PORTFOLIO SELECTION AND ASSET PRICING WITH DYNAMICALLY INCOMPLETE MARKETS AND TIME-VARYING FIRST AND SECOND MOMENTS

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Portfolio Selection and Asset Pricing with Dynamically Incomplete Markets and Time-Varying First and Second Moments[1]

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Nontechnical Summary

This paper reexamines and simplifies the theory of optimal dynamic portfolio management. We propose a new characterization of those changes in investment opportunities against which investors need to protect themselves. We show how the relevant investment opportunities can be described in terms of the capital market line, or the efficient mean-variance frontier, which are well-known concepts from the static analysis of optimal portfolio management.

The classical analysis of Merton (1973) found that the investor's optimal portfolio can be decomposed into a number of funds. First of all, the investor should hold the so-called growth-optimal portfolio, which is a portfolio constructed so as to maximize the long-run expected rate of return. Second, depending on his risk aversion, he should balance the investment in the growth-optimal portfolio by putting a part of his wealth into a short-term low-risk investment such as a money market account. Finally, he should hold a number of hedge funds to protect himself against the possibility that the investment opportunities in the market may deteriorate.

The investment opportunities in the market can be described in statistical terms by short-term expectations, variances and covariances of the returns to the available securities. Variances are important because it is essential to make a trade-off between risk and return. Covariances carry information about how best to diversify among the various securities. Merton envisioned that investors would hedge against any possible change in these expectations, variances and covariances.

We show that investors will, in fact, at most hedge against changes that affect the global trade-off between risk and return. This trade-off is described by what we call the instantaneous capital market line or mean-variance efficient frontier. It shows, at each instant of time, the maximum expected return the investor can get, as it depends on how much risk he is willing to accept, where risk is measured by the standard deviation of his portfolio.

The capital market line has played a central role in static mean-variance models of optimal portfolio selection, and our analysis shows that it is equally important in a dynamic framework. The instantaneous version of the capital market line is defined like the capital market line in static models, except

that it is based on short-term rather than investment-horizon expectations and risk parameters.

Our point is that there is plenty of scope for changes in the statistical properties of securities returns without any change in the capital market line, and such changes do not give rise to hedging demands. We formulate our result in the form of a two fund separation theorem, which says that if the instantaneous capital market line is constant, then investors will simply hold a possibly time varying combination of two funds that span the capital market line: the riskless asset and the growth-optimal portfolio. We interpret the theorem as saying that if investors deviate from the two fund strategy and hold hedge funds in addition to the riskless asset and the logarithmic portfolio, then they do so to hedge against changes in the capital market line.

The theorem holds even when some risks are not completely hedgeable, and therefore, some contingent claims cannot be replicated through dynamic trading strategies using the existing securities.

Based on the two-fund separation theorem, we argue that the most relevant concept of the investment opportunity set is the capital market line, and therefore changes in the investment opportunity set should be equivalent to changes in the slope or position of the capital market line.

Our two fund separation theorem applies to a single investor who makes some minimal assumptions about the securities markets but does not make any assumptions about how other investors behave. However, if we assume that all investors behave like we describe, then we can aggregate over investors and derive a market-wide capital asset pricing model.

Our capital asset pricing model is a single-factor model similar to the original Sharpe-Lingner CAPM, except that the first and second moments of security returns may change stochastically over time. The single factor is the return to the market. In other words, the model predicts that the expected return to each security, in excess of the short-term riskless interest rate, depends on its covariance or beta with respect to the market portfolio.

Our version of the CAPM differs from the consumption CAPM of Breeden (1979) in that we do not assume all risks to be perfectly hedgeable. Furthermore, the single factor in our model is the return to the market portfolio, whereas in the consumption CAPM, it is aggregate consumption.

In equilibrium, the market portfolio of risky assets plots on the capital market line, and constancy of the capital market line translates into a linear relation between the mean and the standard deviation of the excess return to the market portfolio. This specification was one of the empirical models proposed by Merton (1980) for estimating the expected return to the market, and it is also consistent with some of the autoregressive conditional heteroscedastic in mean (ARCH-M) and generalized ARCH-M (GARCH-M) specifications that have been used more recently in the empirical literature.

Abstract

In Merton's (1973) intertemporal portfolio selection and capital asset pricing model, investors hold as many hedge funds as there are state variables that drive changes in the first and second moments of asset returns. We simplify Merton's fund separation theorem to show that investors' optimal portfolio holdings will include only hedge funds that hedge against changes in the slope or the position of the instantaneous capital market line. If the capital market line is constant, then investors will hold only the riskless asset and the logarithmic portfolio. This result implies that the capital market line is as important in dynamic analysis as it is in static analysis. The result does not assume that the securities prices are functions of a Markovian vector of state variables. We allow the markets to be dynamically incomplete in the sense that the number of sources of uncertainty is larger than the number of risky assets. Based on our result, we redefine the concepts of the investment opportunity set and changes in the investment opportunity set in terms of the capital market line. Even though our result does not assume equilibrium, we can also aggregate over investors and derive a single factor CAPM with a constant instantaneous capital market line, where the first and second moments of security returns may change stochastically over time and markets are potentially incomplete. This model is consistent with some of the autoregressive conditional heteroscedastic in mean (ARCH-M) and generalized ARCH-M (GARCH-M) specifications that have been used recently in the empirical literature. Our version of the CAPM differs from the consumption CAPM not only by allowing capital market incompleteness but also by the fact that the single factor is the return to the market portfolio rather than aggregate consumption. The model resolves the paradox of Rosenberg and Ohlson (1976) because unlike other single-factor dynamic versions of the Sharpe-Lintner CAPM, it does not imply that the value weights in the market portfolio are constant. The market portfolio will always be proportional to the logarithmic portfolio, but as means, variances and covariances change, so do the value weights in the logarithmic portfolio.

1 Introduction

This paper reexamines the definition and nature of the so-called investment opportunity set and the role of hedge funds in intertemporal portfolio selection and capital asset pricing models with stochastically time-varying first and second moments of securities returns and dynamically incomplete markets.

In the intertemporal portfolio selection model of Merton (1973) "a sufficient set of statistics for the [investment] opportunity set at a given point in time is $\{\alpha_i, \sigma_i, \rho_{ij}\}$" where α_i, σ_i denote the mean and standard deviation of the instantaneous rate of return to security i, and ρ_{ij} the correlation coefficient between instantaneous rates of return to securities i and j. Furthermore, "the dynamics for the changes in the opportunity set over time" are given by a set of Itô processes that describe changes in α_i, σ_i, and potentially ρ_{ij}, and which, together with the securities price processes, form a Markovian vector of state variables[1].

According to this definition, any change in means, variances or covariances is sufficient to generate a change in the investment opportunity set. Merton's formulation suggests that investors will hedge all such changes by including in their optimal portfolio holdings as many hedge funds as there are state variables that describe the dynamics of the returns.

We show that investors will, in fact, at most hedge against changes in instantaneous means, variances and covariances that affect the slope or position of what we call the instantaneous capital market line. This leaves plenty of room for other changes in the first and second moments which do not affect the capital market line and do not give rise to hedge demands.

The capital market line has played a central role in static mean-variance models of optimal portfolio selection and equilibrium, and our analysis shows that it is equally important in continuous time models. By the capital market line we mean the efficient frontier in the presence of a riskless asset, which has intercept equal to the riskless rate and is tangent to the efficient frontier of risky assets only. Since in most of our analysis, we do not impose equilibrium, it is not necessarily true that the tangency point corresponds to the

[1] see page 483 in Merton (1992)

market portfolio. Only in equilibrium do we know that this is the case. The instantaneous version of the capital market line is defined like the capital market line in static models, except that it is based on instantaneous rather than finite horizon means, variances and covariances.

We formulate our result in the form of a two fund separation theorem, which says that if the instantaneous capital market line is constant, then investors will simply hold a possibly time varying combination of two funds that span the capital market line: the riskless asset and the logarithmic portfolio. This theorem holds in a general framework with securities whose rates of return have stochastically time varying first and second moments which, unlike in Merton (1973), do not have to be functions of a Markovian vector of state variables.

