EFFICIENCY VERSUS COMPLETENESS: STRATEGIC USE OF OPTION MARKETS

by

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Efficiency versus Completeness: Strategic Use of Option Markets

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Abstract

Through a simple model of stock and option pricing, this article shows the effects of introducing a special trader whose action triggers a change in the value of the stock. This market participant would be able to realize abnormal profits if the markets were inefficient or efficient but nonsynchronous by trading options. If markets are efficient and synchronous, the strategic decision to exercise or not options at maturity yields arbitrage opportunities, and surprisingly leads to market incompleteness. Inefficient pricing rules are sufficient to prevent this behavior. A simplified pricing model for callable convertible bonds fits to this framework. The subsequent analysis of the US market shows that it tends to be inefficient, but firms do not seem to fully exploit asymmetric information, so that market prices are coherent. In contrast, Japan exhibits characteristics of an efficient but nonsynchronous market, which allows visible arbitrage opportunities that are very profitable to issuing firms. A similar framework may apply to other areas like open market share repurchases and, in a normative approach, to the takeover process.

1 Introduction

In addition to the traditional financial markets, the exchange of derivative securities on organized markets creates complexity through the fact that they relate to existing underlying securities. In most cases, because of the redundancy of their payoff function with the one of some self-financing trading strategies, the pricing of these securities can be performed by arbitrage arguments. The essence of this technique is to find a trading strategy in basic securities, subject to technical regularity conditions, that exactly replicates the payoffs of the contingent claim at every point in time and in all states of the world, but without using the derivative under study itself. The uniqueness of the price found using these arguments, which corresponds to the initial investment required in order to implement the replicating trading strategy, is a property guaranteed in "dynamically complete" markets. The assumption of market completeness is the main premise of the famous Black-Scholes [4] model for the valuation of European stock options.

Another notion, which should not be confused with completeness but involves the pricing of securities too, is the informational efficiency of financial markets. The ability to take advantage of historical, all public, or all public and private information in order to realize abnormal profits characterize the degree of, respectively, weak, semi-strong or strong form of market efficiency. The latest form is extremely difficult to test empirically, but corresponds to a market where information is integrated so quickly into the prices that no gain can be captured by trading on the basis of superior insider information. Models of full revelation of information, such as Grossman [14], [15] Nielsen [30] and Milgrom and Stokey [28] address the question of the formation of an equilibrium under these conditions. A lack of strong form market efficiency implies that the market price of securities is not a sufficient statistic for all information available; consequently, there exists an information asymmetry, and it is theoretically possible to make use of it in order to beat the market.

This issue is among the most controversial of modern finance. Since the paradigm was introduced by Fama [11], deviations from efficiency have been found to be numerous, even for the weak form, but seemed to be due to methodological problems or to disappear as soon as they were emphasized. Nowadays, the dogm of market efficiency is more than challenged. Some so-

called market anomalies, like seasonalities, have become so regular that the dream of systematically beating the market does not look out of reach. However, measurement issues, social and tax externalities and other dimensions still make market inefficiency not definitely accepted either.

The perspective taken by this paper is not critical with respect to this controversy. Jarrow [22] has found that the presence of a large trader on otherwise efficient, but nonsynchronous options and stock markets allows market manipulation strategies. Back [3] finds that the presence of asymmetric information in coexisting option and underlying markets leads the volatility of the underlying to become stochastic; since volatility is not a traded asset, pricing options by arbitrage is impossible. My starting point is more stringent with respect to efficiency: this paper analyzes the issue of completeness when both markets, in isolation and as a whole, are strongly efficient (Jarrow and Back have each found that failure to respect this condition leads to incompleteness) and where every trader is a price-taker in that nobody is able to choose her trading price.

The "imperfectness" introduced by the model is of a third order: all market participants are not equal with respect to the price charged for their orders because trading itself reveals private information, and the manifestation of a particular market actor induces a discontinuity in the updated market expectation of the asset value. Price taking still holds because this trader has to accept the market price corresponding to her trade, and informational efficiency is guaranteed by the fact that nobody can beat the market on the basis of private information. Furthermore, markets are assumed to be frictionless.

The basic model that I present shows that, although it seems that the conditions are filled for the pricing of options by the appropriate replicating trading strategy, arbitrage opportunities are opened through the ability for the special market actor to choose the payoff of the options at maturity; pricing the option accordingly in order to preclude arbitrage opportunities leads to neglect the information created by the trade, and thus not integrate new private information into the price, creating an anomaly in market efficiency. The strategic use of option markets makes it therefore possible to oppose completeness and efficiency, and to take advantage of this incompatibility. This shows the need for going beyond the synchronous markets assumption,

and may indeed lead to the statement that true perfect competition holds: let alone everybody is a price-taker in the strict sense, but the effects of trading decisions are not usable by their author.

I will take a specific, real-life case in order to illustrate the model. The example uses the pricing mechanism of callable convertible bonds: the issuing firm is the special market actor, and it has the possibility to either issue an overpriced bond, or open up arbitrage opportunities to existing shareholders. Some empirical puzzling results corroborate this dual behavior. The comparison between the US and the Japanese markets suggests that they are structurally different. The US market does not seem to be strongly efficient, but works quite well; in Japan, the pricing mechanism is closer the efficient pricing rule, but the lack of market synchroneity seriously alleviates this advantage while the system yields very severe arbitrage opportunities.

The next section formalizes the problem in a simple continuous-time setup, and derives the opposition between efficiency and completeness. Section 3 introduces the analysis of convertible bonds within this framework, and reevaluates some empirical facts in this new light. The fourth section concludes the article.

2 The Basic Model

2.1 Setup

The basic idea of Jarrow's [22] analysis considers that a market participant which would have, due to its size or notoriety, the opportunity to impact on the price of a given asset could, using a derivatives market (preferably futures contracts), realize arbitrage opportunities. My premises are quite different of his assumptions. I am also interested in a special market participant, who is actually a price-taker, but whose action triggers a price reaction by the market. Therefore, the sequence of events is that the participant commits to a trade, but due to its special status, the market fixes the new price of the asset prevailing for the transaction. The issue of efficiency is translated into the pricing rule. The imperfectness is thus not due to size, but to the information content of the commitment.

I equip the market with the complete probability space $(\Omega, \mathcal{F}, P)$. The arrival of information is represented by an augmented filtration $F = \{\mathcal{F}_t\}_{t\in\tau}$ for the time interval $\tau = [0, \infty)$. There are two basic securities: a riskless money market account and a risky security, namely the stock, on which European call and put options are written.

The stochastic process for the money market account is the following:

$$dB(t) = rB(t)dt \tag{1}$$

where r is the instantaneous interest rate, considered as constant. The introduction of a source of risk would not alter the basic results of the model, but would simply add superfluous complexity to the next developments.

The non dividend-paying underlying security is assumed to follow, under the initial probability measure, an ordinary geometric Itô process:

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$$

where $Z(t)$ is a standard Wiener process adapted to F. Since the existence of a numeraire is guaranteed, the corresponding risk-adjusted process is:

$$dS(t) = rS(t)dt + \sigma S(t)dZ^*(t) \tag{2}$$

where $Z^*(t)$ is an adapted Wiener process under the risk-neutral probability measure Q, obtained using Girsanov's theorem (see Nielsen [29]).

A trader, denoted I, is such that her order initiates a price reaction. Without loss of generality, the reaction occurs if the trader increases her initial holding, denoted K_0 in the stock by 1 unit. The premium then is assumed to be a constant α. The pricing rule is the following:

$$S_I(t) = \begin{cases} S(t) & \text{if } K_I(t) - K_0 < 1 \\ S(t)(1+\alpha) & \text{if } K_I(t) - K_0 \geq 1 \end{cases} \tag{3}$$

The market is efficient in that the action by I signals that the security will be worth $S(T)(1+\alpha)$ at some point in time T under her new holding $K_I(t)$. The price process of the security still respects the martingale property:

$$S(t)(1+\alpha) = \mathrm{E}_Q[e^{-r(T-t)}S(T)(1+\alpha)|\mathcal{F}_t]$$

and trader I would pay a fair price for this security. The risk-adjusted expected rate of return on this portfolio is r, and the real expected rate of return is still μ. If the initial holding K_0 corresponds to the proportion of this security held in the portfolio maximizing trader I's utility, then this pricing rule does not add any change in the risk-return relationship, and $K_I(t) = K_0$: the status quo prevails on the market. For the further developments, let us assume that this trader is almost indifferent between the two portfolios, but slightly prefers the absence of trade. This will justify her action as soon as the risk-adjusted expected rate of return becomes greater than r. Moreover, if I's holding comes back to a value lower than $K_0 + 1$, the stock value falls to its previous level.

