PORTFOLIO SELECTION WITH RANDOMLY TIME-VARYING FIRST AND SECOND MOMENTS: THE ROLE OF THE INSTANTANEOUS CAPITAL MARKET LINE

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Portfolio Selection with Randomly Time-Varying First and Second Moments: The Role of the Instantaneous Capital Market Line[1]

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Abstract

In the intertemporal portfolio selection model of Merton (1973), any change in means, variances or covariances of security returns is sufficient to generate a change in the investment opportunity set. Merton's formulation suggests that investors will hedge all such changes by including in their optimal portfolio holdings as many hedge funds as there are state variables that describe the dynamics of returns. In this paper, we show that investors need to hedge only against changes in the random slope and position of the instantaneous capital market line. If the instantaneous capital market line is constant or deterministic, then investors will not hold any hedge funds at all, even though means, variances and covariances of securities returns may be changing randomly over time. Based on these results, we propose a new definition of the investment opportunity set and changes in the investment opportunity set. Our analysis allows for incomplete markets and does not assume that the securities prices are Markovian. It provides a potential theoretical foundation for certain conditional tests of asset pricing models which ignore the intertemporal hedging premia.

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1 Introduction

In the intertemporal portfolio selection model of Merton (1973) "a sufficient set of statistics for the [investment] opportunity set at a given point in time is $\{\alpha_i, \sigma_i, \rho_{ij}\}$" where α_i, σ_i denote the mean and standard deviation of the instantaneous rate of return to security i, and ρ_{ij} the correlation coefficient between instantaneous rates of return to securities i and j. Furthermore, "the dynamics for the changes in the opportunity set over time" are given by a set of Itô processes that describe changes in α_i, σ_i, and potentially ρ_{ij}, and which, together with the securities price processes, form a Markovian vector of state variables[1].

According to this definition, any change in means, variances or covariances generates a change in the investment opportunity set. Merton's formulation suggests that investors will hedge all such changes by including in their optimal portfolio holdings as many hedge funds as there are state variables that describe the dynamics of the returns.

This paper re-examines the definition and nature of the investment opportunity set, as well as the role of hedge funds in intertemporal portfolio selection. We show that investors will only hedge against changes in instantaneous means, variances and covariances that affect the slope or position of the instantaneous capital market line (ICML). This leaves plenty of room for other changes in the first and second moments which do not affect the capital market line and do not give rise to hedging demands.

By the instantaneous capital market line (ICML) we mean the instantaneous efficient frontier which has intercept equal to the instantaneous riskless rate and is tangent to the instantaneous efficient frontier of risky assets only. In other words, it is defined like the capital market line in static models, except that it is based on instantaneous rather than finite horizon means, variances and covariances.

The capital market line has played a central role in static mean-variance portfolio theory, where all optimal portfolios plot along that line. Merton (1971) showed that the two-fund separation result also applies in a dynamic, continuous-time setting, provided that the interest rate and the first and

[1] see page 483 in Merton (1992)

second moments of returns to all securities are constant. This study reveals that the ICML is equally important when means, variances and covariances change randomly over time.

Our results are set in a general framework with securities whose rates of return have stochastically time varying first and second moments which, unlike in Merton (1973), do not have to be functions of a Markovian vector of state variables. However, we do assume that the slope and position of the ICML either are functions of the path of a vector of state variables or are deterministic.

In Theorem 1, the slope and position of the ICML are assumed to be functions of the path of a vector of hedgeable state variables, one of which is proportional to the logarithmic portfolio. In that case, investors will hold a possibly time varying combination of the riskless asset and the hedge funds associated with the state variables, including the logarithmic portfolio. If some state variables have no effect on the ICML, then investors will not hedge against them, even if these state variables do affect the first and second moments of the individual security returns.

In Corollary 1, the ICML is assumed to be constant or deterministic. In that case, investors will not hedge at all against changes in the first and second moments of security returns. They will simply hold a possibly time-varying combination of the riskless asset and the logarithmic portfolio. This result provides a potential theoretical justification for ignoring intertemporal hedging premia in conditional asset pricing tests.

Our results lead us to argue that the most relevant concept of the investment opportunity set is not the set of first and second moments of security returns as in Merton, but the ICML. Correspondingly, changes in the investment opportunity set should be understood as changes in the slope and position of the ICML.

An implication of our new definition of the investment opportunity set is that predictability in security returns does not necessarily imply changes in the investment opportunity set.

Markets are allowed to exhibit the type of incompleteness studied in He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). Specifically, markets may be dynamically incomplete because the number of sources of uncertainty may be larger than the number of risky assets. In such a framework, some risks are not completely hedgeable, and therefore, some contingent claims cannot be replicated through dynamic trading strategies using the existing securities. This implies that an optimal trading strategy may not exist. He and Pearson (1991) and Karatzas et al. (1991) provide conditions for existence of an optimal trading strategy in such an environment.

Our results suggest an alternative to the conditions of He and Pearson (1991) and Karatzas et al. (1991). If the ICML either is driven by a vector of hedgeable state variables, one of which is proportional to the logarithmic portfolio, or is deterministic, then an optimal portfolio exists. Moreover, it is unique, and it has the same form as under market completeness.

The rest of the study is organized as follows. Section 2 outlines the model. In Section 3 we define the ICML. Section 4 discusses optimal portfolio choice in our model and our assumptions on utility functions. Section 5 provides our fund separation result, summarized in Theorem 1, for the case where changes in the position and slope of the ICML are described by a vector of hedgeable state variables. Section 6 considers the case where the ICML is either deterministic or constant. The result is stated in Corollary 1. We conclude in Section 7. A detailed derivation of the theorems is provided in the appendix.

2 The Model

The time horizon is $[0, T]$ for a fixed $T > 0$. The investors' information structure is represented by a filtration $F = (\mathcal{F}_t)_{t\in[0,T]}$ on an underlying probability space $(\Omega, \mathcal{F}, P)$. The interpretation is that $\mathcal{F}_t$ is the information set available to the investors at time t.

There are $N + 1$ basic long-lived securities. Like in many models, including Merton (1973), their price dynamics will be specified without distinguishing between real and nominal terms.

We shall assume that security zero is a *money market account* with constant interest rate. This means that its price process M (price per share) has the form

$$M(t) = M(0)\exp\left\{\int_0^t r\,ds\right\}$$

for some $M(0) > 0$ and some process $r \in \mathcal{L}^1$, the *interest rate*. The notation $r \in \mathcal{L}^1$ means that r is measurable and adapted and that

$$\int_0^T |r|\,ds < \infty$$

with probability one.

The remaining N securities are instantaneously risky. Their prices are given by an N dimensional vector S of Itô processes, which are assumed to be positive. This implies that there exist processes $\mu \in \mathcal{L}^1$ and $\sigma \in \mathcal{L}^2$ such that

$$dS = \mathcal{D}(S)\mu\,dt + \mathcal{D}(S)\sigma\,dW$$

where $\mathcal{D}(S)$ is the diagonal matrix with the vector S along the diagonal.

The notation $\mu \in \mathcal{L}^1$ means that μ is adapted and measurable and satisfies

$$\int_0^T \|\mu\| ds < \infty$$

with probability one. The notation $\sigma \in \mathcal{L}^2$ means that σ is adapted and measurable and satisfies

$$\int_0^T \|\sigma\|^2 ds < \infty$$

with probability one.

