau)N(-d) + \beta(\tau)(m - q)N(-d) + (\beta(\tau)v)N'(d) - r_x N(-d)] \\ d &= \frac{r_x - (\alpha(\tau) + \beta(\tau)(m - q))}{\beta(\tau)v} \end{aligned}$$

For a cap on LIBOR, this call price can simply be multiplied by the principal and the period τ to give the value of the capped position.

The formula for the floor on LIBOR is the one of a put on a yield. However, the usual put-call parity equation does not work in the context of options on yields, basically because the yield is not a traded security. Hence, in order to price the floor on LIBOR, we need the customized put-call parity proposed by Longstaff [10]:

$$\Pi(Y(\tau), T, r_x) = C(Y(\tau), T, r_x) + r_x P(r, t, T) - C(Y(\tau), T, 0) \tag{47}$$

This, because the value of a portfolio that pays off $Y(\tau)$ at time T is not the current value of $Y(\tau)$ but a call option on $Y(\tau)$ with a strike price of 0[8].

This view improves on the mere application of Black-Scholes formula on the interest rate, which is indeed an incorrect way to price the swaption. Considering the cap as an option on a yield allows to derive a precise formulation. The advantage of this approach is that the future physical settlement, which happens at time s, is directly integrated into the yield of the bond. The analysis that we propose here works on the "payoff" at time T, whose discounting back to time t can be soundly performed in spite of the fact that it is not actually a traded asset.

Interestingly, the hedge ratio in terms of yields can be perfectly justified, since we know the functional relationship between P and r from equation (3): with formula (47), the delta of the option can be computed in terms of r, and the one-to-one relationship with bonds can be used in order to find the position that hedges the call.

The extension to the fit of the initial term structure of interest rate proceeds in the same way as before; so is it also for the case of several cap payments, because the portfolio of caps is indeed equivalent to a portfolio of options on yields. The difference with swaptions, however, is that exercising a cap does not imply that all other caps are exercised. Therefore, the value of a portfolio of caps is equal to the sum of the values of the individual caps. Here, the argument is thus different of the one for swaptions: the portfolio of swaptions is indeed equivalent to a single option on a portfolio of bonds with

[8]For a deeper analysis of the properties of options on yields, see Longstaff [10]: most of his results apply here.

different maturities; the portfolio of caps is a porfolio of options on yields on the same periods (because the LIBOR rate is usually checked periodically), but with different striking dates. There is no question of a single option here.

7 Conclusion

This paper brings additional applications to the work proposed by Jamshidian and Longstaff. The swaption is explicitly defined as an option on a bond, and is priced accordingly. There is no substantial theoretical addition to the Vasicek model, but we carefully analyze some implications for bond options and we introduce options on yields in this framework. We also show how the initial interest rate and volatility structures can be added in the model. The sensitivity of the bond option with respect to the volatility parameter in the Gaussian model of the interest rate is considered, with a particular stress on the elasticity of the bond.

Our main concern was to outline the difference in essence that exists between swaptions and caps, and how the latter can be properly handled. Expressing the cap and the floor as options on yields allows to correct for the usual approach of options on interest rates; moreover, it allows to express the option as the discounted expectation of a truncated normal variable. This work can be taken both as a technical tool for practitioners and as a scientific contribution to the field. Furthermore, it sets the basis for a further inquiry in the domain of swaps, for which default risk still needs to be rigorously modeled, especially because it involves bilateral risk.

Finally, it is especially interesting to notice that valuing swaptions and caps as options on interest rates is not a good procedure, but the alternatives proposed here are very different depending on the nature of the option. This stresses that there might be various reasons for which the "blind" use of the Black formula is likely to be inadequate.

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