Yet, information of this pricing rule mechanism (3), represented by the set $\Phi(I, \alpha)$ in case of a trade by agent I is included into the market sigma-algebra $\mathcal{F}_t$ *ex ante*. In contrast, the same sigma-algebra without taking into account this information is denoted $\tilde{\sigma}(\mathcal{F}_t \setminus \tilde{\sigma}(\Phi(I, \alpha))) = \mathcal{G}_t \subset \mathcal{F}_t$ where $\tilde{\sigma}(.)$ denotes the sigma-algebra generated by the sets in the argument. In both cases, the occurrence of a trade reveals information, and pricing rule (3) applies.

2.2 Option Pricing Rules with one Maturity Date

2.2.1 Inefficient Option Market

Under the former assumptions, but without taking into account the potential influence of trader I, the standard Black-Scholes formula applies for a European call option maturing at time T, with strike price K, on the stock:

$$\begin{aligned}
C_0(T) &= [S(T) - K]1_{\{S(T)\geq K\}} \\
C_0(t) &= \mathrm{E}_Q[e^{-r(T-t)}S(T)1_{\{S(T)\geq K\}}|\mathcal{G}_t] - \mathrm{E}_Q[e^{-r(T-t)}K1_{\{S(T)\geq K\}}|\mathcal{G}_t] \quad (4) \\
&= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (5) \\
d_1 &= \frac{\ln(S(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (6) \\
d_2 &= d_1 - \sigma\sqrt{T-t} \quad (7)
\end{aligned}$$

and the European put option is priced using the put-call parity.

The availability of a European call option with maturity T and strike price K, priced by equations (5) to (7) modifies the situation for trader I: her payoff $C_I(T)$ if she holds one of these options is either zero, if the option finishes out of the money, or $(1+\alpha)S(T)$ if the option is exercised, physical delivery is made but no trade is done on the stock market:

$$C_{I0}(T) = [(1+\alpha)S(T) - K]1_{\{S(T)\geq K\}}$$

This payoff at maturity exceeds the regular Black-Scholes value by:

$$\Pi_0(T) = C_{I0}(T) - C_0(T) = \alpha S(T)1_{\{S(T)\geq K\}}$$

and, discounted to t, this yields a risk-adjusted expected excess payoff of:

$$\begin{aligned} \Pi_0(t) &= e^{-r(T-t)}\alpha \mathrm{E}_Q[S(T)1_{\{S(T)\geq K\}}|\mathcal{G}_t] \\ &= \alpha S(t)N(d_1) > 0 \end{aligned} \tag{8}$$

Equation (8) implies that the information $\Phi(I,\alpha)$ is not revealed through the option market, which is thus inefficient. This shows the consequence of the naive pricing of the call option on the expected payoff for trader I. Consistently with the assumption of originally almost indifference between trading or not, this trader will engage in this strategy, because $C_0(t)/B(t) < \mathrm{E}_Q[C_{I0}(T)/B(T)|\mathcal{F}_t]$, and the risk-adjusted expected rate of return is greater than r.

2.2.2 Efficient but Nonsynchronous Markets

The necessary adjustment on the option market should be such that it precludes arbitrage opportunities. It is shown here how information about the effect of trader I is introduced into the option market sigma-algebra, but failing to take into account the possible supplementary trade on the stock market. In order to process with this task, the set $\Phi(I,\alpha)$ is partitioned in two components $\Psi(I,\alpha,M_C)$ and $\Theta(I,\alpha,M_S)$. Ψ stands for information that would be only used on the option market, and Θ stands for information only used on the stock market. The dealer knowing information contained in Θ but not Ψ is aware of the possibility for I to trade on the option market, but neglects the possible impact of this trade on the price process on the stock

market. The information available to the option market is described by the sigma-algebra $\mathcal{H}_t = \tilde{\sigma}(\mathcal{G}_t \bigcup \Psi(I, \alpha, M_C))$. Obviously, $\mathcal{H}_t \subset \mathcal{F}_t$.

As deduced from the previous subsection, the pricing rule $C_0(t)$ does not correspond to the risk-adjusted expectation of the payoff to the option for trader I. Replacing $\mathcal{G}_t$ by $\mathcal{H}_t$ impacts the process for the stock price: upon purchase of the option, the market knows that the terminal value of the stock will be $S(T)(1+\alpha)$ if the option is exercised, and $S(T)$ otherwise. Accordingly, the pricing rule for the option becomes:

$$\begin{aligned} C_1(t,T) &= \mathrm{E}_Q[e^{-r(T-t)}(1+\alpha)S(T)1_{\{S(T)\geq K\}}|\mathcal{H}_t] \\ &\quad -\mathrm{E}_Q[e^{-r(T-t)}K1_{\{S(T)\geq K\}}|\mathcal{H}_t] \qquad (9) \\ &= (1+\alpha)S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (10) \end{aligned}$$

where d_1 and d_2 are unchanged.

The stock price process, in order to obey the martingale property, shifts up and integrates the information:

$$\begin{aligned} S_1(t) &= \mathrm{E}_Q[e^{-r(T-t)}[(1+\alpha)S(T)1_{\{S(T)\geq K\}} + S(T)1_{\{S(T)<K\}}]|\mathcal{H}_t] \\ &= S(t)(1+\alpha N(d_1)) > S(t) \qquad (11) \end{aligned}$$

Considering the possibility for I to trade on the stock market, she has an additional opportunity which is overlooked by the option market: at maturity, for values of the stock such that $S(T) \geq K/(1+\alpha)$, it is optimal to exercise the option (paying K) and receive a stock worth $S(T)(1+\alpha) > K$. Hence, the action of exercising out of the money options drives them in the money. By this strategic behavior, a new profit perspective opens up for the trader:

$$C_{I1}(T) = [(1+\alpha)S(T) - K]1_{\{S(T)\geq \frac{K}{1+\alpha}\}}$$

This payoff exceeds the regular payoff implicit in pricing rule C_1 by:

$$\Pi_1(T) = C_{I1}(T) - C_1(T) = (1+\alpha)S(T)(1_{\{S(T)\geq \frac{K}{1+\alpha}\}} - 1_{\{S(T)\geq K\}})$$

and, discounted to t, this yields a risk-adjusted expected excess payoff of:

$$\begin{aligned} \Pi_1(t) &= e^{-r(T-t)}(1+\alpha)\mathrm{E}_Q[S(T)(1_{\{S(T)\geq \frac{K}{1+\alpha}\}} - 1_{\{S(T)\geq K\}})|\mathcal{H}_t] \\ &= (1+\alpha)S(t)(N(d_1^*) - N(d_1)) > 0 \qquad (12) \\ d_1^* &= \frac{\ln \frac{S(t)(1+\alpha)}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad (13) \end{aligned}$$

Again, a profitable tradig strategy exists for trader I and, subsequently, arbitrage opportunities are available for the other market partcipants who would possess information $\Phi(I, \alpha)$. A third level of treatment of information is necessary in order to prevent this situation.

2.2.3 Efficient and Synchronous Markets

The results presented in this subsection are equivalent, in the simple setup of this model, to the ones induced by the sufficient condition of synchronous markets proposed in Jarrow [22]. All market actors are assumed to be endowed with the full information sigma-algebra $\mathcal{F}_t$. The goal is to suppress arbitrage profits made possible by equation (10). Trader I is now supposed to exercise the option whenever $S(T) \geq K/(1+\alpha)$. Adaptation of pricing rule (10) yields an arbitrage-free pricing formula:

$$\begin{aligned} C_I(t,T) &= \mathrm{E}_Q[e^{-r(T-t)}(1+\alpha)S(T)1_{\{S(T)\geq \frac{K}{1+\alpha}\}}|\mathcal{F}_t] \\ &\quad -\mathrm{E}_Q[e^{-r(T-t)}K1_{\{S(T)\geq \frac{K}{1+\alpha}\}}|\mathcal{F}_t] \qquad (14) \\ &= (1+\alpha)S(t)N(d_1^*) - Ke^{-r(T-t)}N(d_2^*) \qquad (15) \end{aligned}$$

where d_1^* is given in (13) and $d_2^* = d_1^* - \sigma\sqrt{T-t}$. Subscript I corresponds to the necessary pricing rule adjustment due to the presence of trader I.

