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Ayse KOCABIYKOGLU
Ioana POPESCU
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by

Ayse Kocabiykoglu*

and

Ioana Popescu**

** Assistant Professor of Management, Bilkent University, Bilkent, Ankara 06800, Turkey, aysekoca@bilkent.edu.tr

*** Associate Professor of Decision Sciences, The Booz Allen Hamilton Term Chaired Professor in Strategic Revenue Management at INSEAD, Boulevard de Constance, 77305 Fontainebleau Cedex, France, ioana.popescu@insead.edu

Revenue Management with General Stochastic Price-Sensitive Demand

Ayşe Kocabıyıkoğlu
Department of Management, Bilkent University, Bilkent, Ankara 06800, Turkey, aysekoca@bilkent.edu.tr

Ioana Popescu*
INSEAD, Decision Sciences Area, Blvd. de Constance, 77300 Fontainebleau, France, ioana.popescu@insead.edu

We extend classical, static two-fare class revenue management models to account for demand sensitivity to price, and optimize pricing decisions. We first characterize the sensitivity of the optimal protection level with respect to price, in a standard revenue management setting. We then simultaneously optimize price and capacity allocation decisions for the high-end segment, and characterize the uniqueness of the optimal solution. We find that firms with larger capacity should expect lower revenue rates, set higher protection levels, and lower high end prices in a monopolistic environment. We explore several models, with or without demand and resource substitution. Our results hold under general models of stochastic price-dependent demand. A unifying condition for our results is the monotonicity of the elasticity of the rate of lost sales, a concept introduced by Kocabıyıkoğlu and Popescu (2007). Several heuristics for coordinating pricing and allocation decisions are proposed, leading to bounds on expected revenues. Numerical experiments indicate that jointly optimizing the price and protection level for high fare customers leads to significant profit benefits over a hierarchical decision process.

1. Introduction

Broadly speaking, revenue management investigates sales strategies that extract maximum profits from a limited inventory across market segments with heterogeneous, uncertain demands. Revenue management models and practice have traditionally focused on optimal inventory allocation decisions, treating price and demand as exogenous. In practice, this can be explained by the separation of the marketing (including pricing) and revenue management functions in organizations, as well as by the technical and operational difficulties inherent in the implementation of a coordinated price-availability decision support system. This paper investigates the effects of coordinating price and

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allocation decisions for the high end market segment, in a simple, classical two-fare class revenue management setting.

Recent advances in information technology have enabled companies to acquire valuable information about customers' preferences and price-sensitivity. Putting this knowledge into practice can dramatically affect profitability, through pricing and inventory decisions. The benefits of managing pricing decisions, relative to other profit levers, are generally well established. McKinsey & Company, a consultancy, estimated that "for the average S&P 1500 company, a price increase of 1% would generate an increase in profits of 8-12%, that is an impact 50% greater than a 1% cut in variable costs and 300% greater than a 1% increase in volumes (McKinsey Quarterly 2003). The importance of coordinating pricing and inventory decisions has also been widely acknowledged in the revenue management context (McGill and van Ryzin 1999), and generally in the operations literature. In a broad spectrum survey, Fleischmann, Hall and Pyke (2004) observe that "Pricing decisions have a direct effect on operations and visa versa. Yet, the systematic integration of operational and marketing insights is in an emerging stage, both in academia and in business practice." In a revenue management context, this motivates the need for a systematic understanding of the impact, interaction and sensitivity of capacity allocation and pricing decisions, under a unified framework for modeling uncertain, price-sensitive demand.

Revenue management approaches are common in capacitated service industries such as airlines, hotels, car rentals, events, TV advertising etc., where demand is responsive to price changes. In some of these settings, such as music and sporting events, low end prices are kept low for fairness or image considerations, but high end prices are actively managed. In other settings, the low end market is highly competitive, with little degree of pricing power, relative to the high end segment. In fact, the first revenue management initiative, American Airlines' Ultimate Super Saver Programme, was specifically designed to (conditionally) match low fare competitor People Express on the low end segment, while reserving capacity (so called *protection level*) for higher margin sales. To date, major airlines continue to offer low fare products, on a limited basis, to compete against low cost competitors such as Southwest and Easy Jet. In the high end market, however, airline price dispersion is very high (reaching up to 700%, Donofrio 2002), and competition is less stark. High fare customers are typically business customers, sensitive to service (e.g. schedule, network) and flexibility, and less likely to shop around (e.g. higher switching costs due to loyalty programmes). This offers the firm a wider degree of price flexibility, and makes pricing for this segment an important profit lever. Overall, this motivates us to focus on optimizing allocation and pricing decisions for the high end segment in this paper.

Classical revenue management models, such as the *expected marginal seat revenue (EMSR)* model (see Littlewood 1972 or Belobaba 1987), assume prices are fixed and optimize the allocation of a fixed capacity between two (high and low valuation) customer segments, where higher valuation customers purchase later in the horizon. This paper proposes a static *Pricing and Revenue Management (PRM)* model, which extends the standard ($EMSR$) model by incorporating pricing decisions. Price is a key determinant of demand, which motivates us to model high end demand as a stochastic function of price. Our goal, in this context, is to characterize the structure of the optimal joint pricing and allocation policy for the *(PRM)* and related models, and obtain sensitivity results. Considering that ($EMSR$) is fundamentally the base model for revenue management, and the oldest revenue management model still in practical use, it is at least surprising that price sensitivity and optimization have not been previously analyzed in this context. This paper aims to fill this gap in the literature.

Our results hold for general stochastic price-sensitive demand models. Part of our contribution is to provide general, unifying demand conditions in terms of an elasticity measure of stochastic demand: the price elasticity of the rate of lost sales. This elasticity measure is a function of both price and quantity introduced in Kocabıyıkoğlu and Popescu (2007), and referred henceforth to for simplicity as *sales elasticity*.

We begin by investigating price sensitivity of the optimal protection level policy. In the classical ($EMSR$) setting, where demand sensitivity to price is not modeled, an increase in high-fare price commands a larger portion of capacity protected for high end customers. When demand is price-sensitive, this effect is countered by a decrease in high-fare demand, so the relationship becomes ambiguous, and often reversed in the (PRM) model. We show that the monotonicity of the protection level in price is guaranteed by price monotonicity of the sales elasticity, or alternatively, by a lower bound of 1 on the sales elasticity along the optimal allocation path.

We further investigate joint pricing and allocation decisions. Similar elasticity conditions allow us to obtain a unique solution for the *(PRM)* model. Specifically, we show that if the sales elasticity is increasing in price, this problem is concave, hence easy to solve, and admits a unique solution. Alternatively, a lower bound of $\frac{1}{2}$ on elasticity along the optimal allocation path guarantees the existence of a unique optimum. These results extend to allow for demand substitution effects. Our results indicate that larger capacity firms should expect lower revenue rates, e.g. lower RAS (revenue per available seat) for airlines and lower REVPAR (revenue per available room) for hotels. They should also set higher protection levels and lower high end prices, if the sales elasticity is monotone in price and quantity.

Results of a similar nature are obtained for a modified, partitioned allocation model, where resource substitution is not allowed. For example, a relevant application is setting prices and capacity decisions for the economy and business cabins. This model further serves to provide heuristics coordinating pricing and allocation decisions, and to obtain bounds on expected revenues for the *(PRM)* model.

All our results hold under fairly general models of stochastic, price-sensitive demand, satisfying monotone sales elasticity conditions. These conditions are the same which ensure analogous structural results in a simpler, price-setting newsvendor framework, as identified in Kocabıyıkoğlu and Popescu (2007). This class of demand models includes for example additive-multiplicative models with linear and iso-elastic price-dependence, and increasing failure rate (IFR) risk. In this respect, our results are similar to the deterministic demand conditions proposed by Ziya, Ayhan and Foley (2004).

The literature on revenue management is vast. The most comprehensive reference to date is the book by Talluri and van Ryzin (2004). Bitran and Caldentey (2003) provide a review of the literature combining pricing and revenue management decisions. A broader scope survey of dynamic pricing by Elmaghraby and Keskinocak (2003) brings together academic literature and industry practice. There is a vast amount of research studying either pricing or capacity allocation problems (separately) in the revenue management literature, but limited work addresses the integration of these two types of decisions. Our paper extends the capacity allocation models of Littlewood (1972) and Belobaba (1987) by adding the pricing dimension. Weatherford (1997) considers a formulation similar to ours in a multi-product environment, where demands are normally distributed with mean demand linearly increasing in price. Numerical results investigate the computational effort required to integrate pricing and allocation decisions, but no analytical results are obtained.

While this work focuses exclusively on a static setting, several authors consider the coordination of pricing and allocation decisions in a multi-period setting. Bertsimas and de Boer (2005) propose heuristics and bounds for a multi-period problem, with aggregate additive-multiplicative price dependent demand. They also provide conditions under which the static problem has a unique solution. Their setup does not allow resource substitution among product classes, similar to our model in Section 6. Our results suggest how theirs could be extended for more general demand models. Several papers model demand as a non homogeneous Poisson arrival process, with price and time-dependent intensity. Among these, Maglaras and Meissner (2006) show that the capacity control problem (where a price taking firm allocates its fixed capacity) and the dynamic pricing problem (where the firm sets prices at each time period) can be reduced to a common formulation, where aggregate capacity consumption (the sum of demand rates for all products at a given time period) is the control variable. Feng and Xiao (2006) show that the optimal price and capacity control policy is based on a sequence of threshold points that incorporate inventory, price and demand intensity. Bi-level optimization models for joint pricing and allocation are studied by several authors, notably Cote et al. (2003).

Our work is related to a vast literature on price setting newsvendor models, where a firm simultaneously sets the optimal order quantity and the price for a single product. Reviews of this literature can be found in Petruzzi and Dada (1999) and Yano and Gilbert (2006); seminal papers include Karlin and Carr (1962), Mills (1959), Young (1978) and Zabel (1970). Most of this literature models demand as an additive and/or multiplicative function of price. Kocabıyıkoğlu and Popescu (2007) study the newsvendor with pricing problem for general stochastic price-sensitive demand models, and obtain unifying results based on the monotonicity of the sales elasticity. Our results are similar, and can be viewed as extending theirs in a revenue management setting. Although our model is more complex, allowing multiple products, with or without demand and resource substitution, the same general demand conditions (monotone sales elasticity) as in the newsvendor setting are sufficient to characterize the optimal solution.