The theorem holds even under market incompleteness along the lines of He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). Markets may be dynamically incomplete because the number of sources of uncertainty may be larger than the number of risky assets. In such a framework, some risks are not completely hedgeable, and therefore, some contingent claims cannot be replicated through dynamic trading strategies using the existing securities. We do not need to use the device of completing the markets by introducing fictitious securities like He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991).

Based on the two-fund separation theorem, we argue that the most relevant concept of the investment opportunity set is the capital market line, and therefore changes in the investment opportunity set should be equivalent to changes in the slope or position of the capital market line.

Even though our two fund separation theorem assumes that the capital market line is constant, its importance does not depend on the hypothesis that this is empirically true. We interpret the theorem as saying that *if* investors deviate from the two fund strategy and hold hedge funds in addition to the riskless asset and the logarithmic portfolio, then they do so to hedge against changes in the capital market line.

Like other mutual fund separation results, our two fund separation theorem is not an equilibrium result, since it does not assume market clearing. It applies to a single investor who makes some minimal assumptions about

the securities markets but does not make any assumptions about how other investors behave.

The two fund separation theorem allows us subsequently to aggregate over investors and derive a single factor CAPM where the first and second moments of security returns may change stochastically over time and markets are potentially incomplete.

Our version of the CAPM differs from the consumption CAPM of Breeden (1979) in that the consumption CAPM assumes complete markets whereas our model does not. Furthermore, the single factor in our model is the return to the market portfolio, whereas in the consumption CAPM, it is aggregate consumption.

In equilibrium, the market portfolio of risky assets plots on the capital market line, and constancy of the capital market line translates into a linear relation between the mean and the standard deviation of the excess return to the market portfolio. This specification was one of the empirical models proposed by Merton (1980) for estimating the expected return to the market, and it is also consistent with some of the autoregressive conditional heteroscedastic in mean (ARCH-M) and generalized ARCH-M (GARCH-M) specifications that have been used more recently in the empirical literature.

Finally, the model resolves the paradox of Rosenberg and Ohlson (1976). The paradox can be restated as follows in the present context. If the interest rate and the means, variances and covariances of returns to the risky securities are constant, then the Sharpe-Lintner CAPM implies that the value weights in the market portfolio are constant. In the absence of random changes in the supplies of the securities, this means that all the prices must be perfectly correlated. This is inconsistent both with the Sharpe-Lintner model and with reality.

Our model resolves the paradox in the sense that it yields the single-factor CAPM equation without implying that the value weights in the market portfolio are constant. The market portfolio will always be proportional to the logarithmic portfolio, but as means, variances and covariances change, so do the value weights in the logarithmic portfolio.

The rest of the study is organized as follows. Section 2 outlines the model. Section 3 defines the instantaneous capital market line and our concept of

the investment opportunity set. Section 4 states our main results. Section 5 discusses the single-factor CAPM with changing means, variances and covariances. Sections 6 explains and derives our results, based on a series of propositions and lemmas which are proved in the Appendix. Section 7 concludes.

2 The Model

Our analysis is carried out within a continuous time trading model which may exhibit a dynamic market incompleteness similar to the one analyzed by He and Pearson (1991) and by Karatzas et al. (1991), because there are fewer risky securities than sources of risk. This section summarizes the notation and main assumptions underlying the model.

The time horizon is $[0, T]$ for a fixed $T > 0$. The investors' information structure is represented by a filtration $F = (\mathcal{F}_t)_{t \in [0,T]}$ on an underlying probability space $(\Omega, \mathcal{F}, P)$. The interpretation is that $\mathcal{F}_t$ is the information set available to the investors at time t.

There are $N + 1$ basic long-lived securities. Like in many models, including Merton (1973), their price dynamics will be specified without distinguishing between real and nominal terms.

We shall assume that security zero is a *money market account* with constant interest rate. This means that its price process M (price per share) has the form

$$M(t) = M(0) \exp\{rt\}$$

for some $M(0) > 0$ and some constant r, the *interest rate.*

The remaining N securities are instantaneously risky. Their prices are given by an N dimensional vector S of Itô processes, which are assumed to be positive. This implies that there exist processes $\mu \in \mathcal{L}^1$ and $\sigma \in \mathcal{L}^2$ such that

$$dS = \mathcal{D}(S)\mu\, dt + \mathcal{D}(S)\sigma\, dW$$

where $\mathcal{D}(S)$ is the diagonal matrix with the vector S along the diagonal.

The notation $\mu \in \mathcal{L}^1$ means that μ is adapted and measurable and satisfies

$$\int_0^T \|\mu\| ds < \infty$$

with probability one. The notation $\sigma \in \mathcal{L}^2$ means that σ is adapted and measurable and satisfies

$$\int_0^T \|\sigma\|^2 ds < \infty$$

with probability one.

We refer to μ and σ as the *mean vector* and the *dispersion matrix* of the instantaneous rates of return to the N risky securities. We allow them to change stochastically over time. Unlike in Merton (1973) and many subsequent models, they are not assumed to be functions of a Markovian vector of state variables. In this sense, our framework is more general than Merton's.

The process W is a K dimensional Wiener process (a vector of K independent one-dimensional Wiener processes) with respect to the filtration $F = (\mathcal{F}_t)$. We assume that $K \geq N$. Markets may be incomplete, in the sense that there may be many more Wiener processes than there are instantaneously risky securities ($K \gg N$). This is the type of market incompleteness analyzed by He and Pearson (1991) and Karatzas et al. (1991).

The K Wiener processes are sources of instantaneous variations in the N risky securities prices. Over extended time intervals, the information in the filtration F may influence the securities prices also because it affects the stochastic parameters μ and σ.

Unlike He and Pearson (1991) and Karatzas et al. (1991), we allow for the possibility that the information set at a point in time may contain even more information than what can be obtained by observing the paths of the K components of W up to that time. Technically speaking, the filtration F may be larger (finer) than the augmented filtration generated by W, so long as W is a Wiener process with respect to F. This means that investors may have access to such additional information, they may make their trading strategies contingent on it, and the parameters μ and σ may vary over time in a way that depends on it.

To compress the notation, write

$$\bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$

$$\bar{\mu} = \begin{pmatrix} Mr \\ \mathcal{D}(S)\mu \end{pmatrix}$$

and

$$\bar{\sigma} = \begin{pmatrix} 0 \\ \mathcal{D}(S)\sigma \end{pmatrix}$$

The processes $\bar{S}$ and $\bar{\mu}$ have dimension $N+1$, and the matrix valued process $\bar{\sigma}$ has dimension $(N+1) \times K$. With this notation, the Itô process $\bar{S}$ has differential

$$d\bar{S} = \bar{\mu}\, dt + \bar{\sigma}\, dW$$

A *trading strategy* is an adapted measurable $(N+1)$-dimensional row vector valued process $\bar{\Delta} = (\Delta_0, \Delta)$. The interpretation is that $\bar{\Delta}(t)$ is the position held at time t: for each security $i = 0, \ldots, N$, $\bar{\Delta}_i(t)$ is the number of units of security i held at time t. The *value process* of a trading strategy $\bar{\Delta}$ is the process $\bar{\Delta}\bar{S}$.

The set of trading strategies $\bar{\Delta}$ such that $\bar{\Delta}\bar{\mu} \in \mathcal{L}^1$ and $\bar{\Delta}\bar{\sigma} \in \mathcal{L}^2$, will be denoted $\mathcal{L}(\bar{S})$.

A trading strategy $\bar{\Delta}$ in $\mathcal{L}(\bar{S})$ is *self-financing* if it satisfies the *budget constraint:*

$$\bar{\Delta}(t)\bar{S}(t) = \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S}$$

or in differential form,

$$d(\bar{\Delta}\bar{S}) = \bar{\Delta}\, d\bar{S}$$

A *portfolio strategy* is an adapted measurable N dimensional row vector valued process $\tilde{\theta}$. The interpretation is that $\tilde{\theta}$ tells us the fractions of wealth invested in the various risky securities, while the remaining fraction, $1 - \tilde{\theta}\iota$, is invested in the money market account. Here, ι is the N dimensional column vector all of whose entries are one.