The stock price is affected by this trade on the option market, and experiences an upward jump in order to compensate for the implied increase in expected value. To comply with the no-arbitrage requirement, the following process for the stock price replaces the original one:

$$\begin{aligned} S_I(t) &= \mathrm{E}_Q[e^{-r(T-t)}[(1+\alpha)S(T)1_{\{S(T)\geq \frac{K}{1+\alpha}\}} + S(T)1_{\{S(T)< \frac{K}{1+\alpha}\}}]|\mathcal{F}_t] \\ &= S(t)(1+\alpha N(d_1^*)) > S_1(t) > S(t) \qquad (16) \end{aligned}$$

and, in a similar spirit to the findings of Back [3], it turns out that the drift and volatility appearing in the SDE corresponding to this process are stochastic, because of the dependence over $S(t)$ which is not observable anymore. From Itô's lemma, equation (16) yields, in the risk-adjusted measure:

$$\begin{aligned} \frac{dS_I(t)}{S_I(t)} &= \Big[\, r + \tfrac{\alpha S(t)}{\sigma\sqrt{T-t}S_I(t)} \Big((r+\tfrac{1}{2}\sigma^2)N'(d_1^*) + \tfrac{\sigma N''(d_1^*)}{2\sqrt{T-t}} \\ &\quad + \tfrac{\ln(S/K)N'(d_1^*)}{2(T-t)} - \tfrac{\sigma^3}{4} \Big) \Big]\, dt + \Big[\, \sigma + \tfrac{\alpha N'(d_1^*)S(t)}{\sqrt{T-t}S_I(t)} \Big]\, dZ^*(t) \end{aligned}$$

This formula is the one to use for the pricing of options maturing before T, as shown later. From T on, it considerably simplifies because $N(d_1^*) = 0$ or 1, and the first and second derivatives are zero. Therefore, $S_I(T)$ follows again an ordinary geometric Brownian motion, with the same instantaneous drift and volatility. In addition, $S_I(T) = S(T)$ if the sample path of $S(T)$ lead to a value below $K/(1+\alpha)$, and $S(T)(1+\alpha)$ otherwise. In other words, the following surprising result obtains:

$$\Pr_Q[S_I(T) \in [\frac{K}{1+\alpha}, K] | \mathcal{F}_t] = 0$$

and, since $S_I(t)$ is the observed price at time while $S(t)$ is a "shadow price" corresponding the the original price process, this indicates that no observed terminal price should lie in the interval. In this market, it is thus impossible to exercise rationally out of the money options.

From simply trading on the option and stock markets, and exercising the option whenever it is in the money, nobody is able to realize arbitrages based on these pricing rules. The next developments show how the existence of different maturities causes the failure of this statement, and ultimately results in incompleteness of the market for the security.

2.3 Option Pricing upon Existence of Several Maturitites

The key to the demonstration of incompleteness in such an economy lies in the availability, on the option market, of several maturities. The pricing rule for trader I remains the same, but she has a choice of options maturing between T and T'. The options available to the market have maturities varying from t to T'. To simplify, it is sufficient to consider only four possible maturity dates $T''' < T < T'' < T'$.

The option pricing rule for the option maturing at T is unchanged. The starting point of the analysis is the situation where trader I buys at time t an option maturing at T. Three cases have to be considered for the pricing of options with different time-to-maturity:

- Maturities for which no call option is held by trader I, but maturing

after T.

- Maturities for which no call option is held by trader I, but maturing before T.
- Other maturities for which trader I holds at least one call option.

2.3.1 Maturities for which no Call Option is Held by Trader I, but Maturing After T

The previously derived pricing rule for the stock price, $S_I(t)$, suggests that the terminal value of the original stock price process $S(t)$ determines the initial value of the stock price from T on.

Imagine that some trader holds a call option with maturity $T'' > T$, whereas trader I is long at least one call with maturity T. Then, the stock price process to consider is $S_I(t)$. The option price at time t is given in the following Proposition:

Proposition 1 *The price at time t of a call option with strike price K'' maturing at $T'' > T$ when trader I is long at least one option with strike price K maturing at T is given by the formula:*

$$\begin{aligned} C_{-I}(t,T,T'') &= S(t)(1+\alpha)M(d_1^*, b_1^*; \rho) - K''e^{-r(T''-t)}M(d_2^*, b_2^*; \rho) \\ &+ S(t)M(-d_1^*, b_1; \rho) - K''e^{-r(T''-t)}M(-d_2^*, b_2; \rho) \end{aligned} \tag{17}$$

where

$$\begin{aligned} d_1^* &= \frac{\ln\frac{S(t)(1+\alpha)}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ b_1^* &= \frac{\ln\frac{S(t)(1+\alpha)}{K''} + (r+\frac{1}{2}\sigma^2)(T''-t)}{\sigma\sqrt{T''-t}} \\ b_1 &= \frac{\ln\frac{S(t)}{K''} + (r+\frac{1}{2}\sigma^2)(T''-t)}{\sigma\sqrt{T''-t}} \\ d_2^* &= d_1^* - \sigma\sqrt{T-t} \\ b_2^* &= b_1^* - \sigma\sqrt{T''-t} \end{aligned}$$

$$
\begin{aligned}
b_2 &= b_1 - \sigma\sqrt{T'' - t} \\
\rho &= \sqrt{\frac{T-t}{T''-t}}
\end{aligned}
$$

with $M(x, y; \rho)$ denoting the cdf of a standard bivariate normal distribution taken at values x and y when the correlation between x and y is ρ. The subscript $-I$ means that the pricing rule assumes the existence of trader I, who holds at least one call option maturing at T but no call maturing at T''.

Proof. The terminal payoff of this option is:

$$
\begin{aligned}
C_{-I}(T'', T, T'') &= [S_I(T'') - K'']1_{\{S_I(T'') \geq K''\}} \\
&= [S_I(T'') - K'']1_{\{S_I(T'') \geq K''\}}(1_{\{S(T) \geq \frac{K}{1+\alpha}\}} + 1_{\{S(T) < \frac{K}{1+\alpha}\}}) \\
&= [(1+\alpha)S(T'') - K'']1_{\{S(T'') \geq \frac{K''}{1+\alpha}\}}1_{\{S(T) \geq \frac{K}{1+\alpha}\}} \\
&\quad + [S(T'') - K'']1_{\{S(T'') \geq K''\}}1_{\{S(T) < \frac{K}{1+\alpha}\}}
\end{aligned}
$$

and, discounted to t, this yields:

$$
\begin{aligned}
C_{-I}(t, T, T'') &= \mathrm{E}_Q[e^{-r(T''-t)}[(1+\alpha)S(T'') - K'']1_{\{S(T'') \geq \frac{K''}{1+\alpha}\}}1_{\{S(T) \geq \frac{K}{1+\alpha}\}}|\mathcal{F}_t] \\
&\quad + \mathrm{E}_Q[e^{-r(T''-t)}[S(T'') - K]1_{\{S(T'') \geq K''\}}1_{\{S(T) < \frac{K}{1+\alpha}\}}|\mathcal{F}_t]
\end{aligned}
$$

The first term, using the law of iterated expectations, can be rewritten:

$$
\mathrm{E}_Q[e^{-r(T-t)}1_{\{S(T) \geq \frac{K}{1+\alpha}\}}\mathrm{E}_Q[e^{-r(T''-T)}[(1+\alpha)S(T'') - K'']1_{\{S(T'') \geq \frac{K''}{1+\alpha}\}}|\mathcal{F}_T]|\mathcal{F}_t]
$$

It corresponds to the risk-adjusted expectation of the payoff at time T of the positive term of a compound option on a call maturing at T''. The strike prices of the compound and underlying options are $\frac{K}{1+\alpha}$ and $\frac{K''}{1+\alpha}$, respectively. The pricing formula for such an option is provided in Geske [12]. The same reasoning holds for the second term, leading to formula (17). □

2.3.2 Maturities for which no Call Option is Held by Trader I, but Maturing Before T

The relevant stock price for the valuation of such options is described in equation (16): the fact that the stock price at T may shift upwards if the

option held by I is exercised impacts on the price of options maturing before this deadline. The next Proposition provides the pricing formula:

Proposition 2 *The price at time t of a call option with strike price K''' maturing at $T''' < T$ when trader I is long at least one option with strike price K maturing at T is given by the formula:*

$$C_{-I}(t,T,T''') = S(t)[N(a_1) + \alpha M(a_1, d_1^*; \rho')] - K'''e^{-r(T'''-t)}N(a_2) \qquad (18)$$

where

$$\begin{aligned} a_1 &= \frac{\ln \frac{S(t)}{S^*} + (r + \frac{1}{2}\sigma^2)(T''' - t)}{\sigma\sqrt{T''' - t}} \\ a_2 &= a_1 - \sigma\sqrt{T''' - t} \\ S^* &= \arg[S(T''') + \alpha N(d_1^*(T''', T)) - K''' = 0] \\ \rho' &= \sqrt{\frac{T''' - t}{T - t}} \end{aligned}$$

with $M(x, y; \rho')$ denoting the cdf of a standard bivariate normal distribution taken at values x and y when the correlation between x and y is ρ, and $d_1^*(T''', T)$ corresponds to (13) where the option is evaluated at T''' and matures at T. The subscript $-I$ means that the pricing rule assumes the existence of trader I, who holds at least one call option maturing at T but no call maturing at T'''.