The rest of the paper is organized as follows. The demand model and underlying assumptions are presented in Section 2. The impact of price on the optimal protection level, in a classical ($EMSR$) setting, is investigated in Section 3. A systematic relationship is characterized by the monotonicity of the stochastic sales elasticity, which is a key driver of the results in the paper. Demand models with monotone elasticity are presented in Section 4. Section 5 studies the joint Pricing and Revenue Management (PRM) problem, provides sensitivity results and an extension to substitution effects. Section 6 studies the Partitioned Allocation (PA) model, a base case for the (PRM) problem, where resource substitution is not allowed; we characterize the optimal solution and compare it with the (PRM) model. Section 7 provides several price-allocation heuristics, and obtains analytical bounds on the optimal revenue obtained from the (PRM) model. Section 8 presents numerical experiments which demonstrate significant improvements in revenue from integrating pricing and allocation decisions. We also illustrate the sensitivity of the optimal decisions and revenues to various market factors. Finally, Section 9 concludes the paper.

2. The Model

In the standard ($EMSR$) model (Belobaba 1987), a revenue maximizing firm optimizes the allocation of a fixed quantity of a fully flexible resource between two market segments with given prices (low and high fare) and uncertain, price independent demands; customers with higher willingness to pay arrive later in the horizon. In reality, however, prices impact demand; in particular, demand for major application areas of revenue management, such as airline travel, car rental etc. is quite sensitive to price changes (see Talluri and van Ryzin, 2004 Chapter 7). To capture this relationship, we model high fare demand as $\mathbf{D}(p)$, a stochastic function of its price.

Let C denote the fixed capacity, and x, the protection level. We denote the average contribution margin of low fare class as $\bar{p}$, with $p > \bar{p}$. Throughout the paper, the parameters of the low fare class are denoted by a bar (overline). The expected revenue, for a given high fare price p and protection level x, is defined as:

$$R(p,x) = \bar{p}\mathbb{E}\left[\min\left(\bar{\mathbf{D}}, C - x\right)\right] + \mathbb{E}\left[r(p, \max\left(x, C - \bar{\mathbf{D}}\right))\right], \tag{1}$$

where the expected revenue for the high-end class is given by:

$$r(p,x) = p\mathbb{E}[\min(\mathbf{D}(p), x)]. \tag{2}$$

The *(EMSR)* model, studied in Section 3, solves, for a fixed price p, the allocation problem:

$$(EMSR) \qquad \max_{x} R(p,x). \tag{3}$$

The pricing and revenue management problem *(PRM)*, studied in Section 5, is modeled as:

$$(PRM) \qquad \max_{x,p} R(p,x). \tag{4}$$

This model is extended to allow for demand substitution effects in Section 5.3. Section 6 studies a version of this problem that does not allow resource substitution.

The first term in (1) is the revenue from the low price product. Sales to this class are constrained by demand and the *booking limit*, $C - x$. Let $\bar{\mathbf{\Omega}}(x) = (\bar{\mathbf{D}} \leq C - x)$ define the event that low fare demand does not exceed the booking limit. The second term in (1) is the revenue from the high price product; inventory available to class 1 is uncertain and may exceed the protection level, if $\bar{\mathbf{\Omega}}$ occurs. An equivalent expression for $r(p,x)$ is

$$r(p,x) = pxq(p,x) + \mathbb{E}\left[\pi\left(p, \mathbf{Z}\right); \mathbf{\Omega}\right], \tag{5}$$

where $q(p,x) = 1 - F(p,x) = P(\mathbf{D}(p) \geq x)$ denotes the probability of excess demand for a given protection level x and price p, and $\mathbf{\Omega} = \mathbf{\Omega}(p,x) = (\mathbf{D}(p) \leq x)$ is the event that high fare demand does not exceed x.

For homogeneity of notation, we define all the functions related to class 2 demand $(\bar{q}, \bar{f}, \bar{F}, \bar{z})$ on a mirrored x-axis, with the origin at C. For example, $\bar{q}(x) = 1 - \bar{F}(x) = P(\bar{\mathbf{D}} \geq C - x)$. Furthermore, demands for the two classes, $\mathbf{D}$ and $\bar{\mathbf{D}}$ are assumed to be independent.

The price-sensitive stochastic high fare demand is modeled as:

$$\mathbf{D}(p) = d(p, \mathbf{Z}), \tag{6}$$

where $\mathbf{Z}$ is a random variable which captures the demand *risk* (the uncertainty about market conditions), and has a continuously differentiable density function ϕ and cumulative distribution function Φ. The riskless demand $d(p, z)$ is decreasing in price p, strictly increasing in z, and twice differentiable in p and z.[1] Monotonicity of demand in z allows to uniquely define the inverse function $z(p, x)$ such that $d(p, z(p, x)) = x$. We assume that the riskless (or pathwise) unconstrained revenue $\pi(p, z) = pd(p, z)$ is strictly concave in p for any realization of z, i.e. $2d_p(p, z) + pd_{pp}(p, z) < 0$, where partial derivatives are denoted by corresponding subscripts. This assumption is not necessary for all our results, but simplifies the analysis.

The general demand model (6) encompasses additive-multiplicative models, commonly used in the price setting newsvendor literature (see Young 1978):

$$\mathbf{D}(p) = d(p, \mathbf{Z}) = \alpha(p)\mathbf{Z} + \beta(p), \tag{7}$$

where $\alpha(p), \beta(p)$ are decreasing functions of p. For $\alpha(p) \equiv 1$, this is the additive model (price influences the location of the demand distribution), whereas for $\beta(p) \equiv 0$ this is the multiplicative model (price influences demand scale). Our setup also allows for more general demand formulations, such as $d(p, z) = log(z - bp)$.

The main driver of structural results in this paper is an elasticity concept corresponding to stochastic demand, introduced by Kocabıyıkoğlu and Popescu (2007). This is the *elasticity of the rate of lost sales*, measured as the percentage change in the rate of lost sales with respect to price, and referred to for simplicity as the *sales elasticity.*

[1] Throughout the paper the terms increasing/decreasing, positive/negative are used in their weak sense.

Definition 1. For a given price p and quantity x, the elasticity of the rate of lost sales is given by:

$$\mathcal{E}(p,x) = -\frac{pq_p(p,x)}{q(p,x)} = \frac{pF_p(p,x)}{1-F(p,x)}. \tag{8}$$

The sales elasticity $\mathcal{E}(p,x)$ combines the relative sensitivity of sales with respect to its underlying factors, allocation and price. In Section 4, we provide monotonicity properties of the sales elasticity, which drive our sensitivity and uniqueness results. For a more detailed discussion, and equivalent expressions for the sales elasticity, the reader is referred to Kocabıyıkoğlu and Popescu (2007).

We conclude this section with two assumptions which restrict the range of allowable prices p, or protection levels x. These technical assumptions are used alternatively for some of the results in the paper, in particular in order to obtain a compact expression for Littlewood's rule in terms of the sales elasticity in Section 3.

Assumption A $x \in X = [0, x_U]$, *where* x_U *is such that* $q(\bar{p}, x_U) = 1$.

Assumption B $p \in P = [p_L, p_H]$, *where* $p_L = \arg\max d(p, \bar{\Phi}^{-1}(\bar{p}/p))$.

Specifically, Assumption B implies that p_L solves:

$$p_L^2 \varphi\left(\bar{\Phi}^{-1}(\bar{p}/p_L)\right) = -\frac{d_z\left(p_L, \bar{\Phi}^{-1}(\bar{p}/p_L)\right)}{d_p\left(p_L, \bar{\Phi}^{-1}(\bar{p}/p_L)\right)}. \tag{9}$$

Assumption A states that the firm does not protect more inventory for the high end class than the low fare price $\bar{p}$ can clear. Assumption B states that the firm does not consider high fare prices below a certain level $p_L \geq \bar{p}$. In practice, these assumptions are not very restrictive, because the relevant range of high end prices is significantly higher than the low fare price (see e.g. Donofrio 2002). Section 4 further discusses these assumptions for some special forms of demand.

Throughout the paper, we set without loss of generality $\bar{p} = 1$, to simplify notation.

Table 1 provides a summary of the main notation used throughout the paper.

Table 1. Summary of notation

C	capacity
x	protection level
$\bar{x} = C - x$	booking limit
$p, \bar{p}$	high, low fare price (usually $\bar{p} = 1$)
p_L	lower bound on high fare price set by Assumption B
$\mathbf{D}(p) = d(p, \mathbf{Z})$	high fare (stochastic) demand
$\mathbf{Z}$	random component of $\mathbf{D}(p)$
$z(p, x)$	inverse of the deterministic demand $d(p, z) = x$
$\pi(p, x) = pd(d, z)$	riskless profit
$q(p, x) = P(\mathbf{D}(p) \geq x)$	survival function of $\mathbf{D}(p)$
$f(p, x), F(p, x)$	density, cdf of $\mathbf{D}(p)$
$\phi(z), \Phi(z)$	density, cdf of $\mathbf{Z}$
$\bar{\mathbf{D}}$	low fare (stochastic) demand
$\bar{q}(x) = P(\bar{\mathbf{D}} \geq \bar{x})$	survival function of $\bar{\mathbf{D}}$
$\bar{f}(x), \bar{F}(x)$	density, cdf of $\bar{\mathbf{D}}$ (defined on a mirrored axis at C)
$\mathbf{\Omega}(p, x)$	event that high fare demand does not exceed x
$\bar{\mathbf{\Omega}}(x)$	event that low fare demand does not exceed $\bar{x}$
$\mathcal{E}(p, x)$	sales elasticity (elasticity of the rate of lost sales)

3. Price Sensitivity of the Protection Level

In the standard ($EMSR$) problem, the firm optimizes the allocation of a fixed capacity between low and high fare markets, treating prices and demand as exogenous. In this context, the optimal protection level increases in the high fare price, if the latter does not affect demand; a larger portion of capacity is allocated to the now more profitable class. When the effect of price on demand is considered, however, the relationship between protection level and price is non-trivial. Higher prices lead to higher profit margins, but lower mean demand. Hence, the revenue manager's response to a change in prices is not clearly determined, and depends on demand sensitivity to price. In this section, we identify the elasticity of the rate of lost sales as the driver of a systematic relationship between protection level and price.

We remind the characterization of the optimal protection level, established by Littlewood (1972).

Proposition 1 **(Littlewood, 1972)**. *$R(p, x)$ is quasiconcave in x for any given p. The optimal protection level, $x^*(p)$, is unique and satisfies,*

$$q(p, x) = 1/p. \tag{10}$$

Uniqueness of $x^*(p)$ allows us to denote $f^*(p) = f(p,x)|_{x=x^*(p)}$, the evaluation of any generic function f along the optimal solution path.[2] We also denote $f_x^*(p) = f_x(p,x)|_{x=x^*(p)}$, the derivative of $f(p,x)$ with respect to x evaluated at the optimal quantity. In this notation, used throughout the paper, the derivative always precedes functional evaluation.

