DIFFERENTIABLE UTILITY

by

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97/28/FIN

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February 1997

Abstract

Differentiability is an important property of von Neumann-Morgenstern utility functions which is almost always imposed but has not been translated into behavioral terms is differentiability. In applications, expected utility is usually maximized subject to a constraint, and the maximization is carried out by differentiating the utility function. This paper presents two sets of necessary and sufficient conditions for a risk averse von Neumann-Morgenstern utility function to be differentiable. The first of them is formulated in terms of the equivalent risk premia of small gambles. It says, in brief, that the equivalent risk premium is of a smaller order of magnitude than the risk itself, as measured by the expectation of the absolute value of the risk. The second set of necessary and sufficient conditions is formulated in terms of the probability premium of small lotteries. It says, essentially, that the probability premium for small binary lotteries goes to zero as the size of the lottery goes to zero.

1 Introduction

This paper provides necessary and sufficient conditions for a risk averse von Neumann-Morgenstern utility function to be differentiable.

After von Neumann and Morgenstern (1947) had axiomatized expected utility, it quickly became the dominant paradigm for modeling decision making under uncertainty. It has remained so ever since, even though the late 1980's and early 1990's have seen a considerable interest in alternatives.

There is an extensive literature about axiom systems for expected utility. This literature seeks to formulate the expected utility hypothesis in behavioral terms by showing that it is equivalent to a series of simple and appealing properties of preferences or of choice behavior. The scope of axiom systems has been extended successively from preferences over simple distributions of outcomes (distributions with a finite number of possible outcomes) to continuous and other more general distributions, and from bounded to unbounded utility functions. See the books by Fishburn (1970,1982) and Wakker (1989).

However, when von Neumann-Morgenstern utility functions are used in practice, certain properties are usually imposed on them in addition to the assumption that they exist. Some of these properties are easily understood in terms of preferences or choice behavior. This is the case, for example, of monotonicity and concavity. The observation that concavity of the utility function corresponds to risk aversion goes back at least to Friedman and Savage (1948).

An important property of the utility function which is almost always imposed but has not been translated into behavioral terms is differentiability. In applications, expected utility is usually maximized subject to a constraint, and the maximization is carried out by differentiating the utility function. It is therefore important to know which behavioral assumptions are implicitly imposed on the underlying preferences and choices when the utility function is assumed to be differentiable.

This paper presents two sets of necessary and sufficient conditions for a risk averse von Neumann-Morgenstern utility function to be differentiable. The first of them is formulated in terms of the equivalent risk premia of small gambles. It says, in brief, that the equivalent risk premium is of a smaller order of magnitude than the risk itself, as measured by the expectation of the absolute value of the risk. The second set of necessary and sufficient conditions is formulated in terms of the probability premium of small lotteries. It says, essentially, that the probability premium for small binary lotteries goes to zero as the size of the lottery goes to zero.

We also show that even if the utility function is not risk averse, under a number of alternative regularity condtions, differentiability still implies that the equivalent risk premium of a small risk is of a smaller order of magnitude than the risk itself, and the probability premium of a small binary lottery goes to zero as the size of the lottery goes to zero.

2 The Risk Premium

We consider a decision model where the relevant set of outcomes is an open interval I on the real line. The outcomes may be interpreted as levels of future wealth or future consumption. Typically, $I = \mathbb{R}$ or $I = (0, \infty)$.

The utility function u will be assumed to be strictly increasing, reflecting the idea that more consumption or more wealth is better.

In order to define the equivalent risk premium of a random addition to wealth and state precisely, and preferably in behavioral terms, the conditions under which it exists and is unique, it is convenient first to define the certian equivalent.

If x is an initial wealth level and z is a random addition to wealth, such that $x + z \in I$ with probability one and $u(x + z)$ is integrable, the *certain equivalent* $c(x, z)$ of z at the initial wealth level x is that sure addition to wealth which yields a utility equal to the expected utility of $x + z$:

$$u(x + c(x, z)) = Eu(x + z)$$

Because u is strictly increasing, the certain equivalent is unique if it exists.