If $\bar{\Delta} = (\Delta_0, \Delta)$ is a trading strategy such that the value process $V = \bar{\Delta}\bar{S}$ is positive, then the corresponding portfolio strategy is given by

$$\tilde{\Delta} = \Delta\mathcal{D}(S)/V$$

Conversely, we can recover a unique value process and a unique self-financing trading strategy from knowledge only of the portfolio strategy and the initial value of the trading strategy. If $\tilde{\Delta}$ is a portfolio strategy and $w_0 > 0$ is an initial wealth level, then there is a unique self-financing trading strategy $\bar{\Delta}$ such that $\bar{\Delta}(0)\bar{S}(0) = w_0$, $\bar{\Delta}\bar{S} > 0$, and $\tilde{\Delta}$ is the portfolio strategy corresponding to $\bar{\Delta}$. The value process $V = \bar{\Delta}\bar{S}$ of $\bar{\Delta}$ is the unique Itô process such that $V(0) = w_0$ and

$$\frac{dV}{V} = \left((1 - \tilde{\Delta}\iota)r + \tilde{\Delta}\mu\right) dt + \tilde{\Delta}\sigma\, dW$$

and $\bar{\Delta} = (\Delta_0, \Delta)$ is given by

$$\Delta = \mathcal{D}(S)^{-1}\tilde{\Delta}V$$

and

$$\Delta_0 M + \Delta S = V$$

A *state price process* or *pricing kernel* for $\bar{S}$ is a positive one-dimensional Itô process Π such that $\Pi\bar{S}$ has zero drift.

If Π is a state price process for $\bar{S}$, and if $\bar{\Delta} \in \mathcal{L}(\bar{S})$ is a self-financing trading strategy, then $\Pi\bar{\Delta}\bar{S}$ has zero drift.

A process Π is a state price process if and only if $\Pi(0) > 0$ and

$$\frac{d\Pi}{\Pi} = -r\, dt - \lambda\, dW$$

for some $\Pi(0) > 0$ and some K dimensional row vector valued process $\lambda \in \mathcal{L}^2$ such that

$$\mu - r\iota = \sigma\lambda^\top$$

Since the interest rate r is assumed to be constant, the process μ is determined by σ and λ through this equation. For the purpose of our analysis, we can think of σ and λ as exogenous variables (processes) and μ as endogenous.

The process λ is called the vector of *prices of risk*. The prices of risk are specific to each of the Wiener processes. Each element of λ measures the required excess rate of return on securities per unit of dispersion with respect to the corresponding Wiener process. The total excess rate of return on a security is a linear combination of the security's dispersion coefficients with respect to each of the Wiener processes, where the weights in the linear combination are the prices of risk.

It will be shown in the following section that the process $\sqrt{\lambda\lambda^\top}$ in a certain sense describes the price of risk in the aggregate. Specifically, $\sqrt{\lambda\lambda^\top}$ is the slope of the instantaneous capital market line.

We shall assume that the vector λ of prices of risk has the property that

$$\lambda = \lambda\sigma^\top(\sigma\sigma^\top)^{-1}\sigma$$

This is a fairly innocent assumption, for the following reason. If λ is a vector of prices of risk, then the process

$$\hat{\lambda} = \lambda\sigma^\top(\sigma\sigma^\top)^{-1}\sigma$$

is also a vector of prices of risk, provided that $\hat{\lambda} \in \mathcal{L}^2$. This follows from the fact that

$$\mu - r\iota = \sigma\lambda^\top = \left[\sigma\sigma^\top(\sigma\sigma^\top)^{-1}\right]\sigma\lambda^\top = \sigma\left[\sigma^\top(\sigma\sigma^\top)^{-1}\sigma\lambda^\top\right] = \sigma\hat{\lambda}^\top$$

Moreover, $\hat{\lambda}$ does satisfy the assumption that

$$\hat{\lambda} = \hat{\lambda}\sigma^\top(\sigma\sigma^\top)^{-1}\sigma$$

because

$$\begin{aligned}
\hat{\lambda}\sigma^\top(\sigma\sigma^\top)^{-1}\sigma &= \left[\lambda\sigma^\top(\sigma\sigma^\top)^{-1}\sigma\right]\sigma^\top(\sigma\sigma^\top)^{-1}\sigma \\
&= \lambda\sigma^\top\left[(\sigma\sigma^\top)^{-1}\sigma\sigma^\top\right](\sigma\sigma^\top)^{-1}\sigma \\
&= \lambda\sigma^\top(\sigma\sigma^\top)^{-1}\sigma \\
&= \hat{\lambda}
\end{aligned}$$

So, if λ does not satisfy this assumption, then we can simply replace λ by $\hat{\lambda}$.

3 The Instantaneous Capital Market Line

Recall from mean-variance theory that mean-variance efficient portfolios are portfolios that maximize the expected rate of return given the variance or standard deviation of the rate of return. We can similarly define *instantaneously efficient portfolios* as those that maximize the expected instantaneous rate of return given the standard deviation of the instantaneous rate of return. Their combinations of standard deviation of returns and expected returns plot on a straight line whose intercept with the expected-return axis is the instantaneous interest rate. We call this line the *instantaneous capital market line.*

It also follows from the standard theory that the instantaneously efficient portfolios are the portfolios that are combinations of the money market account and the portfolio $\phi^{\ln}$ given by

$$\phi^{\ln} = \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1}$$

where we note that $\sigma\sigma^\top$ is the covariance matrix of the instantaneous rates of return to the various securities. We call this portfolio the *logarithmic portfolio* because, as is well known and will also follow from the analysis below, it is indeed the optimal portfolio for an investor with a logarithmic utility function.

The logarithmic portfolio should be distinguished from the *tangency portfolio*, which consists of investments in instantaneously risky securities only and plots on the capital market line at the point where it is tangent to the risky-security frontier. The tangency portfolio $\phi^{\tan}$ can be calculated from the logarithmic portfolio $\phi^{\ln}$ by scaling the fractions of wealth invested in risky securities so that they add up to one:

$$\phi^{\tan} = \frac{1}{\phi^{\ln}{}_\iota}\phi^{\ln}$$

The slope of the capital market line is the ratio of excess expected instantaneous rate of return and the standard deviation of the instantaneous rate of return to the logarithmic portfolio. We can calculate this slope as follows.

The excess instantaneous expected rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}(\mu - r\iota) = \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} \sigma\lambda^\top = \lambda\lambda^\top$$

The variance of the instantaneous rate of return to $\phi^{\ln}$ is

$$\begin{aligned} \phi^{\ln}\sigma\sigma^\top\phi^{\ln\top} &= \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} \sigma\sigma^\top \left(\sigma\sigma^\top\right)^{-1} \sigma\lambda^\top \\ &= \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} \sigma\lambda^\top \\ &= \lambda\lambda^\top \end{aligned}$$

and the standard deviation is $\sqrt{\lambda\lambda^\top}$.

Hence, the slope of the instantaneous capital market line is

$$\frac{\phi^{\ln}(\mu - r\iota)}{\sqrt{\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top}}} = \frac{\lambda\lambda^\top}{\sqrt{\lambda\lambda^\top}} = \sqrt{\lambda\lambda^\top}$$

It follows that the instantaneous capital market line is the straight line with intercept r and slope $\sqrt{\lambda\lambda^\top}$. We shall argue that for the purpose of optimal portfolio selection, this capital market line is the most useful concept of the "investment opportunity set."

While the individual elements of the vector λ are prices of risk with respect to the individual Wiener processes, $\sqrt{\lambda\lambda^\top}$ is the price of risk in the aggregate, since it is the slope of the capital market line, or the ratio between instantaneous excess return and instantaneous volatility for portfolios that are instantaneously mean-variance efficient. We can also think of it as the instantaneous Sharpe ratio for instantaneously mean-variance efficient portfolios. It also happens to be the volatility of the state price process.