Because $d(p,z(p,x)) = x$, the optimality condition (10) can be written in terms of z as $P(\mathbf{Z} \geq z^*(p)) = 1/p$. We obtain that

$$\frac{\partial z^*(p)}{\partial p} = \frac{1}{p^2 \phi^*(p)} \geq 0, \tag{11}$$

i.e. under the optimal allocation policy, $z^*(p)$ is increasing in p. The behavior of $z^*(p)$ with respect to p underlines the generally ambiguous nature of the relationship between $x^*(p)$ and p. Indeed, by definition, $x^*(p) = d(p, z^*(p))$, so $\frac{\partial x^*(p)}{\partial p} = d_p(p, z^*(p)) + \frac{\partial z^*(p)}{\partial p} d_z(p, z^*(p))$. Because $d_p \leq 0$ and $d_z \geq 0$, the direction of change in $x^*(p)$ with respect to p is not clear. The next result shows that a bound of 1 on the sales elasticity along the optimal allocation path guarantees that the direct price effect dominates, and $x^*(p)$ decreases in price.

Proposition 2. *$x^*(p)$ is decreasing in p if and only if $\mathcal{E}^*(p) \geq 1$.*

Proof. Optimality of $x^*(p)$ and the implicit function theorem imply $\dfrac{\partial x^*(p)}{\partial p} = \dfrac{-R_{xp}^*(p)}{R_{xx}^*(p)}$. In particular,

$$R_{xx}(p,x) = \bar{f}(x)[pq(p,x) - 1] - \bar{q}(x) p f(p,x), \tag{12}$$

implying via (10), $R_{xx}^*(p) = -\bar{q}^*(p) p f^*(p)$. Because

$$R_{xp}(p,x) = \bar{q}(x)\left[q(p,x) + p d_p\left(p, z(p,x)\right) f(p,x)\right] = \bar{q}(x) q(p,x)(1 - \mathcal{E}(p,x)), \tag{13}$$

we can calculate:

$$\frac{\partial x^*(p)}{\partial p} = -\frac{R_{xp}^*(p)}{R_{xx}^*(p)} = d_p^*(p)\left[1 - \frac{1}{\mathcal{E}^*(p)}\right]. \tag{14}$$

Because $d(p,z)$ is decreasing in p, (14) is negative whenever $\mathcal{E}^*(p) \geq 1$. $\square$

[2] This is a slight abuse of notation; however, the generic argument of f^* makes the evaluation path unambiguous.

The relationship between optimal protection level and high fare price is determined by two, typically opposing, effects. An increase in the high fare price increases the marginal return from protecting more for this class (hence leading to higher protection levels). On the other hand, increasing high-end prices leads to a lower rate of lost sales, and hence lower protection levels. The dominant effect determines the direction of change in $x^*(p)$. When the rate of lost sales is elastic with respect to changes in price along the optimal path (i.e. $\mathcal{E}^*(p) \geq 1$), the decrease in the rate of lost sales dominates the increase in marginal return, leading to lower protection levels. When demand is not a function of price, price changes have no impact on the rate of lost sales, hence the optimal protection level increases in p.

Proposition 2 provides necessary and sufficient conditions for the monotonicity of the optimal protection level in price. An alternative, sufficient condition requires monotonicity of the sales elasticity along the optimal decision path $x^*(p)$, a condition further discussed in Section 4.

Proposition 3. *If $\mathcal{E}^*(p)$ is increasing in p, then $x^*(p)$ decreases in p under Assumption B.*

The result is directly implied by the following lemma, via Proposition 2.

Lemma 1. *$\mathcal{E}^*(p_L) = 1$, where p_L was defined in Assumption B.*

Proof. Rewriting the elasticity definition (8) in terms of $\mathbf{Z}$, we obtain:

$$\mathcal{E}(p,x) = -\frac{\varphi(z(p,x))}{\bar{\Phi}\left(z(p,x)\right)} \frac{p d_p\left(p, z(p,x)\right)}{d_z\left(p, z(p,x)\right)}. \tag{15}$$

Using the critical fractile condition $z^*(p) = \bar{\Phi}^{-1}(1/p)$, we can now write $\mathcal{E}(p_L, x^*(p_L))$ as:

$$\mathcal{E}^*(p_L) = -\frac{\varphi^*\left(p_L\right)}{\bar{\Phi}^*\left(p_L\right)} \frac{p_L d_p^*\left(p_L\right)}{d_z^*\left(p_L\right)} = -\varphi^*\left(p_L\right) p_L^2 \frac{d_p^*\left(p_L\right)}{d_z^*\left(p_L\right)} = 1,$$

where the last equality follows from the definition of p_L in (9). $\square$

Our results so far require the evaluation of the elasticity along the optimal decision path $x^*(p)$. Stronger sufficient conditions for monotonicity of the protection level in price, which do not require any preliminary optimization, are presented next. These conditions amount to monotonicity of the sales elasticity in price, and are obtained via equivalent expressions for Littlewood's rule (10), in terms of the sales elasticity. The next section characterizes demand models with monotone sales elasticity.

PROPOSITION 4. *If $\mathcal{E}(p,x)$ is increasing in p, then $x^*(p)$ is decreasing in p under Assumption A or B.*

The proof of the proposition, provided below, relies on the following lemma:

LEMMA 2. *Under Assumption B, $p_L q(p_L, x^*(p)) \geq 1$.*

Proof. By definition, $p_L = \arg\max d(p, \bar{\Phi}^{-1}(1/p)) = \arg\max d(p, z^*(p)) = \arg\max x^*(p)$. Hence, $x^*(p) \leq x^*(p_L)$ for all $p \geq p_L$. Assumption B further implies that for all $p \geq p_L$, $q(p_L, x^*(p)) \geq q(p_L, x^*(p_L)) = \dfrac{1}{p_L}$. $\square$

Proof of of Proposition 4. We first show the result under Assumption B. The marginal revenue condition (10) states that the following expression, evaluated at $x^*(p)$, equals zero:

$$
\begin{aligned}
pq(p,x) - 1 &= \int_{p_L}^{p} \frac{\partial}{\partial v}(vq(v,x) - 1)dv + p_L q(p_L, x) - 1 \\
&= \int_{p_L}^{p} q(v,x)(1 - \mathcal{E}(v,x))dv + (p_L q(p_L, x) - 1)\,.
\end{aligned}
$$

By Lemma 2, the second term, evaluated at $x^*(p)$, is non-negative. So the first must be non-positive at $x^*(p)$, i.e. $\int_{p_L}^{p} Q(v, x^*(p))dv \leq 0$, where $Q(p,x) = q(p,x)(1 - \mathcal{E}(p,x))$. Because $\mathcal{E}(p,x)$ increases in p, and $q(p,x) \geq 0$, $Q(p,x)$ crosses zero at most once, and from above, as p increases. Therefore $Q(p, x^*(p)) \leq 0$, i.e. $\mathcal{E}^*(p) \geq 1$. The rest follows from Proposition 2.

Similarly, under Assumption A, because $q(1,x) = 1$, we can rewrite:

$$
pq(p,x) - 1 = \int_{1}^{p} \frac{\partial}{\partial v}(vq(v,x) - 1)dv = \int_{1}^{p} q(v,x)(1 - \mathcal{E}(v,x))dv.
$$

By the same argument, the first order condition $\displaystyle\int_{1}^{p} Q(v, x^*(p))dv = 0$ implies $Q(p, x^*(p)) \leq 0$, i.e. $\mathcal{E}^*(p) \geq 1$. $\square$

Remark that, by definition, the lower bound p_L set by Assumption B is the price that achieves the largest protection level $x^*(p)$ (see the proof of Lemma 2). Intuitively, this suggests that p_L is fairly low (as it must generate sufficiently ample high fare demand, to justify the largest protection level), so the lower bound set by Assumption B is not very restrictive. The next section calculates p_L for some common demand models, and provides conditions for these models to have monotone sales elasticity.

4. Demand Models and Elasticity Conditions

Our main results so far, and for the remaining of this paper are driven by monotonicity conditions, or bounds on the sales elasticity. We next identify a general class of demand models for which these conditions are met.

4.1. Conditions for Monotone Elasticity

Kocabıyıkoğlu and Popescu (2007) provide equivalent characterizations of the sales elasticity, as well as conditions under which $\mathcal{E}(p,x)$ is monotone in price p and/or quantity x. For completeness, we replicate here some of their general, easy to check conditions in terms of the riskless demand d, and risk distribution $\mathbf{Z}$. We verify these conditions for additive-multiplicative demand models. For a comprehensive set of conditions we refer the reader to Kocabıyıkoğlu and Popescu (2007).

PROPOSITION 5. *Suppose that $\mathbf{Z}$ has IFR. If $d_{zz} \leq 0$ and $d_{pz} \leq 0$ then $\mathcal{E}(p,x)$ increases in x. If moreover, $d_p + pd_{pp} \leq 0$, then $\mathcal{E}(p,x)$ increases in p.*

Proof. See Kocabıyıkoğlu and Popescu (2007). □

The increasing failure rate assumption on $\mathbf{Z}$ is common in the operations literature, especially in the context of the price setting newsvendor (see e.g. Zabel 1970, Young 1978, Lau and Lau 1988, Ha 2001). Distributions with logconcave survival functions, including Gamma, Uniform, Exponential, Normal and truncated Normal, have increasing failure rates. For more on increasing failure rate distributions, the reader is referred to Barlow and Proschan (1996) or Lariviere (2006).

The conditions on d assume that risk has diminishing marginal impact on riskless demand ($d_{zz} \leq 0$) and the marginal demand stimulation from lowering price is larger at higher risk levels ($d_{zp} \leq 0$). These two conditions are always met for additive-multiplicative demand models, $\mathbf{D}(p) = \alpha(p)\mathbf{Z}+\beta(p)$ (with α, β decreasing in p), so for these models $\mathcal{E}(p,x)$ is increasing in x. The additional condition for monotonicity of elasticity in price, $d_p + pd_{pp} \leq 0$, holds whenever d is concave in p, and also ensures the concavity of π. In particular, this condition amounts to $p\alpha'(p)$ and $p\beta'(p)$ being decreasing in p. These conditions are met for example if α, β are concave in p, in particular linear $(a - bp, a, b > 0)$. We obtain the first part of the following result.

Corollary 1. *Consider the demand model* $\mathbf{D}(p) = \alpha(p)\mathbf{Z} + \beta(p)$, *with* α, β *decreasing in* p.