We want the certain equivalent to exist for every x and every z such that $x + z \in I$ with probability one and $u(x + z)$ is integrable. It is easily seen that it is sufficient to assume that the certain equivalent exists for binary lotteries z, and this assumption is equivalent to continuity of u.

A random variable z will be called a *binary lottery* if it has at most two distinct values.

Recall that the utility function u is said to be *risk averse* if

$$Eu(x+z) \leq u(x)$$

whenever x is in I and z is a binary lottery with $Ez \leq 0$ and such that $x + z \in I$ with probability one. It is well known that u is risk averse if and only if it is concave. But if it is concave, then it is continuous. So we can conclude that if u is risk averse, then certain equivalents exist.

If x is an initial wealth level and z is an integrable random addition to wealth such that $x+z \in I$ with probability one and $u(x+z)$ is integrable, define the *equivalent risk premium* $\pi(u)(x, z)$ or $\pi(x, z)$ of z at the initial wealth level x by

$$u(x + Ez - \pi(x, z)) = Eu(x + z)$$

which is equivalent to

$$\pi(x, z) = Ez - c(x, z)$$

The equivalent risk premium is unique if it exists. The equivalent risk premium exists for every initial wealth level x and every integrable random addition to wealth z such that $x + z \in I$ with probability one and $u(x + z)$ is integrable, if and only if u is continuous.

Pratt (1964) showed that the equivalent risk premium characterizes the degree of risk aversion, in the sense that one utility function is more risk averse than another if and only if it always requires a higher equivalent risk premium.

Theorem 1 below characterizes differentiability of a risk averse utility function at a given point x. It is a behavioral characterization in the sense that it translates differentiability into statements about the equivalent risk premium. These statements can be directly interpreted in terms of preferences and choice behavior. They say that for small risks or gambles, the equivalent risk premium is of a smaller order of magnitude than the risk itself, as measured by the expectation of the absolute value of the risk.

Let $0 < s < 1$. A random variable z is a *binary s-lottery* if it has only two distinct possible values a, b, with $a > b$ and $p(\{z = a\}) = 1 - p(\{z = b\})$.

Theorem 1 *Let u be a strictly increasing and risk-averse utility function on I. Let x be a point in I and let $0 < s < 1$. The following three statements are equivalent.*

1. *If (z_n) is a sequence of binary s-lotteries with zero mean such that for all n, $x + z_n \in I$ with probability one and $E|z_n| \neq 0$, and such that $\|z_n\|_\infty \to 0$, then*
$$\frac{\pi(x, z_n)}{E|z_n|} \to 0$$
2. *u is differentiable at x*
3. *If (x_n) is a sequence in I such that $x_n \to x$, and if (z_n) is a sequence of integrable random variables such that for all n, $x_n + z_n \in I$ with probability one, $u(x_n + z_n)$ is integrable, and $E|z_n| \neq 0$, and such that $\|z_n\|_\infty \to 0$, then*
$$\frac{\pi(x, z_n)}{E|z_n|} \to 0$$

All proofs are in Section 4.

All three statements in Theorem 1 are equivalent, but (1) has been designed as a sufficient condition which appears as weak as possible, while (3) appears as a strong necessary condition. In (1), to make the sufficient condition weak, convergence of the ratio of the risk premium to the expected absolute value of z_n is assumed only for sequences (z_n) of binary s-lotteries with zero mean, with a fixed probability s, and only at the fixed initial wealth level x. However, if indeed u is differentiable at x, then it follows from (3) that convergence holds for more general sequences (z_n) of random variables, and not only at a fixed initial wealth level x but also when the initial wealth is not fixed but converges to the fixed level x.

Even if u is not risk averse, if u is differentiable at x, it will typically still be true that the equivalent risk premium per unit of risk (measured as expectation of absolute value) goes to zero as the amount of risk goes to zero. Several sets of regularity conditions under which this is true are spelled out in Proposition 1 below.

In (1) of Proposition 1, the initial wealth level is fixed. Statement (2) is a part of Theorem 1 and will in fact be used in the proof of the theorem. In (3), the initial wealth level is not fixed, but u is assumed to be twice continuously differentiable.