We shall explore the consequences of the condition that the capital market line is constant. This means that both the interest rate r and the slope $\lambda\lambda^\top$ are constant. The condition is equivalent to a constant ratio between the expected excess rate of return and standard deviation of the tangency portfolio.

A constant slope $\sqrt{\lambda\lambda^\top}$ does not require that the elements of λ stay constant. They may change according to virtually any adapted processes so long as their sum-of-squares is constant. At the same time, all the elements of the matrix σ may change in virtually any non-previsible way so long as the matrix continues to have full rank.

4 Two Fund Separation

In this section, we shall show that under quite attractive and general conditions on the utility function, if the instantaneous capital market line is constant, then there exists a unique optimal portfolio strategy, and the optimal portfolio at each instant is on the instantaneous capital market line. In other words, the optimal portfolio is a combination of the money market account and the logarithmic portfolio even when the instantaneous means, variances and covariances vary over time.

This implies that if investors do hold hedge funds, then these funds hedge against changes in the slope and intercept of the capital market line and not against general changes in the instantaneous means, variances and covariances.

When the capital market line is constant, it turns out that we do not need to use the device of introducing fictitious securities to complete the markets as in Karatzas et al. (1991) and He and Pearson (1991).

In order to keep the model simple and focus on the main issues, which are stochastically time varying first and second moments and market incompleteness, we restrict ourselves to a model with a finite time horizon and with only final consumption[2].

Let $w_0 > 0$ be the investor's initial wealth level, and let u be his utility function, defined on the positive half-line $(0, \infty)$.

The investor chooses a self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ subject to the constraints $\bar{\Delta}(0)\bar{S}(0) = w_0$ and $\bar{\Delta}\bar{S} > 0$, so as to maximize the expected utility $Eu(\bar{\Delta}(T)\bar{S}(T))$ of final payoff.

We shall make sure that $Eu(\bar{\Delta}(T)\bar{S}(T))$ is well defined in the sense that $u(\bar{\Delta}(T)\bar{S}(T))$ is integrable above for all self-financing trading strategies satisfying the constraints.

A self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ is *optimal* given initial wealth

[2]We expect that as is usually the case in models like this, the results can be generalized to a model with a flow of consumption over time and with a finite or an infinite time horizon.

$w_0 > 0$ if $\bar{\Delta}(0)\bar{S}(0) = w_0$, $\bar{\Delta}\bar{S} > 0$, $u(\bar{\Delta}(T)\bar{S}(T))$ is integrable above, and if for every self-financing trading strategy $\bar{\delta} \in \mathcal{L}(\bar{S})$ such that $\bar{\delta}(0)\bar{S}(0) = w_0$ and $\bar{\delta}\bar{S} > 0$, $u(\bar{\delta}(T)\bar{S}(T))$ is integrable above with

$$Eu(\bar{\Delta}(T)\bar{S}(T)) \geq Eu(\bar{\delta}(T)\bar{S}(T))$$

Choosing a self-financing trading strategy with positive value process is equivalent to choosing the corresponding portfolio strategy. So, a portfolio strategy $\tilde{\Delta}$ is *optimal* given initial wealth w_0 if $\bar{\Delta}$ is optimal given initial wealth w_0, where $\bar{\Delta}$ is the unique self-financing trading strategy with initial value $\bar{\Delta}(0)\bar{S}(0) = w_0$ and such that the portfolio strategy corresponding to $\bar{\Delta}$ is $\tilde{\Delta}$.

Assuming that the utility function u is twice differentiable with $u'' < 0$, let $R_R(u)(x)$ denote the coefficient of relative risk aversion at the wealth level x.

The following theorem is our main result. It is a two-fund separation theorem for an intertemporal portfolio selection model with time-varying first and second moments, where the slope and intercept of the instantaneous capital market line is constant.

Theorem 1 *Assume that*

1. *the interest rate r is constant,*
2. *the slope $\sqrt{\lambda\lambda^\top}$ of the instantaneous capital market line is a positive constant,*
3. *the utility function u is twice continuously differentiable with $u' > 0$, $u'' < 0$, $u'(x) \to 0$ as $x \to \infty$, and $u'(x) \to \infty$ as $x \to 0$, and*
4. *the relative risk aversion is bounded below away from zero at high wealth levels: there exist constants $\gamma > 0$ and $x_0 > 0$ such that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$.*

Then for each initial wealth level $w_0 > 0$ there is a unique optimal portfolio strategy $\tilde{\Delta}$. It is a combination of the money market account and the logarithmic portfolio: it has the form

$$\tilde{\Delta} = \alpha\phi^{\ln}$$

for some one-dimensional process $\alpha \in \mathcal{L}^2$.

It is important to note that in Theorem 1, σ, μ or λ can change over time, so long as the relation

$$\mu - r\iota = \sigma\lambda^\top$$

is satisfied and r and $\lambda\lambda^\top$ remain constant[3]. The processes σ, μ and λ do not have to be functions of a Markovian vector of state variable processes. To our knowledge, the result in the theorem has previously been shown only in the case of constant μ and σ. This case follows from Merton (1973).

We interpret the theorem as saying that if investors deviate from the two fund strategy and hold hedge funds in addition to the riskless asset and the logarithmic portfolio, then they do so to hedge against changes in the capital market line. Our result simplifies Merton's separation theorem, since it implies that state variables which affect the instantaneous means, variances and covariances but do not affect the capital market line will not give rise to hedge funds.

It follows that the most relevant concept of an "investment opportunity set" is the instantaneous capital market line. Changes in the investment opportunity set can be understood as changes in the slope and the intercept of the instantaneous capital market line. Recall that Merton (1973) defined changes in the investment opportunity set as any changes in instantaneous means, variances or covariances.

An implication of our analysis which is interesting for empirical purposes is that security returns and moments may be predictable while the investment opportunity set is constant. Thus, one should be cautious in interpreting predictability as evidence of a changing investment opportunity set.

Our separation theorem makes quite general and attractive assumptions on the utility function. We require relative risk aversion to be bounded from below away from zero at high levels of wealth. This is an economically meaningful and reasonable assumption. For example, the classical paper by Arrow (1965) argued that the utility functions could reasonably be assumed to have decreasing absolute but increasing relative risk aversion. If relative risk aversion is increasing, then certainly it will be bounded away from zero at high wealth levels. We also require marginal utility to go to infinity as

[3]The theorem can easily be generalized to the case where $\lambda\lambda^\top$ changes over time in a purely deterministic manner.

final wealth or consumption goes to zero. This ensures that the optimal final consumption will be positive with probability one. It rules out some utility functions with hyperbolic absolute risk aversion[4].

Note that the potential market incompleteness in our model has no effect on optimal portfolio holdings. The form of the optimal portfolio holdings would be the same if the last $K - N$ Wiener processes did not exist and if the filtration F were equal to the (augmented) filtration generated by the first N Wiener processes. By exhibiting conditions under which market incompleteness does not matter, the theorem will help identify those situations where market incompleteness is truly important.

Note that even with a constant capital market line, the value weight α invested in the logarithmic portfolio will in general change over time in response to changes in the investor's wealth and resultant changes in the risk aversion of his indirect utility function. Changes in α can be interpreted as sliding up and down the instantaneous capital market line.

With additional regularity conditions, we would be able to interpret the value weight α invested in the logarithmic portfolio as the risk tolerance of the investor's indirect utility function. See Cox and Huang (1989) for a discussion of the relevant regularity conditions.

Instead of pursuing the general case, we will treat the case of constant relative risk aversion in the following proposition. In this case, the weights α can be interpreted as the risk tolerance not only of the investors' indirect utility function of wealth, but also of his direct utility function, because that will be the same.