Proof: The terminal payoff of this option is:

$$\begin{aligned} C_{-I}(T''', T, T''') &= [S_I(T''') - K''']1_{\{S_I(T''') \geq K'''\}} \\ &= [\alpha \mathrm{E}_Q[e^{-r(T-T''')}S(T)1_{\{S(T) \geq \frac{K}{1+\alpha}\}}|\mathcal{F}_{T'''}] \\ &\quad + S(T''') - K''']1_{\{S_I(T''') \geq K'''\}} \end{aligned}$$

where the first term between brackets represents the positive part of the payoff at time T''' of a compound option on a call option on the stock maturing at T. The second and third terms represent the payoff of an ordinary call option on the stock, which can itself be considered as a call option on the stock with infinite maturity and exercise price 0; therefore, these terms can also be considered as the payoff on a compound option. Discounted back to t at the risk-free rate, both options can be priced using the formula for the compound option proposed by Geske [12], yielding formula (18). □

2.3.3 Other maturities for which Trader I Holds at least One Call Option

Exercise might occur in two circumstances: either the previous holding of trader I in the stock is greater, either lower than K_0+1. For the considered maturity T', if $K_I \geq K_0+1$, then the stock price is already equal to $S(T')(1+\alpha)$, and the call option with strike price K' is exercised whenever $S(T')(1+\alpha) \geq K'$. The same rule holds when the stock price process is the original one, so that the two situations yield the same result. Thus, equation (15) holds.

The relationship between these prices is summarized in the next Corollary:

Corollary 1 *Let $t < T''' < T < T'' < T'$, and all options have the same strike price. The following inequaltities hold:*

$$C_{-I}(t,T,T'') < C_I(t,T'')$$
$$C_{-I}(t,T,T''') < C_I(t,T''')$$
$$C_{-I}(t,T,T''') < C_I(t,T) < C_{-I}(t,T,T'') < C_I(t,T')$$

Proof: Since all options have the same strike price, $C_I(T,T) < C_{-I}(T,T,T'')$ because the payoff at time T of the option maturing at time T'' if $S(T) \geq K/(1+\alpha)$ is a call option for which the time value is positive and which costs nothing, whereas the payoff of $C_I(T)$ is just the intrinsic value of this call. Moreover, $C_{-I}(T'',T,T'') < C_I(T'',T')$ and $C_{-I}(T''',T,T''') < C_I(T''',T)$, both because the time value of the option maturing last is greater and the process of the underlying is more favorable for this option, since it is shifted upwards by a factor $1+\alpha$ for all sample paths versus only the paths respecting $S(T) \geq K/(1+\alpha)$ for the option with subscript $-I$. To show that $C_{-I}(T'',T,T'') \leq C_I(T'',T'')$, and $<$ with positive probability, it is again sufficient to notice that the underlying process of the observed stock price is more favorable for the first option. A similar reasoning holds for $C_{-I}(T''',T,T''') \leq C_I(T''',T''')$, and $<$ with positive probability. Using the same discount factor for all options allows to transfer all inequalities to time t. □

2.4 Source of Incompleteness and Remedy

2.4.1 Creation of Arbitrage Opportunities on Efficient Markets

The previous pricing rules correspond to a complete market. The price of all securities reflect, under the unique risk-adjusted probability measure, the expected payoff discounted at the risk-free rate. This supposes that trader I acts accordingly, and has no interest in using her characteristics to affect the price. If the stock were the only security exchanged in this economy, no other strategy would be imaginable, because the payoff to this security cannot be manipulated by its holder. Yet, introducing a call option opens the way to manipulating strategies. The idea is the following: the pricing of the option supposes that trader I exercises her right to buy the stock at expiration. If she sells the security instead, the premium α on the stock price is lost, because it is intimately attached to its holding by I. Let alone this will affect the stock price process, but also the payoff of all options with longer maturities not held by trader I; the value of other options held by I is unchanged. The special trader can use this mispricing for arbitrage purposes. She does so provided that the final position in the underlying asset is at least $K_0 + 1$, and is accounted for with respect to the initial payoff; costless arbitrages are available for all other traders.

Here is a simple strategy yielding arbitrages. Imagine that trader I has bought one call with strike K and maturity T. In addition, she sequentially enters the following (costless) trading strategy: first, buy one option with strike K and maturity T, and sell $1+\gamma$ options with strike K and maturity T'', $T < T'' < T'$, then buy $\gamma \geq 1$ options with strike K and maturity T'. The maturity T' is chosen so that $C_I(t,T) + \gamma C_I(t,T') - (1+\gamma)C_{-I}(t,T,T'') = 0$, where the options bought buy trader I respect (15) and the options sold respect Proposition 1. Inequality $T < T'' < T'$ derives from Corollary 1. Sequential trades ensure that no alternative arbitrage-free pricing rule can be used in order to price the options first bought and sold.

Two trading strategies are considered at time T: either trader I considers exercising her option, or she sells it. This implies that the only relevant cases are the one for which $S(T) \geq K/(1+\alpha)$, because otherwise both strategies produce identical outcomes: the pricing rule at time t corresponds in this

case to the risk-adjusted expected payoff in this price range.

If the options are exercised, the value of the portfolio to trader I is:

$$G(T) = 2[S(T)(1+\alpha) - K] + \gamma C_I(T,T') - (1+\gamma)C_I(T,T'') \tag{19}$$

because the stock price process is $S(T)(1+\alpha)$ upon exercise. These payoffs correspond to the original pricing rules.

If the options are sold, the stock price process from T on becomes similar to the one described in equation (16) where T' replaces T and T replaces t in the new process. The payoff obtained by the call option maturing at T'' is unchanged, since trader I still earns the premium at that maturity date. Therefore, if $S(T) \geq K$, the payoff becomes:[1]

$$H(T) = 2[S(T) - K] + \gamma C_I(T,T') - (1+\gamma)C_{-I}(T,T',T'') \tag{21}$$

where $C_{-I}(T,T',T'')$ respects equation (18).

The next Proposition justifies the existence of arbitrage opportunities leading to market incompleteness:

Proposition 3 *All values of* $\gamma \geq \max[1, \gamma^*(S(T))]$, *where* $\gamma^*(S(T)) < \infty$ *is defined by*

$$\begin{aligned}\gamma^*(S(T)) &= 2\alpha S(T)[S(T)((1+\alpha)N(d_1^*) - N(a_1) - \alpha M(a_1, d_1^*, \rho')) \\ &\quad -Ke^{-r(T''-T)}(N(d_2^*) - N(a_2))]^{-1} - 1\end{aligned} \tag{22}$$

and the arguments of N *and* M *are defined in expressions (15) and (18) with the corresponding maturities, lead trader* I *to choose the strategy yielding payoff* $H(T)$ *for a given value* $S(T) > K$. *There exists a finite value* $\gamma^{**} = \max_{\bar{S} \geq S(T) \geq K}[\max[1, \gamma^*(S(T))]]$ *such that whenever* $\gamma \geq \gamma^{**}$, *this strategy is chosen for all values of* $S(T) \leq \bar{S}$. *Since it does not correspond to the options pricing rules, arbitrage opportunities arise with positive probability on the market, which is therefore incomplete.*

[1] If $K > S(T) \geq K/(1+\alpha)$, this becomes:

$$H(T) = \gamma C_I(T,T') - (1+\gamma)C_{-I}(T,T',T'') \tag{20}$$

which will yield a similar result to the one exposed in the body of the text.