In Theorem 1 and in (1)–(3) of Proposition 1, the risks z_n converge to zero in the sense that $\|z_n\|_\infty \to 0$. According to (4) of Proposition 1, if u is not only twice continuously differentiable but the second derivative is bounded, then we can allow the risks z_n to converge to zero in a weaker sense.

Proposition 1 *Let u be a strictly increasing continuous utility function on I. Let x be a point in I. Let (x_n) be a sequence in I such that $x_n \to x$. Let (z_n) be a sequence of integrable random variables such that for all n, $x_n + z_n \in I$ with probability one and $E|z_n| \neq 0$. Suppose one of the following four sets of conditions holds:*

1. *u is differentiable at x with $u'(x) > 0$, $x_n = x$ for all x, and $\|z_n\|_\infty \to 0$*
2. *u is differentiable at x and concave, and $\|z_n\|_\infty \to 0$*
3. *u is twice continuously differentiable with $u'(x) > 0$, and $\|z_n\|_\infty \to 0$*
4. *u is twice continuously differentiable with $u'(x) > 0$ and u'' bounded, and the variables z_n are square integrable with*

$$\frac{E|z_n^2|}{E|z_n|} \to 0$$

Then

$$\frac{\pi(x_n, z_n)}{E|z_n|} \to 0$$

In Proposition 1, if $Ez_n = 0$ for all n, then we can also conclude that

$$\frac{\pi(x_n, z_n)}{\sigma(z_n)} \to 0$$

This follows from the fact that, by Jensen's inequality,

$$\sigma(z_n) = \sqrt{E(z_n^2)} \geq E|z_n|$$

3 The Probability Premium

If x is an initial wealth level and $h > 0$ is a number such that $x-h, x+h \in I$, define the *probability premium* $p(u)(x,h)$ or $p(x,h)$ of h at x as the difference between the probability of the good outcome $x+h$ and the probability of the bad outcome $x-h$ required in a symmetric binary lottery between $x+h$ and $x-h$ if the decision maker is to be indifferent between the lottery and the initial wealth level x:

$$u(x) = \frac{1}{2}[1 + p(x,h)]u(x+h) + \frac{1}{2}[1 - p(x,h)]u(x-h)$$

Pratt (1964) showed that the equivalent risk premium characterizes the degree of risk aversion, in the sense that if u and v strictly increasing and continuous utility functions on I, then u is more risk averse than v if and only if for every initial wealth level x and every number $h > 0$ such that $x-h, x+h \in I$, $p(u)(x,h) \geq p(v)(x,h)$.

Our definition of the probability premium $p(x,h)$ corresponds to that of Pratt. The function studied by Arrow (1965) is, in our notation,

$$p[x+h, x, x-h] = [p(x,h) + 1]/2$$

Theorem 2 is a second characterization of differentiability of a risk averse utility function at a given point x. It is a behavioral characterization in the sense that it translates differentiability into statements about the probability premium. These statements can be directly interpreted in terms of preferences and choice behavior. They say that the probability premium for small binary lotteries goes to zero as the size of the lottery goes to zero.

Theorem 2 *Let u be a strictly increasing, risk averse utility function on I. Let x be a point in I. The following three statements are equivalent.*

1. *$p(x,h) \to 0$ as $h \to 0$, $h > 0$*
2. *u is differentiable at x*
3. *$p(y,h) \to 0$ as $y \to x$ and $h \to 0$, $h > 0$*

All three statements in Theorem 2 are equivalent, but (1) has been designed as a sufficient condition which appears as weak as possible, while (3) appears as a strong necessary condition. In (1), to make the sufficient condition weak, convergence of the probability premium to zero is assumed only at the fixed initial wealth level x. However, if indeed u is differentiable at x, then it follows from (3) that convergence holds also when the initial wealth is not fixed but converges to the fixed level x.

Even if u is not risk averse, if u is differentiable at x, it will typically still be true that the probability premium goes to zero as the size of the lottery goes to zero. Several sets of regularity conditions under which this is true are spelled out in the Proposition 2 below.