We refer to μ and σ as the *mean vector* and the *dispersion matrix* of the instantaneous rates of return to the N risky securities. We allow them to change stochastically over time. Unlike in Merton (1973) and many subsequent models, they are not assumed to be functions of a Markovian vector of state variables.

The process W is a K dimensional Wiener process (a vector of K independent one-dimensional Wiener processes) with respect to the filtration $F = (\mathcal{F}_t)$. We assume that $K \geq N$. Markets may be incomplete, in the sense that there may be many more Wiener processes than there are instantaneously risky securities ($K \gg N$). This is the type of market incompleteness analyzed by He and Pearson (1991) and Karatzas et al. (1991).

The K Wiener processes are sources of instantaneous variations in the N risky securities prices. Over extended time intervals, the information in the filtration F may influence the securities prices also because it affects the stochastic parameters μ and σ.

Unlike He and Pearson (1991) and Karatzas et al. (1991), we allow for the possibility that the information set at a point in time may contain even more information than what can be obtained by observing the paths of the K components of W up to that time. Technically speaking, the filtration F may be larger (finer) than the augmented filtration generated by W, so long as W is a Wiener process with respect to F. This means that investors may have access to such additional information, they may make their trading strategies contingent on it, and the parameters μ and σ may vary over time in a way that depends on it.

To compress the notation, write

$$\bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$

$$\bar{\mu} = \begin{pmatrix} Mr \\ \mathcal{D}(S)\mu \end{pmatrix}$$

and

$$\bar{\sigma} = \begin{pmatrix} 0 \\ \mathcal{D}(S)\sigma \end{pmatrix}$$

The processes $\bar{S}$ and $\bar{\mu}$ have dimension $N+1$, and the matrix valued process $\bar{\sigma}$ has dimension $(N+1) \times K$. With this notation, the Itô process $\bar{S}$ has differential

$$d\bar{S} = \bar{\mu}\, dt + \bar{\sigma}\, dW$$

A *state price process* or *pricing kernel* for $\bar{S}$ is a positive one-dimensional Itô process Π such that $\Pi\bar{S}$ has zero drift.

A process Π is a state price process if and only if $\Pi(0) > 0$ and

$$\frac{d\Pi}{\Pi} = -r\, dt - \lambda\, dW$$

for some $\Pi(0) > 0$ and some K dimensional row vector valued process $\lambda \in \mathcal{L}^2$ such that

$$\mu - r\iota = \sigma\lambda^{\top}$$

The process λ is called the vector of *prices of risk*. The prices of risk are specific to each of the Wiener processes. Each element of λ measures the required excess rate of return on securities per unit of dispersion with respect to the corresponding Wiener process. The total excess rate of return on a security is a linear combination of the security's dispersion coefficients with respect to each of the Wiener processes, where the weights in the linear combination are the prices of risk.

Assume that

$$(\mu - r\iota)^{\top} \left(\sigma\sigma^{\top}\right)^{-1} (\mu - r\iota) \in \mathcal{L}^1$$

Under this assumption, we can always make the following specific choice of λ:

$$\lambda = (\mu - r\iota)^{\top} \left(\sigma\sigma^{\top}\right)^{-1} \sigma$$

With this choice, clearly

$$\lambda\sigma^{\top} = (\mu - r\iota)^{\top}$$

and

$$\begin{aligned}
\lambda\lambda^{\top} &= (\mu - r\iota)^{\top} \left(\sigma\sigma^{\top}\right)^{-1} \sigma\sigma^{\top} \left(\sigma\sigma^{\top}\right)^{-1} (\mu - r\iota) \\
&= (\mu - r\iota)^{\top} \left(\sigma\sigma^{\top}\right)^{-1} (\mu - r\iota) \in \mathcal{L}^1
\end{aligned}$$

which implies that $\lambda \in \mathcal{L}^2$.

3 The Instantaneous Capital Market Line

Recall from mean-variance theory that mean-variance efficient portfolios are portfolios that maximize the expected rate of return given the variance or standard deviation of the rate of return. We can similarly define *instantaneously efficient portfolios* as those that maximize the expected instantaneous rate of return given the standard deviation of the instantaneous rate of return. Their combinations of standard deviation of returns and expected returns plot on a straight line whose intercept with the expected-return axis is the instantaneous interest rate. We call this line the *instantaneous capital market line*(ICML).

It also follows from the standard theory that the instantaneously efficient portfolios are the portfolios that are combinations of the money market account and the portfolio $\phi^{\ln}$ given by

$$\phi^{\ln} = \lambda\sigma^{\top}\left(\sigma\sigma^{\top}\right)^{-1} = (\mu - r\iota)^{\top}\left(\sigma\sigma^{\top}\right)^{-1}$$

where we note that $\sigma\sigma^{\top}$ is the covariance matrix of the instantaneous rates of return to the various securities. We call this portfolio the *logarithmic portfolio* because, as is well known and will also follow from the analysis below, it is indeed the optimal portfolio for an investor with a logarithmic utility function.

The logarithmic portfolio should be distinguished from the *tangency portfolio*, which consists of investments in instantaneously risky securities only and plots on the capital market line at the point where it is tangent to the risky-security frontier. The tangency portfolio $\phi^{\tan}$ can be calculated from the logarithmic portfolio $\phi^{\ln}$ by scaling the fractions of wealth invested in risky securities so that they add up to one:

$$\phi^{\tan} = \frac{1}{\phi^{\ln}\iota}\phi^{\ln}$$

Note that with the specific choice we have made for the prices of risk λ,

$$\lambda = (\mu - r\iota)^{\top}\left(\sigma\sigma^{\top}\right)^{-1}\sigma = \phi^{\ln}\sigma$$

The slope of the capital market line is the ratio of excess expected instantaneous rate of return and the standard deviation of the instantaneous rate of return to the logarithmic portfolio. We can calculate this slope as follows.

The excess instantaneous expected rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}(\mu - r\iota) = (\mu - r\iota)^{\top}\left(\sigma\sigma^{\top}\right)^{-1}(\mu - r\iota) = \lambda\lambda^{\top}$$

The variance of the instantaneous rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}\sigma\sigma^{\top}\phi^{\ln\,\top} = \lambda\lambda^{\top}$$

and the standard deviation is $\sqrt{\lambda\lambda^{\top}}$. Hence, the slope of the ICML is

$$\begin{aligned}\frac{\phi^{\ln}(\mu - r\iota)}{\sqrt{\phi^{\ln}\sigma\sigma^{\top}\phi^{\ln\,\top}}} &= \frac{\lambda\lambda^{\top}}{\sqrt{\lambda\lambda^{\top}}} \\ &= \sqrt{\lambda\lambda^{\top}}\end{aligned}$$

It follows that the ICML is the straight line with intercept r and slope $\sqrt{\lambda\lambda^\top}$. While the individual elements of the vector λ are prices of risk with respect to the individual Wiener processes, $\sqrt{\lambda\lambda^\top}$ is the price of risk in the aggregate. We can also think of it as the instantaneous Sharpe ratio for instantaneously mean-variance efficient portfolios. It also happens to be the volatility of the state price process.