Proof: The sale of the option at time T incurs a loss of αS per option. But from Corollary 1, $C_{-I}(T, T', T'') < C_I(T, T'')$, which implies that the third term of $H(T)$ is greater than of $G(T)$. The strategy yielding $H(T)$ can become more profitable than strategy leading to $G(T)$ whenever the following inequality becomes true:

$$(1+\gamma)(C_I(T, T'') - C_{-I}(T, T', T'')) > 2\alpha S(T)$$

with the condition $\gamma \geq 1$ in order to ensure the existence of the premium at time T'. Both sides are positive provided $\gamma > -1$, and the left-hand side is linearly increasing in γ. Therefore, for each value of $S(T)$, there exists a finite value of γ for which this becomes an equality, and all greater values of γ respect the inequality. Replacing $C_I(T, T'')$ by its value given in equation (15) and $C_{-I}(T, T', T'')$ by formula (18) yields this critical value of γ expressed in (22). Maximizing over an arbitrary interval of values of $S(T)$ yields γ^{**}, and ensures that this strategy is chosen with nonzero probability. Since this strategy is as costly as the one yielding $G(T)$, trader I will implement it, and the actual options payoffs will not correspond to the ones used for risk-neutral valuation. Because all information is public, this mispricing can be used by other traders in order to realize arbitrage opportunities, and thus the martingale measure Q is not unique (Harrisson and Kreps [16]). The market is incomplete. □

Again, since the cost of implementing strategy leading to $G(T)$ is equal to the risk-adjusted discounted expected payoff of this strategy, getting $H(T) > G(T)$ induces trader I to engage in this operation, hence creating actual arbitrage opportunities. This result is important, because it is perfectly coherent with respect to the assumptions and yet results in the surprising conclusion that the most efficient markets are incomplete. The natural question to ask is whether there exists a remedy to this situation, since the ultimate level of efficiency seems to have been considered here.

2.4.2 Market Inefficiency as a Remedy for Incompleteness

The source of market incompleteness does not directly lie in the possibility for trader I to choose the payoff of her option at maturity, but in the incentive to steal value from options sold to other investors who relied on a too high

price with respect to the modified one. This trader has indeed the power to set two different prices for the same security; no arbitrage would be created if she could not find any advantage in one of the two possible payoffs regardless of the state of the world.

In order to prevent trader I from misleading the market, it must force this particular participant to make sure that exercising her option at maturity yields the maximal gain whenever it is rational to exercise (i.e. $S(T) \geq K/(1+\alpha)$). The inefficiency of the option market is a sufficient characteristic that ensures this result. Consider that the market sigma-algebra at time t is $\mathcal{G}_t$, and thus no operation on the option market reveals information $\Phi(I, \alpha)$. In this case, it has been shown earlier that the purchase of an option maturing at T for trader I yields a profit of:

$$\Pi_0(t) = \alpha S(t) N(d_1)$$

The maturity and strike price that maximize this profit are, respectively, $T = t$ and $K \leq S(t)/(1+\alpha)$. Choosing $K = S(t)$ would signify to buy an at the money option at maturity, for a price of 0. The profit is strictly positive. However, it is not an arbitrage profit because it corresponds to a positive cost K and the stock cannot be sold after the operation for a price $S(t)(1+\alpha)$ because this value is contingent on trader I holding this security. In this case, the trader is simply able to beat the market, and has no incentive to delay the realization of trading gains by taking a strategy leading to a payoff like $H(T)$ before: the possible profit per option purchased is already maximized.

Replacing $\mathcal{F}_t$ by $\mathcal{G}_t$ secures the absence of arbitrage opportunities for other market traders, since ignorance of information $\Phi(I, \alpha)$ results in revelation only through the impact of an exercise of the option on the stock market. It it then too late to use the option market because the shift in the stock price process is irreversible and precedes the release of information.

This assumption can be replaced by a form of market inefficiency where the information Φ is not included in the sigma algebra of the market for options with maturities greater than some prespecified T: in this case, the previous findings obtain, but trader I's profit is maximized for options maturing at T. If nobody, including trader I, is aware of Φ, then no trade takes place on this basis and the Black-Scholes assumptions hold.

3 Analysis of Callable Convertible Bonds

The model developed in the previous section is in essence applicable to the issue of pricing callable convertible bonds, but needs some preliminary changes in order to account for the specificity of this security. The results drawn from the previous analysis may subsequently throw a new light on some issues raised by numerous studies on the convertibles market.

3.1 Model Adjustments

The convertible bond is a security whose payoff is contingent on the stock price process, because it is a bond which can be converted into equity at some point in time: the decision to exercise the option is made whenever the conversion value of the bond is higher than the ongoing value of the convertible. It involves a mix of a bond and of a call option on the stock.

When the convertible bond is callable, the issuing company has to opportunity to redeem the security with cash. In fact, it has been shown by Brennan and Schwartz [5] and Ingersoll [19] that the optimal call policy for the firm is to exercise this right as soon as the conversion value of the bond (the corresponding value in stocks) is equal to the call price, i.e. the call option attached to the bond becomes in the money. In this case, securityholders choose to convert the bond, and the firm indeed sells its own stock in exchange of the market value of the convertible. This possibility can hence be viewed as a put option given to the company.

The pricing of callable convertibles is very complex when it is assumed that all options are of the American type, and that the company exhibits credit risk. In order to implement my model, I need several simplifying assumptions that make it tractable and allow to interprete its implications, provided that relaxing them does not alter the quality of the conclusions obtained.

As before, the interest rate is assumed to be constant, and the original stock price process is still depicted by a geometric Brownian motion. The starting equations are thus (1) and (2) in the risk-neutral probability measure. The face value of the convertible is F, and it is assumed to pay no coupon. The investment value is therefore the discounted value of F. The specific issue of

the pricing of corporate defaultable debt is therefore avoided, and only the option-like features of the security are put under study. The analysis can for instance be conducted for convertible preferred stock.

The maturity of the bond is T_F. The investor who purchases the convertible security gets the investment value and a call option on one stock (again without loss of generality) with the same maturity, whose exercise involves the conversion of the bond into a unit of common stock. I assume no dividend, so that the call can be taken as an American option without modification of the pricing formula.[2] The convertible is call-protected until some time T_P, $T_P \leq T_F$. This time corresponds to the first possible exercise date of the right for the company to force conversion. For the purpose of the modelling, it is sufficient to consider that the company can exercise its conversion right only at that moment. However, the same firm can issue other convertible bonds with the same conditions except a longer deadline of call protection T'_P, which actually yields here results equivalent to considering that there are several possible exercise dates for the option given to the firm, but without complicating the pricing mechanisms.[3]

The special trader I in the model is the firm. It is already well-documented that the decision of the firm to call convertible debt hurts existing shareholders (see Mikkelson [26], Ofer and Natarajan [31] and Cowan, Nayar and Singh [6]), an effect which is often hypothesized as a negative signal, in the spirit of Harris and Raviv [17]. The alternative view of a substitution effect between debt and equity, tested by Datta and Inskandar-Datta [8], can also be supported, but it turns out that forced conversions are associated with a significant loss in the market value of all the firm's securities. The negative signal effect is assumed to be translated in a negative premium α in the stock value after dilution. The initial stock price process is assumed to already account for the dilution factor.

[2]The time value of the call is zero due to the exponential increase of the strike price, but its volatility value is always positive, so that it is always optimal to keep this option alive rather than exercising it, unless it is forced.

[3]The firm actually holds an American option whose maturity window is $[T_P, T^*]$ for some T^*, and its pricing formula is much more complex than the one of the European put. This availability of many exercise dates is surrounded by the possibility of issuing many convertibles with each put option having a different maturity.

If the market is strongly efficient, the information about the discount α is assumed to be common knowledge, and affects the stock price at announcement of the forced conversion. Therefore, the firm which announces this operation at T_P will float shares worth $(1-\alpha)S(T_P)$ and receive the same amount; this is the reverse case as in the basic model. Nevertheless, the firm can only force conversion when the intrinsic value of the conversion right is positive to the bondholders; otherwise they require cash.