In (1) of Proposition 2, the initial wealth level is fixed. Statement (2) is a part of Theorem 2 and will in fact be used in the proof of the theorem. In (3), the initial wealth level is not fixed, but u is assumed to be twice continuously differentiable.

Proposition 2 *Let u be a strictly increasing continuous utility function on I. Let x be a point in I.*

1. *If u is differentiable at x with $u'(x) > 0$, then $p(x,h) \to 0$ as $h \to 0$, $h > 0$.*
2. *If u is differentiable at x and concave, then $p(y,h) \to 0$ as $y \to x$ and $h \to 0$, $h > 0$.*
3. *If u is twice continuously differentiable with $u'(x) > 0$, then $p(y,h) \to 0$ as $y \to x$ and $h \to 0$, $h > 0$.*

4 Proofs

We prove Proposition 1 first because it is used in the proof ot Theorem 1.

PROOF OF PROPOSITION 1:

First, we show that $c(x_n, z_n) \to 0$.

Let $u^-(y)$ denote the left derivative of u at y. In case (1), it is defined at $y = x$, and $u^-(x) = u'(x)$. In case (2), $u^-(y)$ is defined at all interior points y of I, and it is continuous at $y = x$ with $u^-(x) = u'(x)$. In cases (3) and (4), $u^-(y)$ is defined at all interior points y of I, and it is continuous with $u^-(y) = u'(y)$.

Define the function $B_1(y, h)$ by

$$B_1(y,h) = \frac{u(y+h) - u(y) - u^-(y)h}{h^2}$$

when $h \neq 0$ and

$$B_1(y,h) = 0$$

when $h = 0$. In case (1), it is defined for $y = x$ and for all h such that $x + h \in I$. In cases (2)–(4), it is defined at all interior points $y \in I$ and all h such that $y + h \in I$.

Now,

$$B_1(y,h)h = \frac{u(y+h) - u(y)}{h} - u^-(y)$$

when $h \neq 0$ and

$$B_1(y,h)h = 0$$

when $h = 0$.

In case (1), it is clear that $B_1(x,h)h \to 0$ as $h \to 0$. We will show that $B_1(y,h)h \to 0$ as $(y,h) \to (x,0)$ in cases (2)–(4).

In case (2), because u is concave,

$$u^-(y) \geq \frac{u(y+h) - u(y)}{h} \geq u^-(y+h)$$

when $h > 0$, and

$$u^-(y) \leq \frac{u(y+h) - u(y)}{h} \leq u^-(y+h)$$

when $h < 0$. Hence,

$$|B_1(y,h)h| \leq |u^-(y+h) - u^-(y)| \to 0$$

In cases (3) and (4), Taylor's formula with remainder term says that for every (y, h) such that y and $y+h$ are interior points in I, there exists $t(y,h) \in [0,1]$ such that

$$B_1(y,h) = \frac{1}{2}u''(y + t(y,h)h)$$

Since u'' is continuous,

$$B_1(y,h) \to \frac{1}{2}u''(x)$$

and

$$B_1(y,h)h \to 0$$

as $(y,h) \to (x,0)$.

In cases (1)–(3), given $\epsilon > 0$, since $\|z_n\|_\infty \to 0$, it follows that

$$|B_1(x_n,z_n)z_n| \leq \epsilon$$

with probability one, for sufficiently large n. Hence,

$$|E[B_1(x_n,z_n)z_n^2]| \leq E|B_1(x_n,z_n)z_n^2| \leq \epsilon E|z_n|$$

for sufficiently large n, and

$$\frac{E[B_1(x_n,z_n)z_n^2]}{E|z_n|} \to 0$$

In particular,

$$E[B_1(x_n,z_n)z_n^2] \to 0$$

In case (4), it follows from the Taylor representation of B_1 above that B_1 is globally bounded by some positive constant K. Hence,

$$\frac{E[B_1(x_n,z_n)z_n^2]}{E|z_n|} \leq K\frac{E|z_n^2|}{E|z_n|} \to 0$$

In particular,

$$E\left[B_1(x_n, z_n)z_n^2\right] \to 0$$

In all cases, since $Ez_n \to 0$,

$$\begin{aligned} u(x_n + c(x_n, z_n)) &= Eu(x_n + z_n) \\ &= u(x_n) + (Ez_n)u^-(x_n) + E\left[B_1(x_n, z_n)z_n^2\right] \\ &\to u(x) \end{aligned}$$

Since u is strictly increasing and continuous, it has to be that

$$x_n + c(x_n, z_n) \to x$$

and hence $c(x_n, z_n) \to 0$.