In the following two sections, we prove that investors will optimally hedge only changes in moments that lead to changes in the slope and position of the ICML. We shall therefore argue that for the purpose of optimal portfolio selection, the capital market line is the most useful concept of the "investment opportunity set."

4 Optimal Portfolios and Utility Functions

We define a *trading strategy* as an adapted measurable $(N+1)$-dimensional row vector valued process $\bar{\Delta} = (\Delta_0, \Delta)$. The interpretation is that $\bar{\Delta}(t)$ is the position held at time t: for each security $i = 0, \ldots, N$, $\bar{\Delta}_i(t)$ is the number of units of security i held at time t. The *value process* of a trading strategy $\bar{\Delta}$ is the process $\bar{\Delta}\bar{S}$.

The set of trading strategies $\bar{\Delta}$ such that $\bar{\Delta}\bar{\mu} \in \mathcal{L}^1$ and $\bar{\Delta}\bar{\sigma} \in \mathcal{L}^2$, will be denoted $\mathcal{L}(\bar{S})$. A trading strategy $\bar{\Delta}$ in $\mathcal{L}(\bar{S})$ is *self-financing* if it satisfies the *budget constraint:*

$$\bar{\Delta}(t)\bar{S}(t) = \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S}$$

A *portfolio strategy* is an adapted measurable N dimensional row vector valued process $\tilde{\theta}$. The interpretation is that $\tilde{\theta}$ tells us the fractions of wealth invested in the various risky securities, while the remaining fraction, $1-\tilde{\theta}\iota$, is invested in the money market account. Here, ι is the N dimensional column vector all of whose entries are one.

If $\bar{\Delta} = (\Delta_0, \Delta)$ is a trading strategy such that the value process $V = \bar{\Delta}\bar{S}$ is positive, then the corresponding portfolio strategy is given by

$$\tilde{\Delta} = \Delta \mathcal{D}(S)/V$$

Conversely, we can recover a unique value process and a unique self-financing trading strategy from knowledge only of the portfolio strategy and the initial value of the trading strategy. If $\tilde{\Delta}$ is a portfolio strategy and $w_0 > 0$ is an initial wealth level, then there is a unique self-financing trading strategy $\bar{\Delta}$ such that $\bar{\Delta}(0)\bar{S}(0) = w_0$, $\bar{\Delta}\bar{S} > 0$, and $\tilde{\Delta}$ is the portfolio strategy corresponding to $\bar{\Delta}$. The value process $V = \bar{\Delta}\bar{S}$ of $\bar{\Delta}$ is the unique Itô process such that $V(0) = w_0$ and

$$\frac{dV}{V} = \left((1 - \tilde{\Delta}\iota)r + \tilde{\Delta}\mu\right) dt + \tilde{\Delta}\sigma\, dW$$

and $\bar{\Delta} = (\Delta_0, \Delta)$ is given by

$$\Delta = \mathcal{D}(S)^{-1}\tilde{\Delta}V$$

and

$$\Delta_0 M + \Delta S = V$$

If Π is a state price process for $\bar{S}$, and if $\bar{\Delta} \in \mathcal{L}(\bar{S})$ is a self-financing trading strategy, then $\Pi\bar{\Delta}\bar{S}$ has zero drift.

In order to keep things simple, we restrict ourselves to a model with a finite time horizon and with only final consumption[2].

Let $w_0 > 0$ be the investor's initial wealth level, and let u be his utility function, defined on the positive half-line $(0, \infty)$.

The investor chooses a self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ subject to the constraints $\bar{\Delta}(0)\bar{S}(0) = w_0$ and $\bar{\Delta}\bar{S} > 0$, so as to maximize the expected utility $Eu(\bar{\Delta}(T)\bar{S}(T))$ of final payoff.

A self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ is *optimal* given initial wealth $w_0 > 0$ if $\bar{\Delta}(0)\bar{S}(0) = w_0$, $\bar{\Delta}\bar{S} > 0$, $u(\bar{\Delta}(T)\bar{S}(T))$ is integrable above, and if for every self-financing trading strategy $\bar{\delta} \in \mathcal{L}(\bar{S})$ such that $\bar{\delta}(0)\bar{S}(0) = w_0$ and $\bar{\delta}\bar{S} > 0$, $u(\bar{\delta}(T)\bar{S}(T))$ is integrable above with

$$Eu(\bar{\Delta}(T)\bar{S}(T)) \geq Eu(\bar{\delta}(T)\bar{S}(T))$$

[2]We expect that as is usually the case in models like this, the results can be generalized to a model with a flow of consumption over time and with a finite or an infinite time horizon.

Choosing a self-financing trading strategy with positive value process is equivalent to choosing the corresponding portfolio strategy. So, a portfolio strategy $\tilde{\Delta}$ is *optimal* given initial wealth w_0 if $\bar{\Delta}$ is optimal given initial wealth w_0, where $\bar{\Delta}$ is the unique self-financing trading strategy with initial value $\bar{\Delta}(0)\bar{S}(0) = w_0$ and such that the portfolio strategy corresponding to $\bar{\Delta}$ is $\tilde{\Delta}$.

Assuming that the utility function u is twice differentiable with $u'' < 0$, let $R_R(u)(x)$ denote the coefficient of relative risk aversion at the wealth level x.

We shall impose the following assumption on the investor's utility function:

Assumption 1 *The utility function u is twice continuously differentiable with $u' > 0$, $u'' < 0$, and $u'(x) \to 0$ as $x \to \infty$.*

Assumption 1 implies that marginal utility goes to zero at high wealth levels: $u'(x) \to \infty$ as $x \to 0$. The assumption is quite general and attractive. It is satisfied by the logarithmic and power utility functions.

In Corollary 1, we shall also impose the following assumption.

Assumption 2 *There exist constants $\gamma > 0$ and $x_0 > 0$ such that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$.*

Assumption 2 says that the relative risk aversion is bounded below away from zero at high wealth levels. This is an economically meaningful and reasonable condition. For example, the classical paper by Arrow (1965) argued that the utility functions could reasonably be assumed to have decreasing absolute but increasing relative risk aversion. If relative risk aversion is increasing, then certainly it will be bounded away from zero at high wealth levels. We also require marginal utility to go to infinity as final wealth or consumption goes to zero. This ensures that the optimal final consumption will be positive with probability one.[3]

Finally, in Theorem 1, we shall impose the following assumption, which is a combination of Assumption 2 with an integrability requirement on the state price process.

[3]We expect that our results can easily be generalized to allow for finite marginal utility at zero, but we abstain from doing so since it would divert attention from our main point.

Assumption 3 *$\Pi(T)$ is integrable, and there exist constants $\gamma > 0$ and $x_0 > 0$ such that $R_R(u)(x) \geq \gamma$ for all $x \geq x_0$, and such that*

- *if $\gamma = 1$, then $\ln \Pi(T)$ is integrable below*
- *if $\gamma < 1$, then $\Pi(T)^{\frac{\gamma-1}{\gamma}}$ is integrable.*

5 Optimal Portfolios: Random ICML

In this section, we show that if investors do hold hedge funds, then these funds hedge against changes in the slope and intercept of the capital market line and not against general changes in the instantaneous means, variances and covariances.