Finally, it must be stressed that the firm sets the initial price, but runs the risk of unsuccessfully trying to float overpriced convertibles. The market implicitly fixes the maximum issuing price, and the company is a price-taker with respect to this upper bound.

3.2 Pricing Rules

Under these assumptions, two pricing rules for the convertible bond are proposed: first, the "inefficient market pricing rule" which does not take into account the negative premium upon forced conversion; second, the "efficient market pricing rule" explicitly introducing this discount.[4]

3.2.1 Inefficient Market Pricing Rule

The price of the convertible at time t is given in the following Proposition:

Proposition 4 *The price of a callable convertible bond with maturity T_F, where the firm is able to force conversion at T_P, and where $S(t)$ follows a standard geometric Brownian motion is given by:*

$$P_0^c(t,T_P,T_F) = Fe^{-r(T_F-t)} + C_0(t,T_F) - \pi_0(t,T_P,T_F) \tag{23}$$

$$C_0(t,T_F) = S(t)N(d_1) - Fe^{-r(T_F-t)}N(d_2) \tag{24}$$

$$\pi_0(t,T_P,T_F) = Fe^{-r(T_F-t)}M(a_2,-d_2;\rho) - S(t)M(a_1,-d_1;\rho) \tag{25}$$

[4]The median case, where the convertible market is efficient but not the stock market, is obtained by arguments analogous to the ones developed in the basic model.

where

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S(t)}{Fe^{-r(T_F-T_P)}} + (r+\frac{1}{2}\sigma^2)(T_P-t)}{\sigma\sqrt{T_P-t}} \\
d_1 &= \frac{\ln \frac{S(t)}{F} + (r+\frac{1}{2}\sigma^2)(T_F-t)}{\sigma\sqrt{T_F-t}} \\
a_2 &= a_1 - \sigma\sqrt{T_P-t} \\
d_2 &= d_1 - \sigma\sqrt{T_F-t} \\
\rho &= \sqrt{\frac{T_P-t}{T_F-t}}
\end{aligned}
$$

with $M(x,y;\rho)$ denoting the cdf of a standard bivariate normal distribution taken at values x and y when the correlation between x and y is ρ.

Proof. The noncallable convertible is made of the investment value and a call option on the stock whose strike price is the investment value, whose Black-Scholes valuation formula corresponds to (24). The introduction of the callability provision gives an option to the firm, with payoff at time T_P:

$$\pi_0(T_P) = [Fe^{-r(T_F-T_P)} + C_0(T_P,T_F) - S(T_P)]1_{\{S(T_P)\geq Fe^{-r(T_F-T_P)}\}}$$

Using put-call parity, the terms between brackets are equivalent to a put option on the stock maturing at T_F with a strike price F. Hence, the payoff of this option is a put option on the stock, and Geske's formula applies. □

3.2.2 Efficient Market Pricing Rule

The market should be aware that the stock price depresses by a factor α whenever the firm forces conversion. Yet, upon exercise of its call, the company must take care of the fact that bondholders still should prefer to convert their security into stock rather than to require cash reimbursement. This concern makes the story more complex than in the base case, because the region where the firm can force conversion is necessarily smaller than in the ordinary Black-Scholes case, and thus decreases the value of its compound option.

The investor still holds a call option on the stock, and its pricing should keep on using the original risk-adjusted process for the stock price, because the option reaches maturity if and only if the firm does not force conversion and does not alter the process for the stock price. The corresponding convertible price is given in the next Proposition:

Proposition 5 *The price of a callable convertible bond with maturity T_F, where the firm is able to force conversion at T_P, and where the market expects S to fall by a factor α upon forced conversion is given by:*

$$\begin{aligned} P_I^c(t,T_P,T_F) &= Fe^{-r(T_F-T)} + C_0(t,T_F) - \pi_I(t,T_P,T_F) \\ &\quad -\alpha S(t)N(a_1^*) \qquad (26) \\ \pi_I(t,T_P,T_F) &= Fe^{-r(T_F-t)}M(a_2^*,-d_2;\rho) - S(t)M(a_1^*,-d_1;\rho) \qquad (27) \end{aligned}$$

where

$$\begin{aligned} a_1^* &= \frac{\ln\frac{S(t)(1-\alpha)}{Fe^{-r(T_F-T_P)}} + (r+\frac{1}{2}\sigma^2)(T_P-t)}{\sigma\sqrt{T_P-t}} \\ d_1 &= \frac{\ln\frac{S(t)}{F} + (r+\frac{1}{2}\sigma^2)(T_F-t)}{\sigma\sqrt{T_F-t}} \\ a_2^* &= a_1 - \sigma\sqrt{T_P-t} \\ d_2^* &= d_1 - \sigma\sqrt{T_F-t} \end{aligned}$$

with $M(x,y;\rho)$ denoting the cdf of a standard bivariate normal distribution taken at values x and y when the correlation between x and y is ρ.

Proof. With respect to the previous Proposition, the only difference appears in the payoff for the compound option held by the firm. At time T_P, it is written:

$$\pi_I(T_P) = [Fe^{-r(T_F-T_P)} + C_0(T_P,T_F) - (1-\alpha)S(T_P)]1_{\{(1-\alpha)S(T_P)\geq Fe^{-r(T_F-T_P)}\}}$$

The old strike price of the compound option is multiplied by $1-\alpha$, and a term $\alpha S(T_P)$ is added in the payoff function. The adjustments in formula (25) are straightforward, yielding (27). □

3.2.3 Comparison of the Pricing Rules

The difference between the prices of the convertible bond in the efficient and inefficient markets is:

$$\begin{aligned}\Delta(t) &= P_I^c(t, T_P, T_F) - P_0^c(t, T_P, T_F) \\ &= -\alpha S(t) N(a_1^*) + (\pi_I(t, T_P, T_F) - \pi_0(t, T_P, T_F)) \qquad (28)\end{aligned}$$

This represents the degree of mispricing of the bond in the inefficient market. Thus, accounting for the decrease in the stock price when conversion is forced has two opposite effects. The first one is that it gives the firm the possibility to get α times the stock for free at maturity of its compound option, whenever it is exercised. This loss in the convertible value is translated through a decrease in the initial price. In the meantime, the fact that the firm has to ensure that conversion really takes place when forced raises the strike price of its option from $Fe^{-r(T_F-T_P)}$ to $Fe^{-r(T_F-T_P)}/(1-\alpha)$. Since the compound option is a call, this increase in the strike price reduces its value. If $\Delta = 0$, it means that the market price of the convertible is the same whether the market is efficient or not.

The stock price process is completely different in each hypothesis: in the inefficient case, it remains a geometric Brownian motion, while it corresponds to equation (16) with $1+\alpha$ replaced by $1-\alpha$ in the efficient market case. Let alone the stock price jumps downwards when the callable convertible is issued but also one gets this surprising result:

$$\begin{aligned}&\mathrm{Pr}_Q[S_I(T_P) \in [Fe^{-r(T_F-T_P)}, Fe^{-r(T_F-T_P)}/(1-\alpha)] | \mathcal{F}_t] = \\ &\mathrm{Pr}_Q[S(T_P) \in [Fe^{-r(T_F-T_P)}, Fe^{-r(T_F-T_P)}/(1-\alpha)^2] | \mathcal{G}_t] \qquad (29)\end{aligned}$$

This is the consequence of the fact that the adjusted stock price is the same as the initial one when the conversion is not forced, but is reduced by a factor $1-\alpha$ if the firm believes it can force conversion. The interval considered for the observed stock price $S_I(T_P)$ in the efficient market case can be qualified as the "uncertain range", i.e. the values of the stock price where forcing conversion is profitable to the company if the market is efficient, but unprofitable if the discount $1-\alpha$ has yet to occur. The probability of falling into this interval is, from this equation, higher than with the original inefficient information structure. This comparison shows that the case of convertibles

is much more delicate to handle than the basic model, because the initial callable convertible price is not necessarily different whether the market is efficient or not, and the stock price at expiration of the call protection does not give sufficient information about whether the market is efficient.

3.3 Convertible Bond Markets

The analysis of what happens on different types of convertible bond markets is performed within the preceding framework. The spirit of the analysis is a bit similar to the model of Harris and Raviv [17], who emphasize two possible equilibria in the conversion game of the firm: one where conversion does not convey any signal, and one where it does and incurs a decline in the stock price. While they primarily focus on the signalling equilibrium, whose implications are close to my inefficient market hypothesis, they do not provide the alternative story for an efficient market. The implications of my model are more integrated, in that they explain a variety of phenomena. A credible story of puzzling facts appearing on both the US and the Japanese convertible bond markets is thus proposed hereafter.