In all cases,

$$Eu(x_n + z_n) - u(x_n) = (Ez_n)u^-(x_n) + E\left[B_1(x_n, z_n)z_n^2\right]$$

On the other hand,

$$\begin{aligned} Eu(x_n + z_n) - u(x_n) &= u(x_n + c(x_n, z_n)) - u(x_n) \\ &= c(x_n, z_n)u^-(x_n) + B_1(x_n, c(x_n, z_n))c(x_n, z_n)^2 \end{aligned}$$

Hence,

$$\begin{aligned} E[B_1(x_n, z_n)z_n^2] &= -\pi(x_n, z_n)u^-(x_n) + B_1(x_n, c(x_n, z_n))c(x_n, z_n)^2 \\ &= -\pi(x_n, z_n)u^-(x_n) \\ &\quad +B_1(x_n, c(x_n, z_n))c(x_n, z_n)(Ez_n - \pi(x_n, z_n)) \\ &= -\pi(x_n, z_n)[u^-(x_n) + B_1(x_n, c(x_n, z_n))c(x_n, z_n)] \\ &\quad +B_1(x_n, c(x_n, z_n))c(x_n, z_n)Ez_n \end{aligned}$$

Divide this equation by $u^-(x_n)E|z_n|$:

$$\begin{aligned} &-\frac{\pi(x_n, z_n)}{E|z_n|}\left[1 + \frac{B_1(x_n, c(x_n, z_n))c(x_n, z_n)}{u^-(x_n)}\right] \\ &\quad + \frac{B_1(x_n, c(x_n, z_n))c(x_n, z_n)}{u^-(x_n)}\frac{Ez_n}{E|z_n|} \\ &= -\frac{E[B_1(x_n, z_n)z_n^2]}{u^-(x_n)E|z_n|} \\ &\to 0 \end{aligned}$$

Since

$$\frac{B_1(x_n, c(x_n, z_n))c(x_n, z_n)}{u^-(x_n)} \longrightarrow 0$$

and

$$\left|\frac{Ez_n}{E|z_n|}\right| \leq 1$$

it follows that

$$\frac{\pi(x_n, z_n)}{E|z_n|} \longrightarrow 0$$

□

PROOF OF THEOREM 1:

(2) implies (3): This follows from Proposition 1.

(3) implies (1): This is obvious.

(1) implies (2):

Note that since u is concave, it has derivatives u^+ and u^- from the right and from the left. Let (h_n) and (k_n) be sequences of positive numbers such that $x + h_n \in I$, $x - k_n \in I$, and $sh_n - (1-s)k_n = 0$ for all n and such that $h_n \longrightarrow 0$ (and hence $k_n \longrightarrow 0$). Let z_n be a sequence of binary lotteries that have value h_n with probability s and $-k_n$ with probability $1-s$. Then each z_n is a binary s-lottery with zero mean, $x + z_n \in I$ with probability one,

$$E|z_n| = sh_n + (1-s)k_n = 2sh_n > 0$$

and

$$\|z_n\|_\infty = \max\{h_n, k_n\} \longrightarrow 0$$

Hence,

$$\frac{\pi(x, z_n)}{E|z_n|} \longrightarrow 0$$

Now,

$$\begin{aligned} 0 &\leq \frac{u(x) - u(x-k_n)}{-k_n} - \frac{u(x+h_n) - u(x)}{h_n} \\ &\leq \frac{(1-s)u(x) - (1-s)u(x-k_n)}{-(1-s)k_n} - \frac{su(x+h_n) - su(x)}{sh_n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{u(x) - (1-s)u(x-k_n) - su(x+h_n)}{sh_n} \\
&= \frac{u(x) - Eu(x+z_n)}{sh_n} \\
&= \frac{u(x) - u(x - \pi(x,z_n))}{sh_n} \\
&= 2\frac{u(x) - u(x-\pi(x,z_n))}{\pi(x,z_n)} \frac{\pi(x,z_n)}{E|z_n|} \\
&\to 0
\end{aligned}
$$