We model random changes in the ICML as driven by a number of state variables which are independent Wiener processes relative to the filtration F. We assume that the slope and intercept of the ICML are functions of the paths of the state variables (technically, they are measurable and adapted to the filtration generated by the state variables), but we do not assume that they are functions of the current levels of the state variables[4].

This way of using state variables is inspired by Chamberlain (1988). Indeed, our Theorem 1 below is a version of Chamberlain's mutual fund separation result, but formulated in terms of the logarithmic portfolio, the hedge funds, and the intercept and slope of the ICML[5].

The state variables will be assumed to be hedgeable in the following sense. If B is an m dimensional Wiener process relative to the filtration F, then we say that B is *hedgeable* if there exists an $m \times N$ dimensional measurable

[4]Merton (1973) assumed that the state variables followed a multi-dimensional diffusion process. Our assumption, that they follow an m-dimensional Wiener process, is not as restrictive as it may seem. An m-dimensional diffusion process can, under appropriate regularity conditions, be seen as driven by an m-dimensional Wiener process. If the diffusion coefficient is a continuous function, then the diffusion and the Wiener process generate the same filtration, as shown by Harrison and Kreps (1979).

[5]The case of a deterministic ICML, where $\Pi(T)$ is lognormally distributed, is not covered by Chamberlain's analysis. He assumes that $\Pi(T)$ (or ρ in his notation) is bounded above and below away from zero.

and adapted process b such that $b\sigma \in \mathcal{L}^2$, $b\sigma\sigma^\top b^\top = I$, the $m \times m$ identity matrix, and

$$B(t) = \int_0^t b\sigma W$$

We can interpret b as a vector of m portfolio strategies which are perfectly instantaneously correlated with the elements of B. We call them the *hedge funds* associated with B.

We shall assume that *the logarithmic portfolio is proportional to the first hedge fund.* We take this to mean that there exists a one-dimensional measurable and adapted positive process η such that

$$\phi^{\ln} = \eta b_{1-}$$

where b_{1-} is the first row of b, representing the first hedge fund.

Theorem 1 *Let B be an m dimensional Wiener process relative to F. Assume that*

1. *B is hedgeable with associated hedge funds b*
2. *The logarithmic portfolio $\phi^{\ln} = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1}$ is proportional to the first hedge fund*
3. *The intercept and slope of the ICML are measurable and adapted with respect to F^B*

Then for each utility function u which satisfies Assumptions 1 and 3, and for each initial wealth level $w_0 > 0$, there exists a unique optimal portfolio strategy. It has the form

$$\tilde{\Delta} = \beta b$$

for some $1 \times m$ dimensional measurable and adapted process β.

There is a detailed proof of Theorem 1 in the appendix. The proof uses the following steps. We build a reduced trading model driven by B and F^B, with the same money market account as the original model and m risky securities with values equal to the values of the hedge funds. This reduced model has complete markets. The interest rate and the slope of the ICML are the same in the reduced model as in the original model. Since they are adapted and measurable with respect to F^B, it follows that the state price process is also the same in the reduced model as in the original model. We define a claim Y^* by the first order condition

$$u'(Y^*) = \kappa\Pi(T)$$

where κ is a positive Lagrangian multiplier chosen such that Y^* satisfies the intertemporal budget constraint

$$\Pi(0)w_0 = E[\Pi(T)Y^*]$$

It is shown in Lemma 3 in the appendix that if Y^* can be replicated by a trading strategy with initial value w_0, then that trading strategy is the unique optimal trading strategy. In fact, Y^* can be replicated within the reduced model, which has complete markets, and hence the trading strategy involves only the money market account and the hedge funds.

Remark: *Suppose the interest rate and the slope of the ICML are functions of the paths of a subset of the state variable processes B_k (including the first of them). Then the remaining state variables are superfluous in the sense that they do not affect the slope and intercept of the ICML and that investors will not hold them. Investors will only hold the money market account, the logarithmic portfolio, and hedge funds that hedge against changes in the slope and the position of the ICML.*

Theorem 1 and its associated remark simplify Merton's separation theorem, since they reveal that it is not necessarily optimal to hedge all changes in the instantaneous means, variances and covariances. They also lead us to redefine the concept of an investment opportunity set provided in Merton (1973), and consequently, the concept of changes in the investment opportunity set.

Definition: *The investment opportunity set is the instantaneous capital market line. Changes in the investment opportunity set are described by changes in the intercept and slope of the instantaneous capital market line.*

Theorem 1 is derived in the presence of market incompleteness. As mentioned earlier, the potential market incompleteness is of exactly the same type as in Karatzas et al. (1991) and He and Pearson (1991). Those papers give conditions for existence of an optimal trading strategy which are, however, difficult to interpret economically. The reason why portfolio choice under market incompleteness is difficult is that in general, the optimal final claim becomes a function of the final value of the state price process only after a sophisticated choice has been made among the infinitely many possible state price processes that are consistent with the given securities price processes.

Our analysis reveals that within the same model, a simple and meaningful condition is sufficient to guarantee the existence of a unique optimal portfolio strategy and to characterize it. Specifically, if the capital market line either is driven by hedgeable state variables or, in the simplified case of Corollary 1 below in Section 6, is deterministic, then the optimal portfolio holdings of the investor are unique and will have the same form as in the case of complete markets. This indicates that the capital market line is as important in continuous-time analysis as it is in discrete time.

Theorem 1 does not require that all the instantaneous means, variances and covariances of the securities prices are driven by the state variables. Only the slope and intercept of the ICML are assumed to be driven by the state variables, and so long as they are, the instantaneous moments of the securities returns can follow general processes.

6 Optimal Portfolios: Deterministic ICML

The following corollary says that when the ICML is deterministic or constant, investors do not at all hedge against changes in the first and second moments of security returns. We allow first and second moments to vary randomly over time. As in Theorem 1, they do not have to be functions of the levels or the paths of a Markovian vector of state variables. The only restriction imposed on the dynamics of returns is that the interest rate and the slope of the ICML should be deterministic.

Corollary 1 *Assume that the interest rate is deterministic and that the slope of the ICML is positive and deterministic. Then for each utility function u which satisfies Assumptions 1 and 2, and for each initial wealth level $w_0 > 0$,* *there exists a unique optimal portfolio strategy. It has the form*

$$\tilde{\Delta} = \alpha \phi^{\ln}$$

for some one-dimensional measurable and adapted process α.

The proof of Corollary 1 is provided in the appendix. The idea of the proof is to define a Wiener process B by

$$B(t) = \int_0^t \frac{1}{\sqrt{\lambda \lambda^\top}} \lambda \, dW$$

We show that B is hedgeable with a hedge fund which is proportional to the logarithmic portfolio, and then appeal to Theorem 1.

Note that the ICML will be constant if and only if the state price process has constant relative drift and volatility. This is so because the state price process has relative drift $-r$ and volatility $\sqrt{\lambda \lambda^\top}$, while the ICML has intercept r and slope $\sqrt{\lambda \lambda^\top}$.

In Corollary 1, the value weight invested in the logarithmic portfolio is given by α. It can be shown that α equals the relative risk tolerance of the investor's indirect utility function. This follows from the fund separation theorem in Merton (1973) for the case of constant moments, and from Corollary 1.