(a) If $\mathbf{Z}$ *has IFR, then* $\mathcal{E}(p,x)$ *increases in* x. *If moreover* $p\alpha'(p)$ *and* $p\beta'(p)$ *decrease in* p, *then* $\mathcal{E}(p,x)$ *increases in* p.

(b) If $\mathbf{Z}$ *has IGFR and* $\beta(p) \equiv 0$, *then* $\mathcal{E}(p,x)$ *increases in* x. *If moreover* α *has increasing price elasticity, i.e.* $-p\alpha'(p)/\alpha(p)$ *is increasing in* p, *then* $\mathcal{E}(p,x)$ *increases in* p.

Part (b) is Corollary 1(b) from Kocabıyıkoğlu and Popescu (2007), and does not follow from Proposition 5. In particular, it requires milder conditions on the risk distribution $\mathbf{Z}$, and applies to iso-elastic $(ap^{-b}, a > 0, b > 1)$ multiplicative models, common in the literature. More general conditions, involving increasing generalized failure rate (IGFR) distributions, can be found in Kocabıyıkoğlu and Popescu (2007).

4.2. Conditions on $\mathcal{E}^*(p)$

Alternative sufficient conditions for several results in this paper are formulated in terms of monotonicity, or bounds on $\mathcal{E}^*(p)$. These condition are *a priori* problem-specific, because they require the computation of the optimal allocation $x^*(p)$, and the evaluation of $\mathcal{E}(p,x)$ along this path. In general, monotonicity of $\mathcal{E}(p,x)$ in p does not imply, nor is it implied by monotonicity of $\mathcal{E}^*(p)$. Nevertheless, for a large class of demand models, satisfying the conditions in the previous section, monotonicity of $\mathcal{E}^*(p)$ is guaranteed by the same conditions as monotonicity of $\mathcal{E}(p,x)$ in p.

Proposition 6. *The sufficient conditions of Proposition 5 and Corollary 1 that insure* $\mathcal{E}(p,x)$ *increasing in* p, *also imply* $\mathcal{E}^*(p)$ *increasing in* p.

Proof. From (15), we can write

$$\mathcal{E}^*(p) = -\frac{\varphi\left(z^*(p)\right)}{\bar{\Phi}\left(z^*(p)\right)} \frac{p d_p\left(p, z^*(p)\right)}{d_z\left(p, z^*(p)\right)}. \tag{16}$$

In particular, for $\mathbf{D}(p) = \alpha(p)\mathbf{Z} + \beta(p)$, we have $d_p(p, z^*(p)) = \alpha'(p)z^*(p) + \beta'(p)$ and $d_z(p, z^*(p)) = \alpha(p)$. The result follows from the conditions of the proposition, respectively corollary, and because $\dfrac{\partial z^*(p)}{\partial p} \geq 0$, from (11). $\square$

Table 2 characterizes $\mathcal{E}^*(p)$ for two common, specific forms of demand functions frequently used in the literature (for example in Petruzzi and Dada 1999): additive linear $(a - bp + \mathbf{Z}, a, b > 0)$ and multiplicative iso-elastic $(ap^{-b}\mathbf{Z}, a > 0, b > 1)$. The random component $\mathbf{Z}$ is either Uniform on $[0,1]$, or Exponential with $\lambda = 1$, both of which are IFR. For these models, $\mathcal{E}(p,x)$ is increasing in p and x: the additive linear model satisfies the conditions of Corollary 1(a), whereas the multiplicative iso-elastic model follows from Corollary 1(b).

Table 2. Evaluation of $\mathcal{E}^*(p)$ for special forms of demand $d(p, \mathbf{Z})$

	$a - bp + \mathbf{Z}$	$ap^{-b}\mathbf{Z}$
$\mathbf{Z} \sim Uniform(0,1)$	bp^2	$b(p-1)$
$\mathbf{Z} \sim Exponential(1)$	bp	$b \ln p$

Table 2 indicates that $\mathcal{E}^*(p)$ is increasing in p for all the models considered (this also follows from Proposition 6). Therefore, for these models, a uniform lower bound on the sales elasticity translates into a lower bound on the high-fare price. In particular, a unit lower bound $\mathcal{E}^*(p) \geq 1$, is actually equivalent to the condition $p \geq p_L$, set by Assumption B, because $\mathcal{E}^*(p_L) = 1$ (by Lemma 1). Table 3 obtains relatively simple expressions for $p_L = \arg\max d(p, \bar{\Phi}^{-1}(1/p))$; it is easy to check that indeed $\mathcal{E}^*(p_L) = 1$ for all models.

Table 3. Characterization of p_L for special forms of demand $d(p, \mathbf{Z})$

	$a - bp + \mathbf{Z}$	$ap^{-b}\mathbf{Z}$
$\mathbf{Z} \sim Uniform(0,1)$	$\frac{1}{\sqrt{b}}$	$\frac{1+b}{b}$
$\mathbf{Z} \sim Exponential(1)$	$\frac{1}{b}$	$\sqrt[b]{e}$

5. Joint Pricing and Revenue Management

In this section, we investigate the (PRM) model, which jointly optimizes expected revenue $R(p,x)$ as a function of price, p, and protection level, x, for high fare demand. We present conditions for a unique optimal price and protection level solution (p^{**}, x^{**}) for this problem, and provide sensitivity results with respect to capacity. We first consider the case where the market is perfectly segmented (e.g. by product fences), and then extend the results to allow for substitution effects.

5.1. Perfect Segmentation

In this section, we present three alternative conditions for the (PRM) model to admit a unique solution (p^{**}, x^{**}). A sufficient condition for a unique optimal price and allocation solution for the (PRM) model is the concavity of the expected revenue function in price, along the optimal protection level path, $R^*(p) = R(p, x^*(p))$. We first show that this is guaranteed by an elasticity bound of 1/2 along the optimal allocation path $x^*(p)$.

PROPOSITION 7. *If $\mathcal{E}^*(p) \geq 1/2$, the revenue corresponding to the optimal protection level, $R^*(p)$ is concave in p, and the pricing and revenue management model (PRM) has a unique optimal price-allocation solution.*

Proof. The proof is in Section EC.1 of the E-companion. □

This result allows us to solve Problem (PRM) as a one-dimensional concave optimization problem. Alternatively, the uniqueness of the optimal solution (p^{**}, x^{**}) is guaranteed by the monotonicity of the elasticity of the rate of lost sales in price.

PROPOSITION 8. *The following alternative conditions are sufficient for model (PRM) to have a unique optimal price-allocation solution:*

(a) $\mathcal{E}(p, x)$ is increasing in p, under Assumption A or B; or

(b) $\mathcal{E}^*(p)$ is increasing in p, under Assumption B.

Proof. For part (a), by Propositions 2 and 4, monotonicity of sales elasticity in p implies $\mathcal{E}^*(p) \geq 1$. This guarantees the elasticity bound required by Proposition 7, which completes the proof. Similarly, for part (b), by Propositions 2 and 3, monotonicity of $\mathcal{E}^*(p)$ in p guarantees the elasticity bound required by Proposition 7. □

5.2. Sensitivity Results

This section presents sensitivity results for the optimal revenue, high end price and protection level with respect to capacity. These results translate into sensitivity with respect to the load in the market, defined as the demand to capacity ratio. Clearly, optimal expected revenues increase with capacity. However, we show that optimal revenue per capacity unit decreases with capacity, i.e.

increases with load. Our results indicate that firms with more capacity should set higher protection levels, $x^{**}(C)$, and lower high end prices $p^{**}(C)$, if the sales elasticity is monotone in p and x. We also characterize how the optimal high-end price corresponding to a given protection level responds to changes in total capacity, and in its own allocation.

Let $R^*(C) = R(p^{**}, x^{**}; C)$, denote the optimal revenue obtained from the (PRM) model.

PROPOSITION 9. *The optimal revenue from the* (PRM) *model,* $R^*(C)$, *increases with capacity* C*, whereas the optimal revenue per unit of capacity,* $R^*(C)/C$, *decreases with* C.

Proof. The proof is in Section EC.2 of the E-companion. □

We now investigate the sensitivity of the optimal joint price-allocation solution with respect to capacity. By Littlewood's rule (10), for a given price, the optimal protection level $x^*(p; C) \equiv x^*(p)$ is independent of capacity, C. When the high end price is also optimized, the next result shows that the optimal protection level for the (PRM) problem, $x^{**}(C)$, increases with capacity C, provided that $\mathcal{E}(p,x)$ is monotone in p and x. The same conditions guarantee that higher capacity leads to lower optimal high-end prices, $p^{**}(C)$. Remark that, for a large class of demand models described in Proposition 5, if $\mathcal{E}(p,x)$ is increasing in p, then it is also increasing in x, so, practically, this assumption does not add additional modeling restrictions. In fact, a weaker sufficient condition for these results is $\mathcal{E}(p,x)$ increasing in x and $\mathcal{E}^*(p) \geq 1$, as indicated by the proof of the next proposition.

PROPOSITION 10. *Assume that* $\mathcal{E}(p,x)$ *is increasing in* p *and* x, *under Assumption A or B.*

(a) The optimal class 1 price for the (PRM) model, $p^{**}(C)$, *decreases with capacity* C.

(b) The optimal protection level for the (PRM) model, $x^{**}(C)$, *increases with capacity* C.

Proof. The proof is in Section EC.3 of the E-companion. □

We conclude this section by investigating the sensitivity of the optimal high end price, for a given protection level, with respect to changes in capacity and allocation. First observe that, in general, the revenue function R is concave in p, which allows us to uniquely define the optimal high-end price corresponding to a given protection level x,

$$p^*(x;C) = \operatorname{argmax}_p R(p,x;C). \tag{17}$$

LEMMA 3. *$R(p,x)$ is concave in p.*

Proof. From (1) and (2), we can write

$$\begin{aligned} R(p,x) &= \mathbb{E}\left[\min\left(\bar{\mathbf{D}}, C-x\right)\right] + p\mathbb{E}\left[\min(\mathbf{D}(p), \max\left(x, C-\bar{\mathbf{D}}\right))\right] \\ &= \mathbb{E}\left[\min\left(\bar{\mathbf{D}}, C-x\right)\right] + \mathbb{E}\left[\min(\pi(p,\mathbf{Z}), p\max\left(x,(C-\bar{\mathbf{D}})\right))\right]. \end{aligned}$$

The first term is not a function of p. Because the minimum of two concave functions is concave, concavity of $\pi(p,z)$ in p implies that the second term, hence $R(p,x)$, is concave in p. □

PROPOSITION 11. *Assume that $\mathcal{E}^*(x) \geq 1$.*

(a) For a given protection level x, the optimal class 1 price, $p^*(x;C)$, decreases with x.