Proposition 1 *Assume that the investor has constant relative risk aversion $\gamma > 0$. If $\gamma \neq 1$ assume that $\lambda\lambda^\top$ and the interest rate r are constant. If $\gamma = 1$ assume that $\ln \Pi(T)$ is integrable below. Then there is a unique optimal portfolio strategy $\tilde{\Delta}$. It is independent of w_0 and is given by*

$$\tilde{\Delta} = \frac{1}{\gamma}\phi^{\ln}$$

[4]We expect that our theorem can easily be generalized to allow for HARA utility, but we abstain from doing so since it would divert attention from our main point.

To our knowledge, Proposition 1 has previously been shown only in the case of dynamically complete markets. That case is developed in Examples 6.4 and 6.5 of Karatzas et al. (1991).

The proposition says that the investor holds a combination of the logarithmic portfolio and the money market account, with value weights given by his *relative risk tolerance* $1/\gamma$. For an investor with logarithmic utility (corresponding to $\gamma = 1$) the optimal portfolio strategy is the logarithmic portfolio strategy $\phi^{\ln}$. That was why we used that name for it, of course.

5 The Single-Factor CAPM

In this section, we develop a single factor CAPM where the first and second moments of security returns may change stochastically over time and markets are potentially incomplete. It differs from the consumption CAPM in that the single factor is not aggregate consumption but the return to the market portfolio. Our model is consistent with empirical specifications of the return to the market portfolio based on ARCH-M and GARCH-M models. We observe that the model resolves the paradox of Rosenberg and Ohlson (1976).

Assume that there are n investors $i = 1, \ldots, n$. Investor i has initial wealth $w_0^i > 0$ and utility function u^i. Continue to assume that r and $\lambda\lambda^\top$ are constant, and assume that each investor's utility function u^i satisfies the assumptions of Theorem 1.

It follows from Theorem 1 that each investor has a unique optimal portfolio strategy $\tilde{\Delta}^i$ which has the form

$$\tilde{\Delta}^i = \alpha^i \phi^{\ln}$$

for some one-dimensional process $\alpha^i \in \mathcal{L}^2$.

Let $\tilde{\Delta}$ be the portfolio strategy that describes the market portfolio of risky assets. Let V_i be the value process of investor i's holdings, and let V be the aggregate value:

$$V = \sum_i V_i$$

Market clearing requires that

$$\tilde{\Delta} = \sum_i \frac{V_i}{V} \tilde{\Delta}^i = \sum_i \frac{V_i}{V} \alpha^i \phi^{\ln} = \bar{\alpha} \phi^{\ln}$$

where $\phi^{\ln}$ is the logarithmic portfolio and $\bar{\alpha}$ is defined by

$$\bar{\alpha} = \frac{1}{V} \left(\sum_i \frac{V_i \alpha^i}{V} \right)$$

So, $\tilde{\Delta}$ is also proportional to the logarithmic risky portfolio.

Provided that $\bar{\alpha} \neq 0$, the equation

$$\tilde{\Delta} = \bar{\alpha} \phi^{\ln}$$

implies

$$(\mu - r\iota)^\top = \lambda \sigma^\top = \frac{1}{\bar{\alpha}} \tilde{\Delta} \left(\sigma \sigma^\top \right)$$

This is a version of the CAPM equation. It says that the excess instantaneous expected rate of return to each asset is proportional to the covariance between the rate of return on the asset and the rate of return on the market portfolio $\tilde{\Delta}$.

The important feature of this version of the CAPM equation is that there is no assumption that σ or μ are constant. The covariance matrix can vary stochastically over time, so long as the capital market line stays constant. Yet, unlike in Merton (1973), the equation contains no risk premia on hedge funds which hedge against the stochastic changes in σ and μ.

In general, it is rare to find equilibrium results in models with incomplete markets. The papers by Karatzas et al. (1991) and He and Pearson (1991) do not analyze what happens in equilibrium. In our case, it is the assumption of a constant capital market line which allows us to derive the CAPM.

The consumption CAPM of Breeden (1979) and Duffie and Zame (1989) is also a single factor model which allows for stochastically changing means, variances and covariances. However, it assumes complete markets, while ours does not. Furthermore, in the consumption CAPM, the single factor is aggregate consumption rather than the return to the market portfolio. This

is problematic, because consumption data are sampled less frequently than financial data, and the model has been found to perform poorly in empirical tests. Our model has the advantage that the single factor is the return to the market portfolio.

In equilibrium, the market portfolio of risky assets plots on the capital market line. Hence, the constancy of the capital market line translates into a linear relation between the mean and the standard deviation of the excess return on the market portfolio.

The assumption of a constant capital market line is one of the empirical models proposed by Merton (1980) for estimating the expected return to the market. It is also consistent with some of the autoregressive conditional heteroscedastic in mean (ARCH-M) and generalized ARCH-M (GARCH-M) specifications that have been used in the empirical literature.

The classes of ARCH-M and GARCH-M processes were proposed by Engle, Lilien and Robbins (1987) and Bollerslev, Engle and Woolridge (1988), respectively. These specifications model the expected excess rate of return to a security or to the market portfolio as a function of its standard deviation or variance. Our model corresponds to specifying the expected excess rate of return to the market portfolio as a linear function of its standard deviation. Such a relation has been examined, for example, by French, Schwert and Stambaugh (1987) and by Bodurtha and Mark (1991). French, Schwert and Stambaugh (1987) tested two versions of GARCH-M, where the excess expected rate of return to the market portfolio is an affine function of either the standard deviation or the variance. Bodurtha and Mark (1991) compared three versions of the ARCH-M model, where the excess expected rate of return on the market portfolio is an affine function of either the variance, the standard deviation, or the logarithm of the variance.

In the single-factor Sharpe-Lintner model, if the interest rate r, the mean vector μ and the variance covariance matrix $\sigma\sigma^\top$ are constant, then the market portfolio has to be constant, i.e., the value weights in the market portfolio are constant. In the absence of random changes in the supplies of the securities, this means that all the prices must always change proportionally, and hence, they must be perfectly correlated. This is of course completely inconsistent both with the model and with reality. This was first observed by Rosenberg and Ohlson (1976).

Our model resolves the Rosenberg-Ohlson paradox. Since it allows the covariance matrix to change over time, the value weights of the market portfolio will change, even though the capital market line stays constant. It follows that the instantaneous price changes of the various securities will not be proportional to each other.

6 Explanations and Derivations

In this section, we explain the logic behind our results and outline their derivation. The derivation uses a series of propositions and lemmas that are proved in the Appendix.

As mentioned earlier, we use the so-called martingale approach to optimal portfolio choice. The central idea behind it is that we can explicitly write down the investor's optimal final wealth (consumption) as a function of the final value $\Pi(T)$ of the state price process. All we then need to do is calculate a trading strategy or portfolio strategy which replicates this optimal final wealth. This can be done also in the present case where there are more sources of risk (Wiener processes) than instantaneously risky securities.

The candidate for the optimal final wealth will be a random claim Y^* whose marginal utility $u'(Y^*)$ is proportional to the final state prices $\Pi(T)$ and whose initial value is w_0. It is defined as follows.

Assume that u' is continuous, positive and strictly decreasing, with $u'(x) \to 0$ as $x \to \infty$ and $u'(x) \to \infty$ as $x \to 0$. Then u' has an inverse

$$I : (0, \infty) \to (0, \infty)$$

Assume that $\Pi(T)I(y\Pi(T))$ is integrable for all $y > 0$.

This assumption allows us to define a function

$$h : (0, \infty) \to I\!R$$

by

$$h(y) = E\left[\frac{\Pi(T)}{\Pi(0)} I(y\Pi(T))\right]$$

This function h is continuous and strictly decreasing with $h((0,\infty)) = (0,\infty)$. Hence, it has an inverse

$$\kappa : (0,\infty) \to (0,\infty)$$

which is continuous and strictly increasing with $\kappa((0,\infty)) = (0,\infty)$.

The candidate for the optimal final consumption claim, given initial wealth $w_0 > 0$, is

$$Y^* = I(\kappa(w_0)\Pi(T))$$

It has been constructed in such a way that $E[\Pi(T)Y^*] = \Pi(0)w_0$. The following proposition says that if there is a self-financing trading strategy which replicates Y^*, then it is the unique optimal trading strategy.