The study is perfomed in two stages. First, I present implications of the model arising in an efficient and in an inefficient market. Second, the observation of empirical regularities on the US and the Japanese markets is adapted to the previous comparison. Because no firm-specific factor is retained in the analysis, the findings I obtain are intrinsicly market-wide, and do not include corporate characteristics as determinants of the design and behavior of convertible debt as studied in Lewis, Rogalski and Seward [24].

3.3.1 Inefficient versus Efficient Market

A. Characteristics of an Inefficient Market

What would happen on an inefficient market ? The market does not integrate information $\Phi(I, \alpha)$ into its sigma-algebra at time of bond issuance, so that the right price of the bond is perceived to be $P_0^c(t, T_P, T_F)$. Provided that the firm knows Φ, the following phenomena are likely to appear:

- *The firm issuing callable convertible bonds forces conversion at values of the stock price respecting:*

$$S(T_P) \geq \frac{Fe^{-r(T_F-T_P)}}{1-\alpha} \tag{30}$$

and the stock price falls at announcement of forced conversions, because the firm needs to make sure that the bondholders will convert even when the negative premium at annoucement of forced conversion is realized.

- *The firm includes a call provision in convertible bonds whenever* $\Delta \leq 0$. The firm cannot force conversion at the strike price implied in the "inefficient pricing rule", because if it did when $S(T_P) < \frac{Fe^{-r(T_F-T_P)}}{1-\alpha}$, the decrease in the stock price would be so high at call announcement that no voluntary conversion would occur, and the firm would be forced to redeem the bonds with cash. Therefore, if $\Delta > 0$, the firm would issue underpriced convertibles on an inefficient market.

- *No initial stock price reaction occurs if convertibles are issued at* P_0^c. The market is unable to gather information from the initial convertible price; since it does not know Φ, there is no adjustment in the risk-neutral distribution of the stock price.

- *A temporary downwards stock price adjustment takes place when convertibles are issued at* $P_I^c < P_0^c$ *if the market believes that the convertible is not underpriced.* The value of the convertible is an increasing function of the stock price (its delta is positive). If the market observes an issuing price which is lower than implied by the current stock price, it revises downwards the stock price, and this discount only lasts until the market realizes that the fundamentals of the stock are indeed unchanged. However, if the market believes in mispricing of the convertible, the stock price remains unchanged.

- *Callable convertibles are issued by firms with superior information structure.* In this model, if the market is efficient, the firm has no incentive to issue callable convertibles for the sole reason of taking profit of forced conversion. But market inefficiency introduces a potential gain for a firm privately owning information Φ.

- *The firm has no incentive to resign to exercise its option.* There is no advantage in delaying the decision; hence, the firm has no interest in creating market incompleteness by not exercising an in the money option.

B. Characteristics of an Efficient Market

In contrast, efficient markets for convertible bonds feature completely different characteristics. First of all, only one price P_I^c can be proposed by the firm at issuance, because the market sigma-algebra $\mathcal{F}_t$ includes information Φ. This brings the following consequences:

- *The firm issuing callable convertibles may or may not force conversion at values of the stock price respecting:*

$$S(T_P) \geq Fe^{-r(T_F - T_P)} \tag{31}$$

  because the decrease in the observed stock price occurs prior to forced conversion, but the fact that this inequality is respected does not ensure that the stock price is adjusted; the firm, owning superior information, should know whether the stock price at T_P incorporates the negative premium α or not, as suggested in equation (29).

- *There is no market-related reason why the firm should not include a call provision.* Since the market is efficient, the price of the compound option corresponds to its risk-adjusted expected payoff, so that the decision of the firm is not contingent on its hope to issue overpriced securities. Nevertheless, the awareness of the arbitrage opportunities created by the call provision may affect the management's decision.

- *Convertibles are issued at P_I^c, and a decrease in stock price occurs, provided that markets are synchronous.* This follows from the expected stock price revised downwards following the issue, whose right price is correctly determined at P_I^c.

- *The firm can favor arbitrage by delaying its exercise of the call provision when possible.* This decision results in an increase in the stock price at maturity of the call protection when conversion should be forced, and may be optimal if it lowers the value of other securities issued by the firm, such as put options. It also leads to a rise in stockholders' wealth.

3.3.2 US versus Japanese Market

The pieces of empirical evidence concerning the markets for convertible bonds in the US and in Japan suggest that they exhibit some structural dissimilarities, whose interpretation happens to become very interesting with respect to the above characteristics. I start with the inspection of the more investigated US market; the analysis of the Japanese market, more liquid but less studied, is mainly performed in contrast with the previous one.

A. The US Market for Convertible Bonds

The characteristics of an inefficient market can be applied to the situation of the US convertible bond market.

Since the theoretical derivation of the optimal exercise rule of the call provision as soon as the bond is callable and the conversion value is greater than the call price by Ingersoll [19] and Brennan and Schwartz [5], empirical observations have emphasized that forced conversion usually occurs when the stock price is much higher than the theoretical threshold (Ingersoll [20], Mikkelson [25], Asquith [1]). The most robust explanation for this phenomenon is provided by Jaffee and Shleifer [21], who motivate the presence of a safety premium. Their results strongly suggest that firms have a strong preference for voluntary conversion over cash refund of a called convertible. If the stock price is expected to fall following the call announcement, this explains the prudent behavior of calling firms. The essence of this result is confirmed by Asquith and Mullins [2] and by Asquith [1], who find that any further delay with respect to this modified optimal exercise rule is best explained by cash flow advantages for the company. Consistently, the stock price is found to actually fall at call announcement (see Datta and Inskandar-Datta [8], Mikkelson [26], Ofer and Natarajan [31] and Cowan, Nayar and Singh [6]). The underlying story for the inefficient market is corroborated by these converging analyses. It corresponds to the signalling equilibrium proposed by Harris and Raviv [17], but the appearance of this equilibrium is made possible through the subsistance of asymmetric information up to the moment when the firm announces the call.

The issue of including a call provision into a convertible security is extremely delicate, since the reasons for this decision go far beyond the market impli-

cations depicted above. The role of convertible debt as a backdoor equity financing device, emphasized by Stein [33], allocates an important role to the presence of a call provision, which is presented as a valuable way for companies to achieve the goal of financing through equity at reduced cost by alleviating the possible adverse selection problem. My model suggests that, if the market is inefficient and the firm holds information Φ, there may be cases where it chooses not to attach a call provision. From 1978 to 1992, studying the NYSE, the AMEX and the NASDAQ, Lewis, Rogalski and Seward [24] find that 94.9% of all convertible bonds are callable. Noncallable convertibles can be found for both investment grade and speculative grade bonds, suggesting that credit quality is not the central reason for not including call provisions. This model is thus able to convincingly interprete this behavior by the awareness of firms that the correct issuing value of the convertible P_I^c may be higher than the price the market would accept if the "safety premium" were not integrated in initial valuation.

The next two characteristics of an inefficient market for convertibles can be associated: Since convertibles are issued only when $P_I^c \leq P_0^c$, the firm can set the initial price in the interval $[P_I^c, P_0^c]$. If the firm believes the market is efficient, it will set the price at P_I^c and realize no profit, and the stock price falls. Otherwise, it should set the price at P_0^c and get a profit of $-\Delta$, corresponding to the underpricing of the bond by the market. The danger for the company then would be actual market efficiency, and thus convertibles would be refused because of overpricing. Empirical evidence suggests that the first explanation is very likely to hold: numerous studies find that the initial stock price falls at the time of issue of convertibles (see Stein [33], Mikkelson and Partch [27], Eckbo [9], Dann and Mikkelson [7]). In the meantime, through a direct study of convertible debt offerings, Kang and Lee [23] are able to detect a mean initial excess return to convertibles of 1.11%. This robust underpricing result has to be faced with the negative abnormal stock returns at issuance. Clearly, this sequence is highly consistent with the hypothesis of underpricing of convertibles by issuers, which raises doubts on the bond market but is quickly understood as a true mispricing rather than the valuation implication of some private information.