Even with a constant capital market line, the value weight α invested in the logarithmic portfolio will in general change over time in response to changes in the investor's wealth and resultant changes in the risk aversion of his indirect utility function. Changes in α can be interpreted as sliding up and down the ICML.

For an investor with logarithmic utility, the optimal portfolio strategy is the logarithmic portfolio strategy $\phi^{\ln}$. That was why we used that name for it, of course.

In the remainder of this section, we shall discuss the significance of Corollary 1 and the degree to which its assumptions are reasonable.

Apart from the case of logarithmic utility, the only previously known result that we are aware of where all investors will simply invest in the money market account and the logarithmic portfolio is Merton's (1973) example where all moments are constant. Corollary 1 generalizes this result, because σ, μ and λ are allowed to change randomly over time, so long as the relation

$$\mu - r\iota = \sigma\lambda^{\top}$$

is satisfied and r and $\lambda\lambda^{\top}$ remain deterministic.

If there are one thousand risky securities, then the case of constant moments requires the constancy of 501,500 parameters, because this is the number of free parameters in the instantaneous excess return vector $\mu - r\iota$ and the instantaneous covariance matrix $\sigma\sigma^{\top}$. Corollary 1 imposes only a two-dimensional restriction on these parameters: the intercept and slope of the ICML should be constant (or equal to deterministic functions of time). In this sense, the corollary has 501,498 more degrees of freedom than the case of constant moments.

A special case of Corollary 1 is where the interest rate and the slope of the ICML are constants. A constant slope $\sqrt{\lambda\lambda^{\top}}$ does not require that the elements of λ stay constant. They may change according to virtually any adapted processes so long as their sum-of-squares is constant. At the same time, all the elements of the matrix σ may change in virtually any non-anticipating way so long as the matrix continues to have full rank. The example below illustrates this. In the example, the elements of λ change randomly because of changes in the vector of instantaneous means, while the elements of σ remain constant.

Example 1 Suppose $N = K$. Suppose the interest rate $r > 0$ and the matrix σ are deterministic constants. Let $x \in I\!R^K$, $x \neq 0$, and suppose

$$\mu = r\iota + \frac{1}{\|W + x\|}\sigma(W + x)$$

Then

$$\lambda = \frac{1}{\|W + x\|}(W + x)^{\top}$$

and $\lambda\lambda^{\top} = 1$. According to Theorem 1, the optimal portfolio is proportional to the logarithmic portfolio. Although the instantaneous means vary over time in a highly random manner, investors will not hold hedge funds to hedge against these changes. Even so, the composition of their portfolio will change stochastically over time, since we find

$$\phi^{\ln} = \lambda\sigma^{-1} = \frac{1}{\|W+x\|}(W+x)^{\top}\sigma^{-1}$$

□

We can well generate other instances of securities price processes which have a constant ICML. The following example exhibits a class of processes where the second moments are random and the vector λ is constant.

Example 2 Consider any process for the dispersion matrix σ. Let it incorporate any kind of time varying conditional second moments that may be considered suitable or used in the empirical literature of conditional asset pricing, such as ARCH or GARCH. Let λ be a constant vector, and define the instantaneous mean return processes μ by

$$\mu = r\iota + \sigma\lambda^{\top}$$

or

$$\mu_i - r = \sigma_{i1}\lambda_1 + \sigma_{i2}\lambda_{i2} + \cdots + \sigma_{iK}\lambda_K$$

for each security i. Then the ICML is constant. □

The assumption of a constant ICML is consistent with specifications that have been used or proposed for empirical work. The assumption means that portfolios on the ICML have expected rates of return that are an affine function of their standard deviation. Combined with a partial equilibrium argument, it implies that the market portfolio is on the ICML, and therefore that its expected rate of return is an affine function of its standard deviation.

This is consistent with Merton (1980), who proposed the assumption of a constant capital market line as one of his empirical models for estimating the expected return to the market.

It is also consistent with some of the literature on applications of ARCH-M and GARCH-M processes. These processes model the expected excess rate of return to a security or to the market portfolio as a function of its standard deviation or variance, in addition to modeling the dynamics of the second moments.

For example, French, Schwert and Stambaugh (1987) tested two versions of GARCH-M, where the excess expected rate of return to the market portfolio is an affine function of either the standard deviation or the variance. Bodurtha and Mark (1991) compared three versions of the ARCH-M model, where the excess expected rate of return on the market portfolio is an affine function of either the variance, the standard deviation, or the logarithm of the variance. In both cases, the specification where the expected rate or return to the market is an affine function of the standard deviation corresponds to our assumption in Corollary 1 of a constant ICML, combined with a partial equilibrium argument to ensure that the market portfolio is on the ICML.

The assumption of a constant slope of the ICML is not easily rejected empirically, as we shall argue now. As noted earlier, the assumption of a constant slope of the ICML in conjuction with the joint hypothesis that the market portfolio is on the ICML will imply that the expectation of the return to the market portfolio is an affine function of its standard deviation. This has been tested by French, Schwert and Stambaugh (1987) and Bodurtha and Mark (1991). However, the point of doing portfolio theory is that one hopes to do better than investing in the market portfolio. Therefore, in the context of portfolio theory, it is not desirable to impose, a priori, the assumption that the market portfolio is on the ICML. The hypothesis that the slope of the ICML is constant, *without* the joint hypothesis that the market portfolio (or some other observed portfolio) is on the ICML, is testable in principle, although we are not aware of any study that derives and implements such a test. The test would probably involve estimating the ICML from data on individual securities. Because of the estimation error, it is unlikely that the test would have enough power to reject the hypothesis. In that sense, one can argue that the assumption of a constant slope of the ICML cannot easily be shown to be empirically unrealistic.

The assumption of a constant interest rate is admittedly easy to reject empirically. However, if the slope of the ICML is constant but the interest rate changes randomly, then investors will only hedge against interest rate risk.

Recall that we have identified the investment opportunity set with the instantaneous capital market line. An implication of our new definition of the concept of the investment opportunity set is that security returns and their moments may be predictable while the investment opportunity set is constant. Thus, one should be cautious in interpreting empirical evidence of predictability as implying that the investment opportunity set is changing over time.

Corollary 1 potentially resolves a prevalent problem in asset pricing tests with time-varying conditional moments. It explicitly allows for time-varying moments, yet there will be no intertemporal hedge funds, and hence, no intertemporal hedging premia in equilibrium. Empirical work often uses time varying conditional moments while ignoring the intertemporal hedge premia that theoretically would be present. For example, Dumas and Solnik (1995) explicitly acknowledge this issue. Corollary 1 provides a theoretical justification for ignoring the intertemporal hedging premia in conditional asset pricing tests, when they explicitly or implicitly assume a constant ICML. Tests where the specification violates the assumption of a constant (or deterministic) ICML cannot be justified by appeal to Corollary 1, however.

7 Conclusions

This study has reexamined the role of intertemporal hedge funds in optimal portfolio selection. We showed that only changes in the slope and position of the instantaneous capital market line (ICML) give rise to hedge funds. Hedge funds that hedge against changes in moments that do not lead to changes in the position and slope of the ICML are superfluous. This result simplifies the fund separation theorem of Merton (1973).