Proposition 2 *Assume that $\Pi(T)I(y\Pi(T))$ is integrable for all $y > 0$. Suppose $\bar{\Delta}$ is a self-financing trading strategy such that $\bar{\Delta}\bar{S} > 0$,*

$$\bar{\Delta}(T)\bar{S}(T) = Y^*$$

and

$$\bar{\Delta}(0)\bar{S}(0) = w_0$$

Then $\Pi\bar{\Delta}\bar{S}$ is a martingale. If $u(Y^*)$ is integrable above, then $\bar{\Delta}$ is the unique[5] *optimal trading strategy given initial wealth w_0.*

Proposition 2 assumes that $\Pi(T)I(y\Pi(T))$ is integrable for all $y > 0$, and that $u(Y^*)$ is integrable above. We need to know that these assumptions will indeed hold under the conditions of Theorem 1 and Proposition 1.

For this purpose, we need to observe that when r and $\lambda\lambda^\top$ are constant, $\Pi(T)$ follows a lognormal distribution. That will follow from the following lemma.

Lemma 1 *If $\sqrt{\lambda\lambda^\top}$ is a positive constant, then the process*

$$B(t) = \frac{1}{\sqrt{\lambda\lambda^\top}} \int_0^t \lambda dW$$

is a Wiener process relative to F.

[5]To be precise, uniqueness means uniqueness almost everywhere with respect to the product measure $P \otimes \lambda$ on $\Omega \times [0,T]$, where P is the probability measure on Ω and λ is Lebesgue measure on $[0,T]$.

When r and $\sqrt{\lambda\lambda^\top}$ are constant, for $0 \leq t \leq T$,

$$\ln \Pi(t) - \ln \Pi(0) = -\left(r + \frac{1}{2}\lambda\lambda^\top\right) t - \sqrt{\lambda\lambda^\top} B(t)$$

Hence, it follows from Lemma 1 that $\Pi(T)$ follows a lognormal distribution.

We have made the assumption that $\Pi(T) I(y\Pi(T))$ is integrable for all $y > 0$. If u is logarithmic, then this assumption is trivially satisfied, and if $\Pi(T)$ is lognormal, a sufficient condition is that the relative risk aversion $R_R(u)(x)$ is bounded below away from zero for sufficiently large $x > 0$. This follows from Proposition 3, applied to $R = \ln \Pi(T)$.

Proposition 3 *Assume that u is twice continuously differentiable. Let R be a positive random variable. Suppose there exist constants $\gamma > 0$ and $x_0 > 0$ such that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$ and such that*

$$R^{\frac{\gamma-1}{\gamma}}$$

is integrable. Then $RI(yR)$ is integrable for all $y > 0$.

We have also made the assumption that $u(Y^*)$ is integrable above. If u is logarithmic, then $u(Y^*)$ will be integrable above provided that $\ln \Pi(T)$ is integrable below. If r and $\sqrt{\lambda\lambda^\top}$ are constant, so that $\Pi(T)$ is lognormally distributed, then $u(Y^*)$ will be integrable above provided that there exist constants $\gamma > 0$ and $x_0 > 0$ such that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$. This follows from Proposition 4, applied to $R = \Pi(T)$.

Proposition 4 *Let $\gamma > 0$ and $x_0 > 0$ be constants. Assume that u is twice continuously differentiable with $u' > 0$. Assume that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$. Let R be a positive random variable. If $\gamma = 1$, assume that $\ln R$ is integrable below. If $\gamma < 1$, assume that*

$$R^{\frac{\gamma-1}{\gamma}}$$

is integrable. Then $u(Y)$ is integrable above for all positive random variables Y such that RY is integrable.

All that remains in order to prove Theorem 1 is to construct a self-financing trading strategy which replicates Y^* in the precise sense of Proposition 2, and which involves only the riskless asset and the logarithmic portfolio.

To this end, we consider a simplified market model with one money market account and one instantaneously risky security, driven by one source of uncertainty, a one-dimensional Wiener process B. The instantaneously risky security will be the logarithmic portfolio. Even though there are incomplete markets in the original model, the simplified model will have complete markets relative to B. It turns out that the optimal payoff Y^* can be replicated in the simplified model at an initial cost equal to the initial wealth w_0. The replicating portfolio strategy can then be re-expressed as a portfolio strategy in the original model.

Let B be the process defined in Lemma 1. Then B is a Wiener process relative to F, and in particular, B is a Brownian motion. Let F^B be the augmented filtration generated by B. Then B is a Wiener process relative to F^B.

Let $S^{\ln}$ be the value process of one unit of account initially invested in the logarithmic portfolio. Then

$$\begin{aligned}
\frac{dS^{\ln}}{S^{\ln}} &= \left((1-\phi^{\ln}\iota)r + \phi^{\ln}\mu\right)dt + \phi^{\ln}\sigma\, dW \\
&= \left(r + \lambda\lambda^\top\right)dt + \lambda\, dW \\
&= \left(r + \lambda\lambda^\top\right)dt + \sqrt{\lambda\lambda^\top}\, dB
\end{aligned}$$

Hence,

$$\ln S^{\ln}(t) - \ln S^{\ln}(0) = \left(r + \frac{1}{2}\lambda\lambda^\top\right)t + \sqrt{\lambda\lambda^\top}B(t)$$

So, $S^{\ln}$ is adapted to the filtration F^B.

The simplified market model has two basic securities, the money market account (with interest rate r) and one instantaneously risky security with price process $S^{\ln}$. The underlying filtration is F^B, and the model is driven by the one-dimensional Wiener process B. This model has dynamically complete markets (with respect to B, but not with respect to W).

The state price process for the simplified model is Π, which is the same as the state price process for the original model. To see this, observe from the

expression

$$\ln \Pi(t) - \ln \Pi(0) = -\left(r + \frac{1}{2}\lambda\lambda^\top\right) t - \sqrt{\lambda\lambda^\top} B(t)$$

that Π is adapted to the filtration F^B. Since the instantaneously risky securities price process $S^{\ln}$ has differential

$$\frac{dS^{\ln}}{S^{\ln}} = \left(r + \lambda\lambda^\top\right) dt + \sqrt{\lambda\lambda^\top}\, dB$$

the price of risk with respect to B is $\sqrt{\lambda\lambda^\top}$. This implies that the state price process in the simplified model is Π.

The payoff $Y^* = I(\kappa(w_0)\Pi(T))$ is measurable with respect to the final information set in the filtration $\mathcal{F}^B$, since it follows from the expression above for $\ln \Pi(t)$ as a function of $B(t)$ that $\Pi(T)$ is a measurable function of $B(T)$. Furthermore, $\Pi(T)Y^*$ is integrable. Since the simplified model has dynamically complete markets, there exists a trading strategy in the simplified model which replicates Y^* and whose value process V has the martingale property,

$$E[\Pi(t)V(t)|\mathcal{F}_s^B] = \Pi(s)V(s)$$

for $0 \leq s \leq t \leq T$.

Notice that the definition of dynamically complete markets and the completeness result that we refer to are simpler than what is usually found in the literature. In particular, we do not at all need the regularity conditions on λ which are necessary in order to make sure that the so-callled risk adjusted probability measure exists. The only integrability condition on the payoff Y that we need is that $\Pi(T)Y^*$ be integrable. For details, see Nielsen (1996).

We need to verify that the initial cost $V(0)$ of replicating Y^* in the simplified model is equal to the initial wealth w_0. Recall that Y^* is constructed in such a way that $E[\Pi(T)Y^*] = \Pi(0)w_0$. But since ΠV is a martingale and $V(T) = Y^*$, we have

$$E[\Pi(T)Y^*] = E[\Pi(T)V(T)] = \Pi(0)V(0)$$

and hence, $V(0) = w_0$.

Let α be the portfolio strategy in the simplified model which corresponds to the trading strategy that replicates Y^*. It is then clear that the trading strategy $\alpha\phi^{\ln}$, implemented with initial wealth $\hat{w}_0$, replicates Y^*.