(b) If moreover $\mathcal{E}(p,x)$ is increasing in x, then $p^*(x;C)$ decreases with the capacity C.

Proof. The proof is in Section EC.4 of the E-companion. □

The first result is the counterpart of Proposition 2, regarding price-sensitivity of the optimal protection level for a given price, $x^*(p;C)$. A unit upper bound on the sales elasticity along the optimal price path, $\mathcal{E}^*(x) \geq 1$, is not directly implied by monotone elasticity. The condition effectively translates into bounds on the protection level, by similar arguments to those presented in Section 4 for $\mathcal{E}^*(p) \geq 1$. However, the expression for these bounds is more complex, due to the relative difficulty of obtaining a closed form expression for $p^*(x)$.

5.3. Substitution Effects

The model described in Section 5.1 implicitly assumes that the market is perfectly segmented into low and high fare customers. Traditionally, airlines have achieved this segmentation between leisure and business customers by designing product fences (restrictions) such as booking less than 14 days prior to departure or not staying over a Saturday night. This allowed them to charge four to five times higher prices for the higher flexibility offered. In other practical settings, however, such as event ticketing, where perfect segmentation is more difficult to achieve, firms offer comparable products and consumers make choices based on price and product characteristics.

This section captures such substitution effects by modeling demand for each product as a function of the price of the other product as well. Keeping the low fare price fixed ($\bar{p}=1$ without loss of generality), high fare demand is given by $\mathbf{D}(p)=d(p,\mathbf{Z})$, and low fare demand by $\bar{\mathbf{D}}(p)=\bar{d}(p,\bar{\mathbf{Z}})$ where $\bar{\mathbf{Z}}$ is a random variable independent of $\mathbf{Z}$ and $\bar{d}(p,z)$ is increasing in p, capturing the substitution effect (changing the price of high fare moves the demand for the two products in opposite directions). In this case, the joint price-allocation problem is formulated as follows:

$$\max_{p,x} R(p,x)=\mathbb{E}\left[\min\left(\bar{\mathbf{D}}(p),C-x\right)\right]+\mathbb{E}\left[r\left(p,\max\left(x,C-\bar{\mathbf{D}}(p)\right)\right)\right], \tag{18}$$

where recall that the revenue from the high-end class, $r(p,x)$, is given by (2).

Assuming diminishing marginal impact of substitute prices on low fare demand, monotone elasticity implies the existence of a unique optimal price-allocation solution for this problem.

Proposition 12. *Assume that* $\bar{d}_{pp}\le 0$. *Then, under the following alternative conditions, Problem (18) has a unique price-allocation solution:*

(a) $\mathcal{E}(p,x)$ *is increasing in* p *under Assumption A or B; or*

(b) $\mathcal{E}^*(p)$ *is increasing in* p *under Assumption B.*

Proof. The proof is in Section EC.5 of the E-companion. □

The additional technical assumption on low fare demand holds, for example, for linear additive demand models $\mathbf{D}(p,\bar{p})=\mathbf{Z}-b_1p+a_1\bar{p}$ and $\bar{\mathbf{D}}(\bar{p},p)=\bar{\mathbf{Z}}-b_2\bar{p}+a_2p$, as well as for iso-elastic multiplicative demand models $\mathbf{D}(p,\bar{p})=p^{-b_1}\bar{p}^{a_1}\mathbf{Z}$ and $\bar{\mathbf{D}}(\bar{p},p)=p^{a_2}\bar{p}^{-b_2}\bar{\mathbf{Z}}$, where $a_i,b_i\ge 0, i=1,2$. When $\mathbf{Z}$ has increasing failure rate, these models also satisfy the monotone elasticity condition of Proposition 12, based on the results of Section 4. Clearly, $d_{zz},d_{pz}\le 0$ for both models, and $d_p+pd_{pp}\le 0$ for the linear model. Monotonicity of $\mathcal{E}(p,x)$ in price for the iso-elastic demand model follows by Corollary 1(b).

6. Partitioned Allocation Model

This section considers a Partitioned Allocation model (PA), where capacity is divided into blocks which can only be sold to a designated market segment. In contrast with the standard revenue management model, this model allows no resource substitution, i.e. resources that are not utilized by one class are not available to other (higher-ranked) classes. Belobaba (1987) considers this problem as a base case for the standard ($EMSR$) model, in an airline context. Bertsimas and de Boer (2005) use this model as an approximation for a multi-period pricing and revenue management problem. In a similar spirit, Section 7 uses the (PA) model as a benchmark and input heuristic for the joint pricing and revenue management model (PRM) studied in the previous section.

The (PA) model is relevant, for example, in deciding the amount of aircraft capacity dedicated for business and economy cabins, and respective prices. Such models are also directly relevant in flexible manufacturing settings (see e.g. Chod and Rudi 2004, Fine and Freund 1990), where firms design production processes that postpone product differentiation, in order to better respond to market conditions. Hence, at the product differentiation stage, inventory is limited by long production lead-times. Benetton, for example, uses the same material for all of its knitted sweaters, which is not dyed to a particular color until the very last stage of the process, in order to better respond to market conditions. At that time, however, the amount of available resources is limited, because of production lead times.

In the partitioned allocation model, the firm's objective is to maximize expected revenues with respect to both products' prices $p, \bar{p}$ and resource allocation between products, x:

$$R[PA] = \max_{p,\bar{p},x} V(p,\bar{p},x) = p\mathbb{E}\left[\min\left(\mathbf{D}(p), x\right)\right] + \bar{p}\mathbb{E}\left[\min\left(\bar{\mathbf{D}}(\bar{p}), C - x\right)\right]. \tag{PA}$$

The expected revenue for this problem can be written as

$$V(p,\bar{p},x) = r(p,x) + \bar{r}(\bar{p},x), \tag{19}$$

where $r(p,x) = p\mathbb{E}\left[\min\left(\mathbf{D}(p), x\right)\right]$ and $\bar{r}(\bar{p},x) = \bar{p}\mathbb{E}\left[\min\left(\bar{\mathbf{D}}(\bar{p}), C - x\right)\right]$ are the expected revenues for each product.

We first provide conditions for the (PA) model to admit a unique solution. Then, we compare the optimal decisions obtained from this model with those of the (PRM) model.

6.1. Conditions for Unique Optimum

The next result shows that the (PA) model has a unique solution and can be solved efficiently, provided that for both demand classes, $\mathbf{D}(p)=d(p,\mathbf{Z})$ and $\bar{\mathbf{D}}(\bar{p})=\bar{d}(\bar{p},\bar{\mathbf{Z}})$, sales elasticities $\mathcal{E}(p,x)$, respectively $\bar{\mathcal{E}}(\bar{p},x)$, are increasing in x. Conditions for elasticity to be monotone in x were discussed in Section 4, and are usually weaker than those for price-monotone elasticity required by the (PRM) model (see e.g. Proposition 5). These conditions also guarantee the monotonicity of the optimal price for each product in its own allocation. Note that, we do not make any assumption regarding dependence of the risk distributions $\mathbf{Z}$ and $\bar{\mathbf{Z}}$, thereby allowing for correlations in the demands for the two products.

Proposition 13. *Assume that $\mathcal{E}(p,x)$ and $\bar{\mathcal{E}}(\bar{p},x)$ are increasing in x.*

(a) Under model (PA), the optimal price for each product, keeping all other variables constant, is decreasing in its own allocation and independent of the other product's price.

(b) The (PA) model has a unique optimal price-allocation solution $(x^P, p^P, \bar{p}^P)$ satisfying

$$\int_0^x q(p,v)(1-\mathcal{E}(p,v))dv=0, \qquad \int_0^{C-x} \bar{q}(\bar{p},v)(1-\bar{\mathcal{E}}(\bar{p},v))dv=0, \tag{20}$$

and the marginal revenue condition

$$pq(p,x)=\bar{p}\bar{q}(\bar{p},x). \tag{21}$$

Proof. The proof is in Section EC.6 of the E-companion. □

6.2. Comparison of Optimal Decisions and Revenues

In this section, we provide local comparisons of the price and allocation decisions produced by the (PRM) and (PA) models; we also argue that global comparisons are generally not possible. For the same price levels, we show that the (PA) model allocates more capacity to the high fare class than the protection level set by the (PRM) model. This is because the (PA) model limits sales to the high fare class, whereas the (PRM) model does not. Further, for the same allocation and low fare price, the high-end price set by the (PA) model exceeds that set by the (PRM) model, under monotone elasticity. Hence, resource substitution leads to lower prices for the high end segment. It also leads to higher expected revenues.

Proposition 14. *The (PA) model allocates a greater portion of capacity to the high-fare class than the protection level set by the (PRM) model, for the same price levels p and $\bar{p}$.*

Proof. The proof is in Section EC.7 of the E-companion. □

The next result compares the optimal prices obtained from the (PA) and (PRM) models.

Proposition 15. *Assume that $\mathcal{E}(p,x)$ is increasing in x. Then the optimal class 1 price set by the (PA) model is higher than the optimal class 1 price set by the (PRM) model, for the same allocation x, and class 2 price $\bar{p}$.*

Proof. The proof is in Section EC.8 of the E-companion. □

Our numerical results in Section 8 (Figures 2 and 3) suggest that these relationships remain valid for the optimal price-allocation pairs $(p^{**},x^{**}),(p^P,x^P)$, set by the (PRM) and the (PA) models, for a given class 2 price $\bar{p}$. Specifically, we observe numerically that $x^P \geq x^{**}$ and $p^P \leq p^{**}$. Finally, it is easy to see that the (PRM) model yields higher optimal expected revenues than the (PA) model, because $V(p,\bar{p},x) \leq R(p,\bar{p},x)$. This follows because the (PRM) model achieves higher expected revenues from the high-end segment, by allowing resource substitution. The next section provides bounds on the revenues generated by the (PRM) model, based on various heuristics for setting the low fare price.

7. Heuristics and Bounds

This section provides heuristics and bounds illustrating the value of combining pricing and capacity allocation decisions in the revenue management context. We propose two-stage price-allocation heuristics to compute the unique solution of the Pricing and Revenue Management model. These heuristics provide analytical bounds on optimal revenues. Two models are used in the first stage to obtain initial prices, and then combined with $(EMSR)$ and respectively (PRM) to compute corresponding optimal revenues. These two price-setting models are (1) the partitioned allocation model (PA), described in Section 6, which divides the available capacity into blocks that can only be sold to the designated class, and (2) the deterministic pricing model (DM), a certainty equivalent approximation which replaces random demands with their means.