since

$$
\frac{u(x) - u(x - \pi(x,z_n))}{\pi(x,z_n)} \to -u^-(x)
$$

Hence,

$$
0 \leq u^-(x) - u^+(x) \leq \frac{u(x) - u(x-k_n)}{-k_n} - \frac{u(x+h_n) - u(x)}{h_n} \to 0
$$

so $u^-(x) = u^+(x)$ and u is differentiable at x with $u'(x) = u^-(x) = u^+(x) > 0$.

□

We prove Proposition 2 before Theorem 2 because Proposition 2 is used in the proof of Theorem 2.

Proof of Proposition 2:

Recall that

$$
u(y) = \frac{1}{2}[1 + p(y,h)]u(y+h) + \frac{1}{2}[1 - p(y,h)]u(y-h)]
$$

Solving this equation for $p(y,h)$ yields

$$
\begin{aligned}
p(y,h) &= \frac{u(y) - u(y-h) - [u(y+h) - u(y)]}{u(y+h) - u(y-h)} \\
&= \frac{[u(y) - u(y-h)]/h - [u(y+h) - u(y)]/h}{[u(y+h) - u(y)]/h + [u(y) - u(y-h)]/h}
\end{aligned}
$$

In case (1), it follows directly that

$$\begin{aligned}
p(x,h) &= \frac{u(x)-u(x-h)-[u(x+h)-u(x)]}{u(x+h)-u(x-h)} \\
&= \frac{[u(x)-u(x-h)]/h-[u(x+h)-u(x)]/h}{[u(x+h)-u(x)]/h+[u(x)-u(x-h)]/h} \\
&\to 0
\end{aligned}$$

In case (2),

$$u^-(y-h) \geq \frac{u(y)-u(y-h)}{h} \geq u^-(y) \geq \frac{u(y+h)-u(y)}{h} \geq u^-(y+h)$$

Hence,

$$0 \leq \frac{u(y)-u(y-h)}{h} - \frac{u(y+h)-u(y)}{h} \leq u^-(y-h)-u^-(y+h) \to 0$$

while

$$\frac{u(y)-u(y-h)}{h} + \frac{u(y+h)-u(y)}{h} \geq 2u^-(y+h) \to 2u^-(x) > 0$$

Hence, $p(y,h) \to 0$.

In case (3), the same argument as in the proof of Proposition 1, using Taylor's formula with remainder term, shows that

$$\frac{u(y+h)-u(y)}{h} - u'(y) = B_1(y,h)h \to 0$$

and, hence,

$$\frac{u(y)-u(y-h)}{h} - u'(y) = B_1(y,-h)(-h) \to 0$$

So,

$$\frac{u(y+h)-u(y)}{h} \to u'(x)$$

and

$$\frac{u(y)-u(y-h)}{h} \to u'(x)$$

This implies that $p(y,h) \to 0$.

$\square$

PROOF OF THEOREM 2:

(2) implies (3): this follows from Proposition 2.

(3) implies (1): this is obvious.

(1) implies (2):

Note that since u is concave, it has derivatives u^+ and u^- from the right and from the left. Observe as in the proof of Proposition 2 that

$$\begin{aligned} p(x,h) &= \frac{[u(x)-u(x-h)]/h-[u(x+h)-u(x)]/h}{[u(x+h)-u(x)]/h+[u(x)-u(x-h)]/h} \\ &\to \frac{u^-(x)-u^+(x)}{u^+(x)+u^-(x)} \end{aligned}$$

If this limit equals zero then u is differentiable at x with $u'(x) = u^+(x) = u^-(x) > 0$.

$\square$

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