This concludes the derivation of Theorem 1.

To prove Proposition 1, we observe that the simplified model has a constant interest rate, a single instantaneously risky security, and dynamically complete markets. Furthermore, in the case where $\gamma \neq 1$, we have assumed that $\sqrt{\lambda\lambda^\top}$ is constant, so that the price of the instantaneously risky security follows a geometric Brownian motion. It is well known that the optimal portfolio strategy in such a model for an investor with constant relative risk aversion γ is to hold the proportion $\alpha = 1/\gamma$ of wealth in the risky security.

7 Conclusions

This study has reexamined the role of hedge funds in the intertemporal portfolio selection and capital asset pricing model of Merton (1973). Using less restrictive assumptions than Merton regarding the security price processes and the completeness of capital markets, we showed that only specific changes in the first and second moments of security returns may give rise to hedge funds in the optimal portfolio selection.

The concept of the investment opportunity set as introduced in Merton (1973) was redefined. We showed that changes in the investment opportunity set are equivalent to changes in the instantaneous capital market line. This result simplifies considerably the fund separation theorem of Merton.

We derived a single factor CAPM with time-varying first and second moments and dynamically incomplete markets. Our model is consistent with one of the empirical specifications proposed in Merton (1980) for estimating the expected return to the market, and with ARCH-M and GARCH-M specifications that have been used in the empirical literature. It is different from the consumption CAPM of Breeden (1979) and Duffie and Zame (1989) in two important ways. First, it allows for market incompleteness, and second, the single factor is the return to the market portfolio rather than aggregate consumption.

Finally, our model resolves the paradox of Rosenberg and Ohlson (1979).

8 Appendix: Proofs

PROOF OF PROPOSITION 2:

First,

$$\begin{aligned} E(\Pi(T)\bar{\Delta}(T)\bar{S}(T)) &= E(\Pi(T)Y^*) \\ &= E(\Pi(T)I(\kappa(w_0)\Pi(T))) \\ &= \Pi(0)h(\kappa(w_0)) \\ &= \Pi(0)w_0 \\ &= \Pi(0)\bar{\Delta}(0)\bar{S}(0) \end{aligned}$$

Since $\bar{\Delta}$ is a self-financing trading strategy such that $\bar{\Delta}\bar{S} > 0$, $\Pi\bar{\Delta}\bar{S}$ is a non-negative stochastic integral. Since $\Pi(T)\bar{\Delta}(T)\bar{S}(T)$ is integrable, it follows that $\Pi\bar{\Delta}\bar{S}$ is a supermartingale. Hence, for $0 \leq t \leq T$,

$$E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t] \leq \Pi(t)\bar{\Delta}(t)\bar{S}(t)$$

and

$$\begin{aligned} \Pi(0)\bar{\Delta}(0)\bar{S}(0) &= E[\Pi(T)\bar{\Delta}(T)\bar{S}(T)] \\ &= E(E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t]) \\ &\leq E[\Pi(t)\bar{\Delta}(t)\bar{S}(t)] \\ &\leq \Pi(0)\bar{\Delta}(0)\bar{S}(0) \end{aligned}$$

which implies that

$$E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t]) = \Pi(t)\bar{\Delta}(t)\bar{S}(t)$$

This shows that $\Pi\bar{\Delta}\bar{S}$ is a martingale.

To show that $\bar{\Delta}$ is optimal, suppose $\bar{\delta}$ is a self-financing trading strategy such that $\Pi(T)\bar{\delta}(T)\bar{S}(T)$ is integrable, $\bar{\delta}(0)\bar{S}(0) = w_0$, and $\bar{\delta}\bar{S} > 0$. Then $\Pi\bar{\delta}\bar{S}$, like $\Pi\bar{\Delta}\bar{S}$, is a supermartingale, and so

$$E[\Pi(T)\bar{\delta}(T)\bar{S}(T)] \leq \Pi(0)\bar{\delta}(0)\bar{S}(0) = \Pi(0)w_0$$

Because u is differentiable and concave,

$$\begin{aligned} u(Y^*) &\geq u(\bar{\delta}(T)\bar{S}(T)) + u'(Y^*)(Y^* - \bar{\delta}(T)\bar{S}(T)) \\ &= u(\bar{\delta}(T)\bar{S}(T)) + \kappa(w_0)\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T)) \end{aligned}$$

Here, $\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T))$ is integrable and

$$E[\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T))] = \Pi(0)w_0 - E[\Pi(T)\bar{\delta}(T)\bar{S}(T)] \geq 0$$

Hence, if $u(Y^*)$ is integrable above, then so is $u(\bar{\delta}(T)\bar{S}(T))$, and

$$Eu(Y^*) \geq Eu(\bar{\delta}(T)\bar{S}(T))$$

This shows that $\bar{\Delta}$ is optimal.

Finally, to show that $\bar{\Delta}$ is the unique optimal trading strategy given initial wealth w_0, suppose both $\bar{\Delta}$ and $\bar{\delta}$ are optimal. Set

$$\bar{\psi} = \frac{1}{2}\bar{\Delta} + \frac{1}{2}\bar{\delta}$$

Then $\bar{\psi}$ is a self-financing trading strategy with positive value process and initial value w_0, and $u(\bar{\psi}(T)\bar{S}(T))$ is integrable above. Because u is strictly concave,

$$\begin{aligned} Eu(\bar{\psi}(T)\bar{S}(T)) &= Eu\left(\frac{1}{2}\bar{\Delta}(T)\bar{S}(T) + \frac{1}{2}\bar{\delta}(T)\bar{S}(T)\right) \\ &\geq \frac{1}{2}Eu(\bar{\Delta}(T)\bar{S}(T)) + \frac{1}{2}Eu(\bar{\delta}(T)\bar{S}(T)) \\ &= Eu(\bar{\Delta}(T)\bar{S}(T)) = Eu(\bar{\delta}(T)\bar{S}(T)) \end{aligned}$$

with strict inequality unless $\bar{\Delta}(T)\bar{S}(T) = \bar{\delta}(T)\bar{S}(T)$ with probability one. But since $\bar{\Delta}$ and $\bar{\delta}$ are optimal, the inequality indeed is not strict, and so $\bar{\Delta}(T)\bar{S}(T) = \bar{\delta}(T)\bar{S}(T)$. By Proposition 2, both $\Pi\bar{\Delta}\bar{S}$ and $\Pi\bar{\delta}\bar{S}$ are martingales. Therefore, for every t with $0 \leq t \leq T$,

$$\begin{aligned} \Pi(t)\bar{\Delta}(t)\bar{S}(t) &= E[\Pi(T)\bar{\Delta}(T)\bar{S}(T)|\mathcal{F}_t] \\ &= E[\Pi(T)\bar{\delta}(T)\bar{S}(T)|\mathcal{F}_t] \\ &= \Pi(t)\bar{\delta}(t)\bar{S}(t) \end{aligned}$$

which implies that $\bar{\Delta}\bar{S} = \bar{\delta}\bar{S}$. Write $\bar{\Delta} = (\Delta_0, \Delta)$ and $\bar{\delta} = (\delta_0, \delta)$. Then

$$\Delta\mathcal{D}(S)\sigma = \bar{\Delta}\bar{\sigma} = \bar{\delta}\bar{\sigma} = \delta\mathcal{D}(S)\sigma$$

Since the matrix $\mathcal{D}(S)\sigma$ has rank N, this implies that $\Delta = \delta$. Since $\bar{\Delta}$ and $\bar{\delta}$ are both self-financing and have the same initial value, it follows that they are identical.