The awareness of information Φ by the firm is obviously a necessary condition for obtaining such results. This suggests the presence of information

asymmetry in favor of issuers. Essig [10] provides evidence of the high level of R&D characterizing issuers of convertibles, which is interpreted by Stein [33] as a reliable proxy for potential information asymmetries. Yet, from the previous paragraph, this does not imply that the firm is aware of its informational advantage when it issues convertibles.

Finally, as Asquith [1] shows, there does not seem to be cases where firms deliberately act in opposition to the optimal call policy of forcing conversion as soon as possible. Managers who choose to delay seem to find an objective reason (safety premium, dividend advantage or call protection), which is not incompatible with this exercise rule, and is pervasively prevalent for the explanation of forced conversions.

The US market for convertibles seems thus to share the characteristics of an inefficient market, although there is no evidence that issuing firms themselves are able to take advantage of this market inefficiency. The initial pricing of convertibles does not correspond to the one the market would have set. Nevertheless, under the current model, it is closer to the risk-adjusted value of the convertible. This leads to the strange conclusion that firms could beat the market, which is otherwise complete, but are not able to do it because they are not aware of their informational advantage over the market. The loss incurred form this weakness is theoretically equal to at most Δ, because the maximum underpricing of the convertible amounts to the price the market would have accepted P_0^c minus the price charged P_I^c. The situation is sustainable, although the market anomaly is theoretically exploitable.

B. The Japanese Market for Convertible Bonds

Empirical evidence concerning the Japanese market is extremely puzzling when compared to the facts emphasized in United States. Thanks to the implications of the duality between efficient and inefficient markets, many a priori surprises become understandable when the market behaves in the efficient way. Yet, the incorporation of information Φ into the market sigma-algebra at time of issuance of corporate bonds brings damageable consequences.

The first characteristic of an efficient market involves the exercise rule for forced conversions. Some, but not all bonds, can be called at a lower level than on an otherwise identical, but inefficient market. The goal of creating market incompleteness may though lead firms to postpone forced conversion,

despite of the violation of the optimal pricing rule of Ingersoll. The exercise rule on an efficient market is therefore much less clear than what should be observed on an inefficient market. In their empirical paper, Greiner, Kalay and Kato [13] mention that there is an informal agreement on the Japanese market that no forced conversion occurs. The authors notice that a promising substitute to calling these bonds would be simple buybacks, and even this possibility seems to be largely overlooked by firms which should otherwise call their bonds under the optimal policy. Clearly, this is far from being in line with the rule on an inefficient market, but does not counter the more complex strategic behavior which is to take place on an efficient market. Although it cannot be considered as positive evidence, the informal prohibition from calling convertibles fits better to the efficient market case.

Curiously, bonds are almost never called in Japan, but the sample of 1,357 bonds used in the Greiner, Kalay and Kato study were all callable. The inclusion of this provision is thus far from corresponding to a backdoor equity financing device or the hope of getting a mispriced compound option. A plausible explanation to this apparent paradox is provided by the argument of creation of market incompleteness that increases potential wealth of insiders by allowing arbitrage opportunities.

The observation of the behavior of callable convertibles compared to the theoretical rate of return they should provide, which corresponds to the stock price process, suggests that convertibles are severely underpriced. The shift between convertible and stock prices is so important that it permits simple arbitrage through voluntary conversions. This phenomenon is analogous to what happens in the basic model when markets are efficient, but nonsynchronous. The issuing price of the convertible bond corresponds to the revised stock price when information Φ is known, but not translated into the stock price process. In my notation, the relevant sigma-algebra is $\mathcal{H}_t \subset \mathcal{F}_t$. The almost parallelism between the theoretical and observed mean bond premia presented in Figure 3 of the Greiner, Kalay and Kato [13] paper is consistent with this explanation. The constant distance between the premia is viewed as the impact of the negative premium α, accounted for on the convertible market but ignored by the stock market. The arbitrage implications of efficient but nonsynchronous markets are more severe than if markets were synchronous, because they combine two market anomalies: let alone the

informed investor is able to beat the market on the basis of nonsynchroneity (as confirmed by the high rate of voluntary conversions), but the call policy of the issuing firm may act as if it were able to eliminate the shift between theoretical and observed premia, creating additional arbitrages.

The last characteristic of efficient markets contaminated by arbitrages is obviously filled in Japan. The fact that convertibles are priced with a discount corresponding to the foreseen stock price depression when the convertible is called, or similarly repurchased by the companies, creates the availability of arbitrage opportunities through delayed forced conversion. The salience of this argument is reinforced by the visibility of the bond mispricing with respect to the stock. The callability of convertibles is necessary for creating the shift; the fact that bonds are never called and scarcely repurchased makes it observable in a long interval of time.

The final concern about the Japanese convertible market is naturally the sustainability of this situation. This nonsynchronous, efficient but incomplete market creates many anomalies. Firms must see an interest in these conditions in order to keep on acting the same way. Here, one has to ask who benefits most from the arbitrages. They mainly lead to voluntary conversions, with two effects for the firm: it manages to raise new equity without announcing forced conversion, and it obtains early conversion of otherwise valuable calls. Companies find thus a source of backdoor equity financing which is, under such circumstances, even cheaper than hypothesized by Stein [33]. As long as the market remains unsynchronous, companies will find an advantage in this strategy of including call provisions; if synchroneity was to be respected, the market would still produce an incomplete price system, but its potential arbitrages would be less visible for the market participants. Yet, firms would still be likely to engage in callable convertible offerings, because the strategic decision of the non-exercise of the call provision still creates discretionary arbitrage opportunities.

Japanese investors seem to integrate information better at the level of the convertible market, but not to link it with the stock market. These nonsynchronous markets cause many arbitrages, and issuing firms have much to gain from this situation. The solution of increasing the information transfers between the stock and bond markets would alleviate, but not suppress, sources of incompleteness. Firms would not be able then to beat the market

on the basis of superior private information, but would simply help smart investors to realize arbitrage profits.

4 Conclusions and Future Research

The model presented in this paper introduces a slight change in market mechanisms on the stock and the option markets that induces surprising and important implications. The introduction of an information about a special market actor whose action influences the stock value, but who is otherwise forced to trade at a fair price, should in ordinary situations result in an efficient market. Even after accounting for the best possible synchroneity between interrelated markets, market incompleteness is not achieved if the actor wishes to create arbitrage opportunities. Quite interestingly, the inefficient market does not exhibit this disturbing characteristic.

This could have remained an aetherical essay, where the theoretical possibility that markets cannot be at the same time efficient and complete would only be pure intellectual speculation. Yet, this special market actor can easily be taken as the firm issuing securities. The analysis of markets for convertible bonds fits particularly well in the framework, provided that some changes in the model and simplifications in the pricing formulas are introduced. With the typical characteristics displayed on efficient and inefficient markets, I have created a valuable tool for analyzing some puzzling empirical evidence detected on the US and Japanese convertible markets. The US market seems to correspond more closely to an inefficient market, but where issuers do not reveal to be able to beat the market. Consequently, the market is complete and there is no systematic exploitation of the market anomaly. In contrast, the Japanese market looks like an efficient, but nonsynchronous market. Efficiency triggers market incompleteness, and it is reinforced by the visibility of arbitrages caused by nonsynchroneity. As a result, firms benefit from a very cheap source of equity financing. It would shrink if the stock and bond markets were synchronous, but would not structurally disappear.

The applications of the model go far beyond the analysis of existing markets such as convertible bonds. Considering that a raider could fill the role of the special trader, the basic framework can be *mutatis mutandis* adapted to the

analysis of hostile takeovers. Although few attempts have been made to use the option markets as potential vehicle for takeovers, my analysis univocally shows its benefits for the informed trader: either she is able to beat the market if inefficient, or she can create arbitrage opportunities if the pricing rules take all information into account. The potentialities of this approach are very important.

Another field of application of this theory concerns share repurchases by companies. The annoucement of an open market share repurchase creates an option for the company (Ikenberry and Vermaelen [18]), and the action of repurchasing usually induces a premium on the stock. If markets are efficient, why then creating such an option, instead of repurchasing directly? An elegant answer is provided by this model, since the firm is able to create arbitrage opportunities by strategic exercise of this option, because it can affect stock prices through this means.

As it can be seen, this paper has the potential to open up a stream of research which may lead to new descriptive insights, such as for convertibles bonds, as well as normative results, which might lead to a new vision of derivative markets, sources of incompleteness rather than market spanning tools as advocated by Ross [32] and the subsequent literature.

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