Because of our finding that investors hedge only against changes in the ICML, we proposed a new definition of the concept of an "investment opportunity set," according to which the investment opportunity set is identical to the ICML. With this definition, predictability in returns does not automatically constitute evidence of a changing investment opportunity set.

It is common in the empirical asset pricing literature to allow for randomly time varying moments of the returns to securities or to the market portfolio, while ignoring the intertemporal hedging premia that should be present in the specification according to Merton (1973). Our analysis provides a potential resolution of this problem. A constant or deterministic ICML implies that the intertemporal hedging premia disappear. Therefore, they can be ignored in empirical tests whose specification is consistent with a constant or deterministic ICML.

All of our results allow for the type of market incompleteness studied in He and Pearson (1991) and Karatzas et al. (1991). We found that market incompleteness does not matter so long as changes in the position and slope of ICML are driven by a vector of hedgeable state variables, one of which is proportional to the logarithmic portfolio. It does not matter in the sense that it does not upset the existence and uniqueness of an optimal portfolio or affect its composition.

Appendix

In this appendix, we prove Theorem 1 and Corollary 1. The proof of Theorem 1 will require a series of auxiliary Lemmas.

PROOF OF THEOREM 1:

Given Assumption 1, u' has an inverse

$$I : (0,\infty) \to (0,\infty)$$

Lemma 1 $\Pi(T)I(y\Pi(T))$ *is integrable for all* $y > 0$.

PROOF:

From Assumption 3, we have the inequality

$$R_R(u)(x) = -\frac{u''(x)x}{u'(x)} \geq \gamma$$

for $x \geq x_0$, which implies

$$(-\ln u'(x))' = R_A(u)(x) = -\frac{u''(x)}{u'(x)} \geq \frac{\gamma}{x}$$

and

$$\begin{aligned}
\ln\left(\frac{u'(x_0)}{u'(x)}\right) &= -\ln u'(x) + \ln u'(x_0) \\
&= [-\ln u'(t)]_{x_0}^{x} \\
&\geq \int_{x_0}^{x} \frac{\gamma}{t}\,dt \\
&= \gamma\,[\ln t]_{x_0}^{x} \\
&= \gamma\,(\ln x - \ln x_0) \\
&= \ln\left(\frac{x^\gamma}{x_0^\gamma}\right)
\end{aligned}$$

Hence,

$$x^\gamma \leq \frac{u'(x_0)x_0^\gamma}{u'(x)}$$

For $0 < z \leq u'(x_0)$, set $x = I(z)$. Then $z = u'(x)$, $I(z) \geq x_0$,

$$I(z)^\gamma \leq u'(x_0) x_0^\gamma z^{-1}$$

and

$$I(z) \leq c z^{-\frac{1}{\gamma}}$$

where

$$c = u'(x_0)^{\frac{1}{\gamma}} x_0$$

If $y > 0$, then

$$\Pi(T) I(y\Pi(T)) \leq \begin{cases} c y^{-\frac{1}{\gamma}} \Pi(T)^{\frac{\gamma-1}{\gamma}} & \text{if } \Pi(T) < u'(x_0)/y \\ \Pi(T) u'(x_0) & \text{if } \Pi(T) \geq u'(x_0)/y \end{cases}$$

It follows from Assumption 3 that right hand side is integrable, because $\Pi(T)$ is integrable, $\Pi(T)^{\frac{\gamma-1}{\gamma}}$ is integrable if $\gamma < 1$, and

$$\Pi(T)^{\frac{\gamma-1}{\gamma}} \leq [u'(x_0)/y]^{\frac{\gamma-1}{\gamma}}$$

if $\gamma \geq 1$ and $R < u'(x_0)/y$.

□

Since $\Pi(T) I(y\Pi(T))$ is integrable for all $y > 0$, we can define a function $h : (0, \infty) \to I\!R$ by

$$h(y) = E\left[\frac{\Pi(T)}{\Pi(0)} I(y\Pi(T))\right]$$

This function h is continuous and strictly decreasing with $h((0,\infty)) = (0,\infty)$. Hence, it has an inverse $\kappa : (0,\infty) \to (0,\infty)$ which is continuous and strictly increasing with $\kappa((0,\infty)) = (0,\infty)$.

Define Y^* by

$$Y^* = I(\kappa(w_0)\Pi(T))$$

Lemma 2 *$u(Y^*)$ is integrable above.*

PROOF:

Let γ be the parameter from Assumption 3, and let u_γ be the utility function on $(0, \infty)$ defined by

$$u_\gamma(x) = \begin{cases} \ln x & \text{if } \gamma = 1 \\ \frac{1}{1-\gamma} x^{1-\gamma} & \text{if } \gamma \neq 1 \end{cases}$$

It has constant relative risk aversion γ.

We will prove first that there exist constants $b > 0$ and a such that

$$u(x) \leq a + b u_\gamma(x)$$

for all $x \geq x_0$. Then we will show that $u(Y)$ is integrable above.

(1):

As in Lemma 1, we find

$$\frac{u'(x)}{u'(x_0)} \leq \frac{x_0^\gamma}{x^\gamma}$$

and

$$u'(x) \leq u'(x_0) x_0^\gamma x^{-\gamma}$$

for $x \geq x_0$. Hence,

$$u(x) - u(x_0) = [u'(t)]_{x_0}^x \leq u'(x_0) x_0^\gamma \int_{x_0}^x t^{-\gamma}\, dt$$

If $\gamma = 1$, then this implies

$$\begin{aligned} u(x) - u(x_0) &\leq u'(x_0) x_0 \left[\ln t\right]_{x_0}^x \\ &= u'(x_0) x_0 \left(\ln x - \ln x_0\right) \\ &= a + b u_\gamma(x) \end{aligned}$$

where $a = -u'(x_0) \ln x_0$ and $b = u'(x_0) x_0$. If $\gamma \neq 1$, we find

$$\begin{aligned} u(x) - u(x_0) &\leq u'(x_0) x_0^\gamma \frac{1}{1-\gamma} \left[t^{1-\gamma}\right]_{x_0}^x \\ &= u'(x_0) x_0^\gamma \frac{1}{1-\gamma} \left(x^{1-\gamma} - x_0^{1-\gamma}\right) \\ &= a + b u_\gamma(x) \end{aligned}$$

where $a = -u'(x_0) x_0 / (1-\gamma)$ and $b = u'(x_0) x_0^\gamma$.

(2):

If $\gamma > 1$, then u is bounded above because $u_\gamma < 0$.

If $\gamma = 1$, then it suffices to show that

$$\ln Y^* = \ln[\Pi(T)Y^*] - \ln \Pi(T)$$

is integrable above. The first term is integrable above because $\Pi(T)Y^*$ is integrable and ln is concave. The second term is integrable above because $\ln \Pi(T)$ is integrable below.