□

PROOF OF LEMMA 1:

Clearly, B is a continuous square integrable martingale with $B(0) = 0$. For $0 \leq s \leq t$,

$$\begin{aligned} E\left[(B(t) - B(s))^2 \mid \mathcal{F}_s\right] &= \frac{1}{\lambda\lambda^\top} E\left[\left(\int_s^t \lambda dW\right)^2 \mid \mathcal{F}_s\right] \\ &= \frac{1}{\lambda\lambda^\top} E\left[\int_s^t \lambda\lambda^\top du \mid \mathcal{F}_s\right] \\ &= \frac{1}{\lambda\lambda^\top}(t-s)\lambda\lambda^\top \\ &= t - s \end{aligned}$$

This shows that B is a Wiener process relative to F.

□

PROOF OF PROPOSITION 3:

The inequality

$$R_R(u)(x) = -\frac{u''(x)x}{u'(x)} \geq \gamma$$

for $x \geq x_0$ implies

$$(-\ln u'(x))' = R_A(u)(x) = -\frac{u''(x)}{u'(x)} \geq \frac{\gamma}{x}$$

and

$$\begin{aligned} \ln\left(\frac{u'(x_0)}{u'(x)}\right) &= -\ln u'(x) + \ln u'(x_0) \\ &= [-\ln u'(t)]_{x_0}^{x} \\ &\geq \int_{x_0}^{x} \frac{\gamma}{t} dt \end{aligned}$$

$$
\begin{aligned}
&= \gamma\,[\ln t]_{x_0}^{x} \\
&= \gamma\,(\ln x - \ln x_0) \\
&= \ln\left(\frac{x^\gamma}{x_0^\gamma}\right)
\end{aligned}
$$

Hence,

$$x^\gamma \leq \frac{u'(x_0)x_0^\gamma}{u'(x)}$$

Now for $0 < z \leq u'(x_0)$, set $x = I(z)$. Then $z = u'(x)$,

$$I(z)^\gamma \leq u'(x_0)x_0^\gamma z^{-1}$$

and

$$I(z) \leq cz^{-\frac{1}{\gamma}}$$

where

$$c = u'(x_0)^{\frac{1}{\gamma}}x_0$$

Now use the fact that

$$R^{\frac{\gamma-1}{\gamma}}$$

is integrable. If $y > 0$, then

$$RI(yR) \leq \begin{cases} cy^{-\frac{1}{\gamma}}R^{\frac{\gamma-1}{\gamma}} & \text{if } R < u'(x_0)/y \\ Ru'(x_0) & \text{if } R \geq u'(x_0)/y \end{cases}$$

This is integrable for all $y > 0$.

□

PROOF OF PROPOSITION 4:

Let u_γ be the utility function on $(0, \infty)$ defined by

$$u_\gamma(x) = \begin{cases} \ln x & \text{if } \gamma = 1 \\ \frac{1}{1-\gamma}x^{1-\gamma} & \text{if } \gamma \neq 1 \end{cases}$$

It has constant relative risk aversion γ.

We will prove first that there exist constants $b > 0$ and a such that

$$u(x) \leq a + bu_\gamma(x)$$

for all $x \geq x_0$. Then we will show that $u(Y)$ is integrable above.

(1):

As in Proposition 3, we find

$$\frac{u'(x)}{u'(x_0)} \leq \frac{x_0^\gamma}{x^\gamma}$$

and

$$u'(x) \leq u'(x_0) x_0^\gamma x^{-\gamma}$$

for $x \geq x_0$. Hence,

$$u(x) - u(x_0) = [u'(t)]_{x_0}^x \leq u'(x_0) x_0^\gamma \int_{x_0}^x t^{-\gamma}\, dt$$

If $\gamma = 1$, then this implies

$$\begin{aligned} u(x) - u(x_0) &\leq u'(x_0) x_0 \left[\ln t\right]_{x_0}^x \\ &= u'(x_0) x_0 \left(\ln x - \ln x_0\right) \\ &= a + b u_\gamma(x) \end{aligned}$$

where

$$a = -u'(x_0) \ln x_0$$

and

$$b = u'(x_0) x_0$$

If $\gamma \neq 1$, we find

$$\begin{aligned} u(x) - u(x_0) &\leq u'(x_0) x_0^\gamma \frac{1}{1-\gamma} \left[t^{1-\gamma}\right]_{x_0}^x \\ &= u'(x_0) x_0^\gamma \frac{1}{1-\gamma} \left(x^{1-\gamma} - x_0^{1-\gamma}\right) \\ &= a + b u_\gamma(x) \end{aligned}$$

where

$$a = -u'(x_0) x_0 \frac{1}{1-\gamma}$$

and

$$b = u'(x_0) x_0^\gamma$$

(2):

If $\gamma > 1$, then u is bounded above because $u_\gamma < 0$.

If $\gamma = 1$, then it suffices to show that

$$\ln Y = \ln[RY] - \ln R$$

is integrable above. The first term is integrable above because RY is integrable and ln is concave. The second term is integrable above because $\ln R$ is integrable below.

If $\gamma < 1$, set $q = 1/(1-\gamma)$ and

$$p = \frac{1}{\gamma}$$

Then

$$\frac{1}{p} + \frac{1}{q} = \gamma + (1-\gamma) = 1$$

It suffices to show that

$$Y^{1-\gamma} = [RY]^{1-\gamma} R^{\gamma-1}$$

is integrable. By Hölder's inequality, it is enough to observe that

$$[RY]^{(1-\gamma)q} = RY$$

and

$$R^{(\gamma-1)p} = R^{\frac{\gamma-1}{\gamma}}$$

are integrable.

□

9 References

1. Arrow, K. J.: "Aspects of a Theory of Risk Bearing," Yrjö Jahnson Lectures, Helsinki (1965). Reprinted in *Essays in the Theory of Risk Bearing* (1971). Chicago: Markham Publishing Co.

2. Cox, J. C. and C. Huang: "Optimal Consumption and Portfolio Policies when Asset Prices follow a Diffusion Process," *Journal of Economic Theory* 49 (1989), 33–83.

3. Bekaert, G. and Campbell R. Harvey: "Time-Varying World Market Integration," *Journal of Finance* 50 (1995), 403–444.

4. Bodurtha, J. N. and N. C. Mark: "Testing the CAPM with Time-Varying Risks and Returns," *Journal of Finance* 46 (1991), 1485–1505.

5. Bollerslev, T., R. F. Engle and J. M. Woolridge: "A Capital Asset Pricing Model with Time-Varying Covariances," *Journal of Political Economy* 96 (1988), 116–131.

6. Breeden, D. T.: "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics* 7 (1979), 265–296.

7. Duffie, D. and W. Zame: "The Consumption-Based Capital Asset Pricing Model," *Econometrica* 57 (1989), 1279–1297.

8. Engle, R. F., D. M. Lilien, and R. P. Robbins: "Estimating Time Verying Risk Premia in the Term Structure: The ARCH-M Model," *Econometrica* 55 (1987), 391–407.

9. Ferson, W. E.: "The Variation of Economic Risk Premiums," *Journal of Political Economy* 99 (1991), 385–415.

10. French, K. R., G. W. Schwert and R. F. Stambaugh: "Expected Stock Returns and Volatility," *Journal of Financial Economics* 19 (1987), 3–29.

11. He, H., and N. D. Pearson: "Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case," *Journal of Economic Theory* 54 (1991), 259–304.

12. Karatzas, I., J. Lehoczky, S. E. Shreve and G. Xu: "Martingale and Duality Methods for Utility Maximization in an Incomplete Market," *SIAM Journal of Control and Optimization* 29 (1991), 702–730.

13. Merton, R.: *Continuous-Time Finance*, Blackwell Publishers, Cambridge, Ma. (1992).

14. Merton, R. C.: "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41 (1973), 867–888. Chapter 15 in Merton (1992).

15. Merton, R. C.: "On Estimating the Expected Return on the Market," *Journal of Financial Economics* 8 (1980), 323–361.

16. Nielsen, L. T.: *Pricing and Hedging of Derivative Securities*, unpublished manuscript, Insead, 1996.

17. Rosenberg, B. and J. Ohlson: "The Stationary Distribution of Returns and Portfolio Separation in Capital Markets: A Fundamental Contradiction," *Journal of Financial and Quantitative Analysis* 11 (1976), 393–402.