7.1. Partitioned Allocation Model

In this section, we use the price input from the (PA) model, described in Section 6, to optimize models $(EMSR)$, respectively (PRM). This leads to the following two heuristics:

$(EMSR-PA)$

(1) Solve (PA) to obtain prices for both classes p^P and $\bar{p}^P$;

(2) Solve $R[EMSR-PA] = \max_x R(p^P, \bar{p}^P, x)$;

(3) Obtain the optimal protection level.

$(PRM-PA)$

(1) Solve (PA) to obtain class 2 price $\bar{p}^P$;

(2) Solve $R[PRM-PA] = \max_{x,p} R(p, \bar{p}^P, x)$;

(3) Obtain the optimal price and protection level for class 1.

The next result shows that the expected revenue obtained from the $(PRM-PA)$ heuristic is higher than expected revenues of the $(EMSR-PA)$ heuristic, and (PA) model.

PROPOSITION 16. $R[PRM-PA] \geq R[EMSR-PA] \geq R[PA]$.

Proof. The first inequality is obvious. We have $R[PA] = \mathbb{E}[r(p^P, x^P)] + \mathbb{E}[\bar{r}(\bar{p}^P, x^P)]$. Because $r(p, x)$ is increasing in x, we obtain,

$$\begin{aligned} R[PA] &\leq \mathbb{E}[r(p^P, \max(C - \bar{\mathbf{D}}(\bar{p}^P), x^P))] + \mathbb{E}[\bar{r}(\bar{p}^P, x^P)] = R(p^P, \bar{p}^P, x^P) \\ &\leq \max_x R(p^P, \bar{p}^P, x) = R[EMSR-PA]. \end{aligned}$$

□

An upper bound for $R[PRM-PA]$ can be obtained by solving the following Infinite Capacity Model (IC), which sets optimal unconstrained prices for each segment. Random demands are replaced by their expectations, but without imposing capacity constraints:

$$R[IC] = \max_{p, \bar{p}} pD(p) + \bar{p}\bar{D}(\bar{p}).$$

PROPOSITION 17. $R[PRM-PA] \leq R[IC]$.

Proof. Let p^*, x^* denote the optimal price-capacity allocation pair obtained from the $(PRM - PA)$ heuristic. From Jensen's inequality, $R[PRM - PA] \leq \hat{R}(p^*, \bar{p}^P, x^*)$. Moreover,

$$\begin{aligned}\hat{R}(p^*, \bar{p}^P, x^*) &\leq p^* D(p^*) + \bar{p}^P \bar{D}(\bar{p}^P)\\ &\leq \max_{p,\bar{p}} pD(p) + \bar{p}\bar{D}(\bar{p}) = R[IC].\end{aligned}$$

□

7.2. Deterministic Pricing Model

Consider now the Deterministic Model (DM), where random demands are replaced by their expectations, and profits are optimized subject to the capacity constraint. This can be interpreted as a perfect information model, where prices and allocations are decided after the uncertainty is realized. This standard approximation is commonly used to provide bounds for the stochastic problem (see e.g. Gallego and van Ryzin 1994 and Bitran and Caldentey 2003). Let $D(p)$ and $\bar{D}(\bar{p})$ denote expected high and low fare demand:

$$\begin{aligned} R[DM] = \max_{p,\bar{p}} \ & pD(p) + \bar{p}\bar{D}(\bar{p}) \\ \text{s.t.} \ & D(p) + \bar{D}(\bar{p}) \leq C. \end{aligned} \tag{DM}$$

We combine the (DM) model with $(EMSR)$ and (PRM) modules, to obtain the $(EMSR - DM)$ and $(PRM - DM)$ heuristics, respectively. These are the analogues of heuristics $(EMSR - PA)$ and $(PRM - PA)$ described in the previous section. The pricing module (DM) replaces (PA), and produces optimal prices p^D and $\bar{p}^D$. The corresponding optimal heuristic values are denoted by $R[EMSR - DM]$ and $R[PRM - DM]$.

We show that, when initial prices are given by the deterministic model, the optimal expected revenue under the (PRM) model is bounded between that given by the $(EMSR)$ model, and the deterministic model revenue. That is, approximating the randomness with the expectation overestimates optimal revenue.

Proposition 18. $R[DM] \geq R[PRM - DM] \geq R[EMSR - DM]$.

Proof. The second inequality is obvious. To show the first, consider a heuristic approach to (PRM), where random demands are replaced by their corresponding means:

$$\hat{R}(p,\bar{p}^D,x) = p\left[\min\left(D(p),\max\left(C-\bar{D}(\bar{p}^D),x\right)\right)\right] + \bar{p}\left[\min\left(\bar{D}(\bar{p}^D),C-x\right)\right],$$

and class 2 price $\bar{p}^D$ is obtained from (DM). Let p^*, x^* denote the optimal price-capacity allocation pair obtained from the $(PRM-DM)$ heuristic. From Jensen's inequality:

$$R[PRM-DM] \le \hat{R}(p^*,\bar{p}^D,x^*) \le \max_{p,x} \hat{R}(p,\bar{p}^D,x) = \hat{R}^*. \tag{22}$$

We now show that $\hat{R}^* = R[DM]$. Let q and $\bar{q}$ denote the number of units actually sold to high and low fare class, respectively, i.e. $q = \left[\min\left(D(p),\max\left(C-\bar{D}(\bar{p}^D),x\right)\right)\right]$ and $\bar{q} = \left[\min\left(\bar{D}(\bar{p}^D),C-x\right)\right]$. Hence, we can rewrite $\hat{R}$ as:

$$\begin{aligned}
\max_{p,q,\bar{q}} \ & \hat{R}(q,\bar{q},p,\bar{p}^D) = pq + \bar{p}^D\bar{q} \\
\text{s.t.} \ & q \le D(p),\ \bar{q} \le \bar{D}(\bar{p}^D) \\
& q + \bar{q} \le C \\
& \bar{q} \le C - x.
\end{aligned}$$

The variable x and the last constraint are redundant, so we obtain problem (DM). $\square$

8. Numerical Results

In this section, we assess, via numerical experiments, the benefit of integrating pricing and allocation decisions in the revenue management context compared to a hierarchical decision process that is common in the industry and literature. We demonstrate significant improvements in revenues relative to the congestion in the market, and demand parameters such as location, scale and variability when price and allocation decisions are coordinated.

Throughout this section, we use an additive-linear demand model: $\mathbf{D}(p) = a - bp + \mathbf{Z}, \bar{\mathbf{D}}(p) = \bar{a} - \bar{b}\bar{p} + \bar{\mathbf{Z}}$. Under this model, $\mathcal{E}(p,x)$ is increasing in p and x, by Corollary 1(a). Furthermore, $\pi(p,z)$ is strictly concave in p. Demand risks $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ have independent Gamma distributions; the Gamma distribution with parameters $\{\rho,\lambda\}$ has logconcave density function given by:

$$\phi(z) = \frac{1}{\Gamma(\rho)}\lambda^\rho z^{\rho-1} e^{-\lambda z}.$$

This choice is motivated by Beckman and Bobkowski (1958), who offer evidence that, among a large set distributions, the Gamma distribution provides the most reasonable fit for airline data (one of the main application areas of revenue management). We use two special cases of the Gamma distribution, namely Exponential ($\rho = 1$), and Erlang (ρ is a positive integer), both of which have explicit expressions for the distribution function.

Because we did not have real data, we generated the required parameters. The capacity of the system, C, was set as the ratio of expected demand to the load in the market, denoted by L. We let L vary between 0.5 and 4. We use the optimal unconstrained prices, i.e. those obtained from the Infinite Capacity Model (IC) of Section 7.1, to obtain C, as

$$C = \frac{D(p^{\infty}) + \bar{D}(\bar{p}^{\infty})}{L}. \tag{23}$$

The low fare price $\bar{p}$ is obtained from either (PA) or (DM) models.

In the following, we compare the revenue, price and capacity allocation obtained from the standard ($EMSR$) model with (PRM), using the heuristics introduced in the previous sections. Specifically, we compare the revenue and capacity allocation obtained from the following three pairs of heuristics: (1) ($EMSR - DM$) and ($PRM - DM$), (2) ($EMSR - PA$) and ($PRM - PA$), (3) (PA) and ($PRM - PA$), with respect to demand and supply parameters. We also compare the optimal high fare price obtained from ($PRM - PA$) and ($PRM - DM$) heuristics.

The relative improvement in revenue from using model (PRM) vs. an alternative model Y (either ($EMSR$) or (PA)), with a pricing heuristic X (either (DM) or (PA)), is denoted

$$\Delta_X^Y = \frac{R[PRM - X] - R[Y - X]}{R[Y - X]}.$$

8.1. Load in the Market

Let $L = \frac{D(p^{\infty}) + \bar{D}(\bar{p}^{\infty})}{C}$ parametrize the average load in the market, where $p^{\infty}, \bar{p}^{\infty}$ are the optimal unconstrained, deterministic prices, i.e. the solutions of $R[IC]$. The actual load faced by the firm depends on the prices it sets and the actual realization of market risk. The latter is obviously determined by the optimal price and allocation decisions. The results in this section are obtained by varying the capacity of the system, C. Demand parameters are $a = 20, b = 0.01$. $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ are exponentially distributed, with $\lambda = 1$.

Table 4. Revenue improvement with respect to load

L	0.3	0.5	0.7	0.9	1	2	3	4
Δ_{DM}^{EMSR}	3.82%	4.72%	5.28%	8.91%	22.59%	23.49%	25.84%	27.01%
Δ_{PA}^{EMSR}	2.71%	3.13%	3.92%	7.11%	17.52%	18.47%	20.47%	21.84%
Δ_{PA}^{PA}	1.91%	2.36%	2.84%	5.86%	10.29%	12.96%	14.58%	17.18%

Table 4 suggests that improvement in expected revenue increases monotonically with load, with gains over existing heuristics exceeding 10% when there is excess demand in the market. The relative change in the improvement tends to decrease as the load increases, suggesting when the congestion in the market is too high, there is less room for improvement. The improvement in the optimal revenue is higher when prices are obtained from the deterministic pricing model (DM), which suggests that the differentiated product model (PA) provides a better approximation for optimal prices.

Figure 1 plots the optimal revenues per unit of capacity obtained from the $(PRM - DM)$ and $(PRM - PA)$ heuristics with respect to the load in the market. The bounds proved in the previous sections are apparent. In this particular example, the (DM) model provides an upper bound for the optimal $(PRM - PA)$ revenue, but this is not universally true (see Figure 9). The optimal revenue per capacity unit from all heuristics increases monotonically in load, illustrating the analytical result proved in Proposition 9. Figure 2 confirms that the optimal high end price increases with load, as predicted by Proposition 10. The assumptions of Proposition 10 are met by the demand function, $\mathbf{D}(p) = 20 - 0.01p + \mathbf{Z}$, used in this section; $\mathcal{E}(p, x) = 0.01p \geq 1$ is increasing in p and independent of x.