If $\gamma < 1$, set $q = 1/(1-\gamma)$ and $p = 1/\gamma$. Then

$$\frac{1}{p} + \frac{1}{q} = \gamma + (1-\gamma) = 1$$

It suffices to show that

$$(Y^*)^{1-\gamma} = [\Pi(T)Y^*]^{1-\gamma}\,\Pi(T)^{\gamma-1}$$

is integrable. By Hölder's inequality, it is enough to observe that

$$[\Pi(T)Y^*]^{(1-\gamma)q} = \Pi(T)Y^*$$

and

$$\Pi(T)^{(\gamma-1)p} = \Pi(T)^{\frac{\gamma-1}{\gamma}}$$

are integrable.

□

The claim Y^* has been constructed in such a way that $E[\Pi(T)Y^*] = \Pi(0)w_0$. The next lemma says that if there is a self-financing trading strategy which replicates Y^*, then it is the unique optimal trading strategy.

Lemma 3 *Suppose $\bar{\Delta}$ is a self-financing trading strategy such that $\bar{\Delta}\bar{S} > 0$, $\bar{\Delta}(T)\bar{S}(T) = Y^*$ and $\bar{\Delta}(0)\bar{S}(0) = w_0$. Then $\Pi\bar{\Delta}\bar{S}$ is a martingale, and $\bar{\Delta}$ is the unique[6] optimal trading strategy given initial wealth w_0.*

[6]To be precise, uniqueness means uniqueness almost everywhere with respect to the product measure $P \otimes \lambda$ on $\Omega \times [0,T]$, where P is the probability measure on Ω and λ is Lebesgue measure on $[0,T]$.

PROOF:

First,

$$
\begin{aligned}
E(\Pi(T)\bar{\Delta}(T)\bar{S}(T)) &= E(\Pi(T)Y^*) \\
&= E(\Pi(T)I(\kappa(w_0)\Pi(T))) \\
&= \Pi(0)h(\kappa(w_0)) \\
&= \Pi(0)w_0 \\
&= \Pi(0)\bar{\Delta}(0)\bar{S}(0)
\end{aligned}
$$

Since $\bar{\Delta}$ is a self-financing trading strategy such that $\bar{\Delta}\bar{S} > 0$, $\Pi\bar{\Delta}\bar{S}$ is a non-negative stochastic integral. Since $\Pi(T)\bar{\Delta}(T)\bar{S}(T)$ is integrable, it follows that $\Pi\bar{\Delta}\bar{S}$ is a supermartingale. Hence, for $0 \le t \le T$,

$$
E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t] \le \Pi(t)\bar{\Delta}(t)\bar{S}(t)
$$

and

$$
\begin{aligned}
\Pi(0)\bar{\Delta}(0)\bar{S}(0) &= E[\Pi(T)\bar{\Delta}(T)\bar{S}(T)] \\
&= E(E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t]) \\
&\le E[\Pi(t)\bar{\Delta}(t)\bar{S}(t)] \\
&\le \Pi(0)\bar{\Delta}(0)\bar{S}(0)
\end{aligned}
$$

which implies that

$$
E[\Pi(T)\bar{\Delta}(T)\bar{S}(T) \mid \mathcal{F}_t]) = \Pi(t)\bar{\Delta}(t)\bar{S}(t)
$$

This shows that $\Pi\bar{\Delta}\bar{S}$ is a martingale.

To show that $\bar{\Delta}$ is optimal, suppose $\bar{\delta}$ is a self-financing trading strategy such that $\Pi(T)\bar{\delta}(T)\bar{S}(T)$ is integrable, $\bar{\delta}(0)\bar{S}(0) = w_0$, and $\bar{\delta}\bar{S} > 0$. Then $\Pi\bar{\delta}\bar{S}$, like $\Pi\bar{\Delta}\bar{S}$, is a supermartingale, and so

$$
E[\Pi(T)\bar{\delta}(T)\bar{S}(T)] \le \Pi(0)\bar{\delta}(0)\bar{S}(0) = \Pi(0)w_0
$$

Because u is differentiable and concave,

$$
\begin{aligned}
u(Y^*) &\ge u(\bar{\delta}(T)\bar{S}(T)) + u'(Y^*)(Y^* - \bar{\delta}(T)\bar{S}(T)) \\
&= u(\bar{\delta}(T)\bar{S}(T)) + \kappa(w_0)\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T))
\end{aligned}
$$

Here. $\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T))$ is integrable and

$$E[\Pi(T)(Y^* - \bar{\delta}(T)\bar{S}(T))] = \Pi(0)w_0 - E[\Pi(T)\bar{\delta}(T)\bar{S}(T)] \geq 0$$

Hence, since $u(Y^*)$ is integrable above by Lemma 2, so is $u(\bar{\delta}(T)\bar{S}(T))$, and

$$Eu(Y^*) \geq Eu(\bar{\delta}(T)\bar{S}(T))$$

This shows that $\bar{\Delta}$ is optimal.

Finally, to show that $\bar{\Delta}$ is the unique optimal trading strategy given initial wealth w_0, suppose both $\bar{\Delta}$ and $\bar{\delta}$ are optimal. Set

$$\bar{\psi} = \frac{1}{2}\bar{\Delta} + \frac{1}{2}\bar{\delta}$$

Then $\bar{\psi}$ is a self-financing trading strategy with positive value process and initial value w_0, and $u(\bar{\psi}(T)\bar{S}(T))$ is integrable above. Because u is strictly concave,

$$\begin{aligned}
Eu(\bar{\psi}(T)\bar{S}(T)) &= Eu\left(\frac{1}{2}\bar{\Delta}(T)\bar{S}(T) + \frac{1}{2}\bar{\delta}(T)\bar{S}(T)\right) \\
&\geq \frac{1}{2}Eu(\bar{\Delta}(T)\bar{S}(T)) + \frac{1}{2}Eu(\bar{\delta}(T)\bar{S}(T)) \\
&= Eu(\bar{\Delta}(T)\bar{S}(T)) = Eu(\bar{\delta}(T)\bar{S}(T))
\end{aligned}$$

with strict inequality unless $\bar{\Delta}(T)\bar{S}(T) = \bar{\delta}(T)\bar{S}(T)$ with probability one. But since $\bar{\Delta}$ and $\bar{\delta}$ are optimal, the inequality indeed is not strict, and so $\bar{\Delta}(T)\bar{S}(T) = \bar{\delta}(T)\bar{S}(T)$. By what has already been shown, both $\Pi\bar{\Delta}\bar{S}$ and $\Pi\bar{\delta}\bar{S}$ are martingales. Therefore, for every t with $0 \leq t \leq T$,

$$\begin{aligned}
\Pi(t)\bar{\Delta}(t)\bar{S}(t) &= E[\Pi(T)\bar{\Delta}(T)\bar{S}(T)|\mathcal{F}_t] \\
&= E[\Pi(T)\bar{\delta}(T)\bar{S}(T)|\mathcal{F}_t] \\
&= \Pi(t)\bar{\delta}(t)\bar{S}(t)
\end{aligned}$$

which implies that $\bar{\Delta}\bar{S} = \bar{\delta}\bar{S}$. Write $\bar{\Delta} = (\Delta_0, \Delta)$ and $\bar{\delta} = (\delta_0, \delta)$. Then

$$\Delta\mathcal{D}(S)\sigma = \bar{\Delta}\bar{\sigma} = \bar{\delta}\bar{\sigma} = \delta\mathcal{D}(S)\sigma$$

Since the matrix $\mathcal{D}(S)\sigma$ has rank N, this implies that $\Delta = \delta$. Since $\bar{\Delta}$ and $\bar{\delta}$ are both self-financing and have the same initial value, it follows that they are identical.