8.2. Demand Variability

We next investigate the improvement in optimal revenue, and the change in optimal prices with respect to demand variability, measured by the coefficients of variation (CV) of $\mathbf{Z}$ and $\bar{\mathbf{Z}}$. In this section, $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ have Erlang distributions, and the results are obtained by varying the

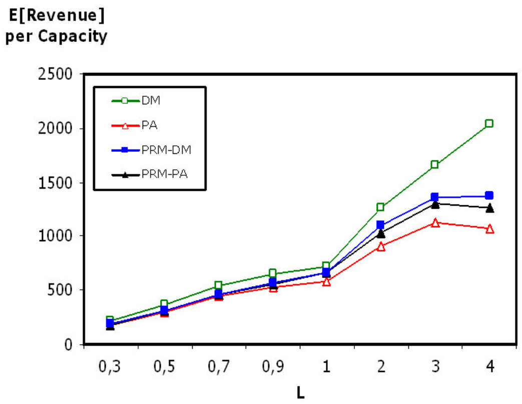

Figure 1 **Revenue vs. load**

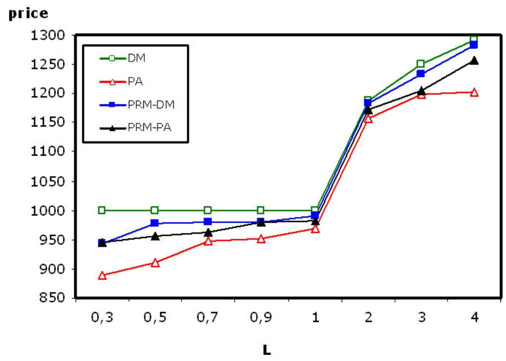

Figure 2 **Class 1 price vs. load**

parameters of the distribution, while everything else is kept constant ($a = 20, b = 0.03, L = 1$). In particular, ρ controls the coefficient of variation of the distribution. Because $\mathbb{E}[\mathbf{Z}] = \frac{\lambda}{\rho}$, λ was varied simultaneously to keep the mean constant.

Table 5 shows the improvement in expected revenue when the CV of $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ were varied together (controlling overall variability in the system). In Table 6, variability in the system comes from class 1, with $CV(\bar{\mathbf{Z}})$ kept constant at 0.33. In Table 7, $CV(\bar{\mathbf{Z}})$ varies between 0.25 and 1, with $CV(\mathbf{Z}) = 0.33$.[3] The improvement in expected revenue is significant, and increases with demand variability. The effect of class 1 variability is stronger.

Table 5. Revenue improvement with respect to overall demand variability

CV	0.25	0.29	0.33	0.37	0.5	0.7	1
Δ_{DM}^{EMSR}	19.56%	19.92%	20.22%	23.47%	25.91%	28.37%	31.41%
Δ_{PA}^{EMSR}	18.03%	18.72%	19.28%	21.49%	23.39%	23.84%	25.03%
Δ_{PA}^{PA}	17.90%	17.93%	18.20%	18.95%	19.47%	19.21%	19.52%

Table 6. Revenue improvement with respect to variability in class 1 demand

$CV(\mathbf{Z})$	0.25	0.29	0.33	0.37	0.5	0.7	1
Δ_{DM}^{EMSR}	21.35%	21.15%	22.34%	24.71%	26.51%	29.79%	30.01%
Δ_{PA}^{EMSR}	20.27%	20.36%	20.42%	21.26%	24.37%	24.52%	25.12%
Δ_{PA}^{PA}	18.12%	18.43%	18.85%	19.23%	20.47%	20.86%	21.12%

Table 7. Revenue improvement with respect to variability in class 2 demand

[3] The evaluation points are determined by admissible values of ρ (positive integers for the Erlang distribution).

$CV(\mathbf{Z})$	0.25	0.29	0.33	0.37	0.5	0.7	1
Δ_{DM}^{EMSR}	15.25%	15.28%	15.64%	16.24%	16.67%	17.29%	17.84%
Δ_{PA}^{EMSR}	12.26%	12.85%	13.84%	15.49%	15.70%	15.87%	16.23%
Δ_{PA}^{PA}	10.21%	10.55%	12.47%	14.23%	14.38%	14.79%	15.63%

Figures 3, 5 and 7 suggest that optimal revenues decrease with demand variability, as the system becomes more difficult to control. None of the (PRM) based heuristics dominates the other, but both dominate (PA). In this case, a constant upper bound is provided by the optimal revenue from the deterministic heuristic (DM).

Figures 4, 6 and 8 suggest that in general, $(PRM - DM)$ sets higher prices than $(PRM - PA)$, but this is not systematic (see Figure 8). The class 1 price tends to decrease with demand variability, suggesting that a discount strategy hedges against market risk.

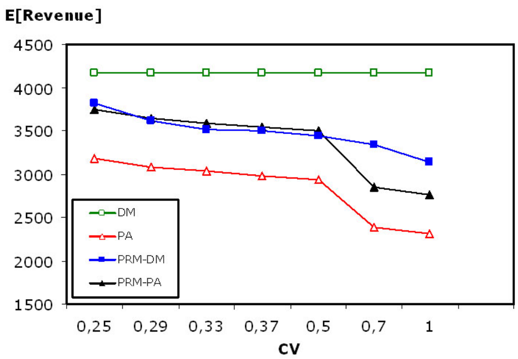

Figure 3 Revenue vs. demand variability

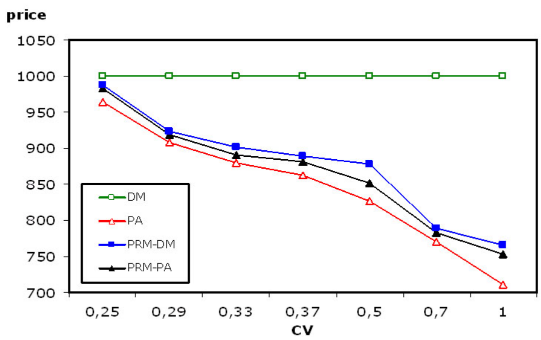

Figure 4 Class 1 price vs. demand variability

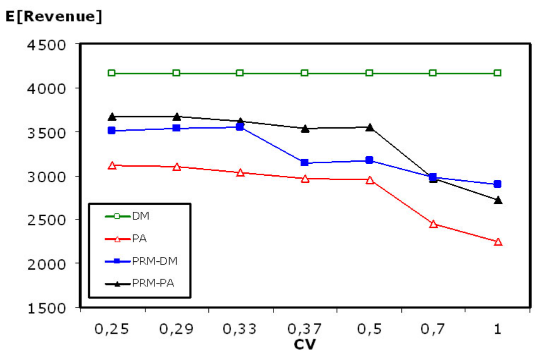

Figure 5 Revenue vs. class 1 variability

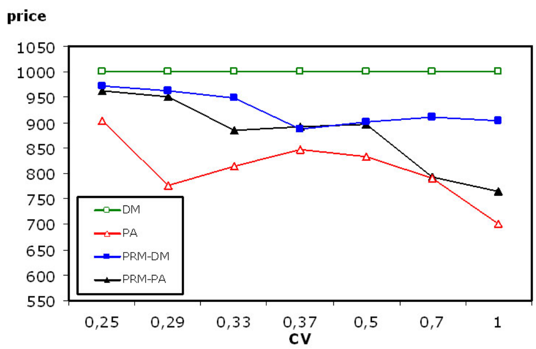

Figure 6 Class 1 price vs. class 1 variability

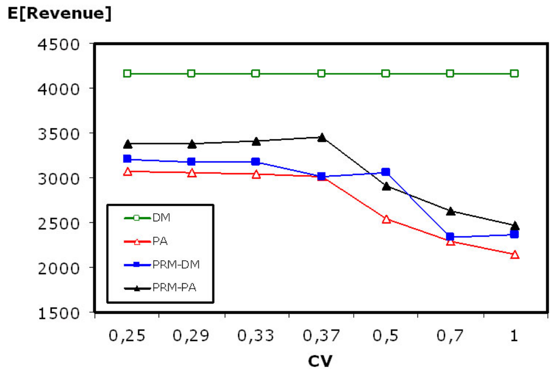

Figure 7 Revenue vs. class 2 variability

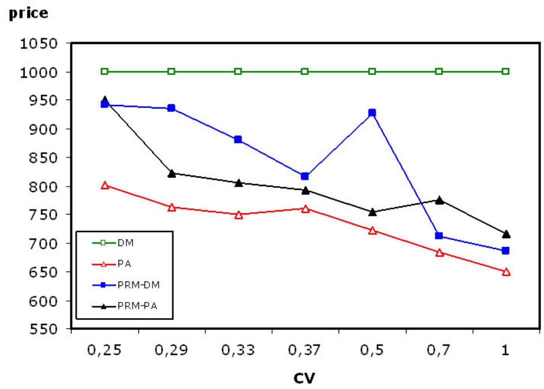

Figure 8 Class 1 price vs. class 2 variability

8.3. Demand Slope

With an additive model $\mathbf{D}(p) = a - bp + \mathbf{Z}$, price influences the mean of the demand distribution, without affecting variability. In this section, we vary the market parameter b between 0.01 and 0.1,[4] and keep L, $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ constant, to control for the effects of congestion and demand variability in the market $(a = 20, \lambda = 2, L = 1)$.

Table 8. Revenue improvement with respect to demand slope

b	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
Δ_{DM}^{EMSR}	22.1%	22.4%	22.8%	23.3%	23.8%	23.5%	21.3%	20.3%	19.2%	16.2%
Δ_{PA}^{EMSR}	17.0%	18.4%	18.9%	20.4%	20.1%	19.2%	18.0%	16.4%	16.4%	15.2%
Δ_{PA}^{PA}	15.2%	15.4%	16.4%	16.3%	16.2%	14.5%	14.2%	13.5%	12.7%	11.2%

The results for the three cases presented in Table 8 are relatively similar. The relative improvements are very important; and appear unimodal. As b increases, the mean of the demand distribution decreases, which can be interpreted as a decrease in the congestion of the system. Hence these results are consistent with those of Section 8.1.