□

Given Lemma 3, it remains to show that the claim Y^* can be replicated by a self-financing trading strategy whose corresponding portfolio strategy has the form stipulated in the theorem.

Recall that

$$B(t) = \int_0^t b\sigma\, W$$

where b is an $m \times N$ dimensional measurable and adapted process b such that $b\sigma \in \mathcal{L}^2$, $b\sigma\sigma^\top b^\top = I$. Each row in B is interpreted as a hege fund.

The assumption that the logarithmic portfolio is proportional to the first hedge fund means that there exists a one-dimensional measurable and adapted positive process η such that

$$\phi^{\ln} = \eta b_{1-}$$

where b_{1-} is the first row of b. The vector of prices of risk is

$$\lambda = \phi^{\ln}\sigma = \eta b_{1-}\sigma$$

and so

$$\lambda\lambda^\top = \eta b_{1-}\sigma\sigma^\top b_{1-}^\top \eta = \eta^2$$

and

$$\eta = \sqrt{\lambda\lambda^\top}$$

In particular, $\eta \in \mathcal{L}^2$, η is measurable and adapted with respect to F^B, and

$$\int_0^t \lambda\, dW = \int_0^t \eta b_{1-}\sigma\, dW = \int_0^t (\eta, 0, \ldots, 0) b\sigma\, dW = \int_0^t (\eta, 0, \ldots, 0)\, dB$$

This process is an Itô process with respect to B and F^B. Now,

$$\begin{aligned} \ln\Pi(t) - \ln\Pi(0) &= \int_0^t \left(-r - \lambda\lambda^\top\right) ds - \int_0^t \lambda\, dW \\ &= \int_0^t \left(-r - \eta^2\right) ds - \int_0^t (\eta, 0, \ldots, 0)\, dB \end{aligned}$$

Since r and η are adapted and measurable with respect to F^B, the process Π is an Itô process with respect to B and F^B.

For $k = 1, \ldots, m$, let b_{k-} denote the k'th row in b, and let Φ be the m dimensional column vector valued process whose k'th element Φ_k is the value process of the portfolio strategy b_{k-}, assuming an initial value of $\Phi_k(0) = 1$. Then

$$\begin{aligned} d\Phi &= \mathcal{D}(\Phi)\left[\left(b\sigma\lambda^\top + r\iota\right)\, dt + b\sigma\, dW\right] \\ &= \mathcal{D}(\Phi)\left[\left(b\sigma\sigma^\top b_{1-}^\top \eta + r\iota\right)\, dt + dB\right] \\ &= \mathcal{D}(\Phi)\left[(\eta + r, r, \ldots, r)^\top\, dt + dB\right] \end{aligned}$$

The process Φ is adapted and measurable with respect to F^B.

Consider the reduced economy driven by B and F^B, where the risky asset prices are Φ_k, $k = 1, \ldots, m$ and the money market account is the same as in the original economy. In the reduced economy, the vector of prices of risk is $(\eta, 0, \ldots, 0)$, and the state price process is Π, like in the original economy. The reduced economy has complete markets.

The claim Y^* is measurable with respect to $\mathcal{F}_T^B$ and $\Pi(T)Y^*$ is integrable. Since the reduced economy has dynamically complete markets, there exists a self-financing trading strategy $\bar{\gamma} = (\gamma_0, \gamma)$ in the reduced economy which replicates Y^* and whose value process

$$V = \gamma_0 M + \gamma\Phi$$

has the property that ΠV is a martingale with respect to F^B. Since Y^* and Π are positive, the value process V is positive. Since ΠV is a martingale, in particular,

$$\Pi(0)V(0) = E[\Pi(T)V(T)] = E[\Pi(T)Y^*] = \Pi(0)w_0$$

and so $V(0) = w_0$.

Now let β be the portfolio strategy in the reduced economy corresponding to the trading strategy $\bar{\gamma}$:

$$\beta = \gamma\mathcal{D}(\Phi)/V$$

Then

$$\begin{aligned} \frac{dV}{V} &= \left[(1 - \beta\iota)r + \beta\left(\eta^\top + r\iota\right)\right]\, dt + \beta\, dB \\ &= \left[\beta\eta^\top + r\right]\, dt + \beta b\, dW \end{aligned}$$

Consider the portfolio strategy in the original economy

$$\tilde{\Delta} = \beta b$$

Start this trading strategy off at the initial wealth level w_0. Let w denote its value process. Then

$$\begin{aligned}
\frac{dw}{w} &= \left(\tilde{\Delta}\sigma\lambda^{\top} + r\right) dt + \tilde{\Delta}\sigma\, dW \\
&= \left(\beta b\sigma(\eta b\sigma)^{\top} + r\right) dt + \beta b\sigma\, dW \\
&= \left(\beta b\sigma\sigma^{\top} b^{\top}\eta^{\top} + r\right) dt + \beta b\sigma\, dW \\
&= \left(\beta\eta^{\top} + r\right) dt + \beta b\sigma\, dW \\
&= \frac{dV}{V}
\end{aligned}$$

Hence, $w = V$, and $\tilde{\Delta}$ replicates Y^* in the original economy. It follows that $\tilde{\Delta} = \beta b$ is the unique optimal portfolio strategy in the original economy.

□

PROOF OF COROLLARY 1:

Set $m = 1$,

$$b = \frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda\sigma^{\top}\left(\sigma\sigma^{\top}\right)^{-1} = \frac{1}{\sqrt{\lambda\lambda^{\top}}}\phi^{\ln}$$

and

$$\eta = \sqrt{\lambda\lambda^{\top}}$$

Then

$$b\sigma = \frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda\sigma^{\top}\left(\sigma\sigma^{\top}\right)^{-1}\sigma = \frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda$$

and

$$b\sigma\sigma^{\top}b^{\top} = \frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda\lambda^{\top}\frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda = 1$$

Define B by

$$B(t) = \int_0^t b\sigma\, dW = \int_0^t \frac{1}{\sqrt{\lambda\lambda^{\top}}}\lambda\, dW$$

Then B is a Wiener process relative to the filtration F, and B is hedgeable with hedge fund b. The logarithmic portfolio is proportional to b, since

$$\phi^{\ln} = \sqrt{\lambda\lambda^{\top}}b = \eta b$$

Since r and η are deterministic, they are, in particular, adapted and measurable with respect to F^B. The final value $\Pi(T)$ is lognormally distributed because r and $\lambda\lambda^{\top}$ are deterministic. Hence, Assumption 2 implies Assumption 3. So, Theorem 2 says that there exists a unique optimal portfolio strategy, and that it has the form $\tilde{\Delta} = \beta b$ for some adapted and measurable one-dimensional process β. But then

$$\tilde{\Delta} = \beta b = \beta \frac{1}{\sqrt{\lambda\lambda^{\top}}} \phi^{\ln} = \alpha \phi^{\ln}$$

where

$$\alpha = \beta \frac{1}{\sqrt{\lambda\lambda^{\top}}}$$

□

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