A key observation from Figure 9 is that $R[PA]$ is not always a lower bound for $R[PRM-DM]$, and $R[DM]$ is not necessarily an upper bound for $R[PRM-PA]$. Figure 10 confirms our insight from previous sections that $(PRM-DM)$ prices typically exceed $(PRM-PA)$. Optimal revenues and prices decrease in b, which controls the elasticity of riskless demand.

Our main findings from these numerical experiments can be summarized as follows:

[4] This range of values were chosen to have a large enough set of feasible prices ($p_{min} \leq p \leq a/b$).

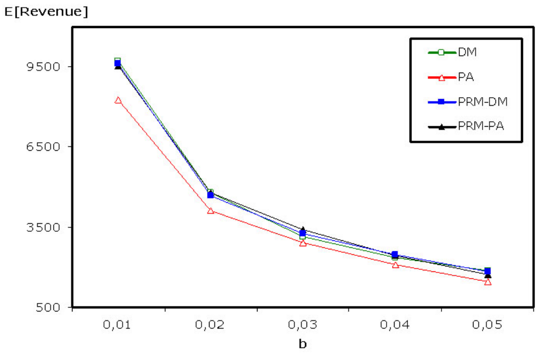

Figure 9 Revenue vs. demand slope

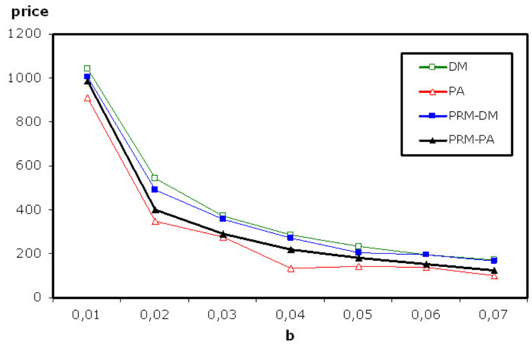

Figure 10 Class 1 price vs. demand slope

1. The value of price optimization is significant, and particularly for high load scenarios (exceeding 17% for loads larger than 1), and for higher demand variability.

2. The optimal high end price increases with load.

3. Demand variability (from both classes) typically decreases expected revenues and prices, but has no clear impact on allocation.

4. It is not conclusive which of the pricing heuristics (DM) or (PA) yields higher revenues when used as input to (PRM). Hence, we suggest that (DM) may be preferable, given its simplicity.

5. The lower and upper revenue bounds of Propositions 18 and 16, given respectively by the (PA) and (DM) models for the corresponding (PRM) based heuristics, are contingent on the pricing heuristic used, i.e. do not apply for both heuristics. For example, one cannot infer that $R[PRM - PA] \geq R[EMSR - DM]$.

6. The $(PRM - DM)$ heuristic typically sets a higher price than $(PRM - PA)$.

Although the numerical results presented here correspond to specific demand models and parameters, our extensive numerical experiments suggest that these insights are robust.

9. Conclusions

This paper extends the traditional revenue management framework to simultaneously optimize high end price and capacity allocation. In this context, we model high end demand as a stochastic function of price, and introduce a new elasticity measure, the elasticity of the rate of lost sales. Structural results are guaranteed by monotonicity properties and bounds on this elasticity measure. General demand models that satisfy these conditions were identified in Kocabıyıkoğlu and Popescu

(2007), and include additive-multiplicative models with linear and iso-elastic price-dependence. Sufficient and easy to verify conditions are presented in Section 4.

In the first part of the paper, we investigated the impact of changes in the high end price on the optimal capacity allocation policy, when demand is price dependent. We identified two, typically opposing effects, which drive this relationship: an increase in high end prices corresponds to higher marginal revenues, but a lower rate of lost sales. We showed that when the rate of lost sales for high-end demand is elastic with respect to price (along the optimal allocation path), the decrease in the rate of lost sales dominates the increase in marginal returns, hence the optimal protection level is decreasing in the high-end price. This is true in particular, if the elasticity of the rate of lost sales is increasing in price.

Monotonicity of $\mathcal{E}(p, x)$ in p also guarantees uniqueness of the optimal solution for the joint pricing and allocation model (PRM). Alternatively, this is also guaranteed by a lower bound of $1/2$ on sales elasticity. These results extend to handle substitution effects. Monotone elasticity conditions also allow to obtain sensitivity results for the optimal revenue rate and price-allocation decision with respect to capacity and load.

Similar results are obtained for a variation of the (PRM) problem, the Partitioned Allocation (PA) model, which does not allow for resource substitution. We showed that this problem has a unique solution if $\mathcal{E}(p, x)$ is monotone in x, typically a weaker condition than monotonicity in p. The same condition guarantees monotonicity of the optimal (PA) prices in their respective allocations, and ensures that the optimal high end price set by the (PA) model is higher than that of the (PRM) model, for a given capacity allocation.

We compared several joint pricing and allocation heuristics that use initial pricing decisions from the (PA) model, as well as a Deterministic Model (DM), as inputs to the (PRM) and $(EMSR)$ models. These heuristics provided lower and upper bounds on the optimal revenue obtained from $(PRM)-$based models. In particular, (PRM) heuristics outperform corresponding $(EMSR)$ heuristics. Approximating random demand with its expectation overestimates the expected revenue achievable from a (PRM) model, whereas the (PA) model underestimates it.

Our numerical experiments demonstrated a significant added value from coordinating pricing and allocation decisions. Revenue improvements as high as 10-20% were observed with a load factor close to 1. The value of price optimization increased with both load and demand uncertainty.

This work focused on the joint optimization of the price and capacity allocation for the high-end market, in a static revenue management setting. Although we provided several heuristics, low-end prices were not optimized. Our modeling choice was practically motivated by the argument that, for most revenue management industries, the firm is either a price taker in the low end market, or low end prices are not actively managed for fairness and image reasons. From a technical standpoint, preliminary work suggests that the general demand conditions used in this paper may not be sufficient to simultaneously optimize both prices, together with the protection level. However, we expect that similar results may be obtained under more restrictive demand assumptions.

In general, we expect that monotone elasticity conditions can be useful in modeling general stochastic demand in revenue management settings, in particular in extending the current results to multiple customer classes, dynamic and competitive settings.

Acknowledgments

We thank James Dana, Guillaume Roels, Garrett van Ryzin and Kalyan Talluri for insightful comments on an earlier version of this paper. The second author gratefully acknowledges research funding from the Booz & Co. Chair in Strategic Revenue Management at INSEAD.

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Selected Proofs

EC.1. Proof of Proposition 7.

To show $R^*(p)$ is concave in p, we first show that $r_p^*(p)$ is decreasing in p if $\mathcal{E}^*(p) \geq 1/2$. From (5),

$$r_p^*(p) = x^*(p)q^*(p) + \mathbb{E}[\pi_p(p,\mathbf{Z});\mathbf{\Omega}^*] = \frac{x^*(p)}{p} + \mathbb{E}[\pi_p(p,\mathbf{Z});\mathbf{\Omega}^*].$$

Differentiating with respect to p, we obtain:

$$\frac{\partial r_p^*(p)}{\partial p} = \frac{1}{p}\frac{\partial x^*(p)}{\partial p} - \frac{x^*(p)}{p^2} + \mathbb{E}[\pi_{pp}(p,\mathbf{Z});\mathbf{\Omega}^*] + \left(d^*(p) + pd_p^*(p)\right)\phi^*(p)\frac{\partial z^*(p)}{\partial p}.$$

Writing $F_p(p,x) = \frac{\partial}{\partial p}P(d(p,\mathbf{Z}) \geq x) = -f(p,x)d_p(p,z(p,x))$ in the elasticity definition, we obtain $\mathcal{E}(p,x) = \dfrac{-pf(p,x)d_p(p,z(p,x))}{q(p,x)}$. Using this expression, equations (11), (14) and $d^*(p) = d(p,z^*(p)) = x^*(p)$, we obtain

$$\frac{\partial r_p^*(p)}{\partial p} = \frac{d_p^*(p)}{p}\left[2 - \frac{1}{\mathcal{E}^*(p)}\right] + \mathbb{E}[\pi_{pp}(p,\mathbf{Z});\mathbf{\Omega}^*]. \tag{EC.1}$$

This is negative because π is concave, $d_p^*(p) \leq 0$ and $\mathcal{E}^*(p) \geq 1/2$, hence $r_p^*(p)$ is decreasing in p.

We next establish concavity of $R^*(p)$. By the envelope theorem,

$$\frac{\partial R^*(p)}{\partial p} = R_p^*(p) = R_p(p,x)|_{x=x^*(p)} = \bar{q}^*(p)r_p^*(p) + \mathbb{E}[r_p(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}^*].$$

To show that this is decreasing in p, we differentiate with respect to p, obtaining:

$$\frac{\partial R_p^*(p)}{\partial p} = \frac{\partial \bar{q}^*(p)}{\partial p}r_p^*(p) + \bar{q}^*(p)\frac{\partial r_p^*(p)}{\partial p} + \mathbb{E}[r_{pp}(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}^*] - r_p^*(p)\bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}. \tag{EC.2}$$

Because $\dfrac{\partial \bar{q}^*(p)}{\partial p} = \bar{f}^*(p)\dfrac{\partial x^*(p)}{\partial p}$, the first and the last (boundary condition) terms cancel. The second term is negative because $r_p^*(p)$ is decreasing in p if $\mathcal{E}^*(p) \geq 1/2$. The third is negative by concavity of π. This concludes the proof. □

EC.2. Proof of Proposition 9.

The first part is obvious. For the second part, the sign of the derivative $\dfrac{\partial}{\partial C}\left(\dfrac{R^*(C)}{C}\right) = \dfrac{CR_C^*(C) - R^*(C)}{C^2}$ is determined by the nominator. To show that this is negative, we calculate each term separately. We write the optimal revenue as:

$$R^*(C) = \mathbb{E}\left[\min\left(\bar{\mathbf{D}}, C - x^{**}\right)\right] + \mathbb{E}\left[r(p^{**}, \max\left(x^{**}, C - \bar{\mathbf{D}}\right))\right] \tag{EC.3}$$

$$= \Big[\bar{q}(x)(C-x) + \mathbb{E}\left[\bar{\mathbf{D}};\bar{\mathbf{\Omega}}\right] + \bar{q}(x)xpq(p,x) + \bar{q}(x)\mathbb{E}\left[\pi(p,\mathbf{Z});\mathbf{\Omega}\right]$$

$$+\mathbb{E}\left[p(C-\bar{\mathbf{D}})q(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}\right] + \mathbb{E}\left[\pi(p,\mathbf{Z});\mathbf{D}\le C-\bar{\mathbf{D}};\bar{\mathbf{\Omega}}\right]\Big]\Big|_{x=x^{**},p=p^{**}}.$$

From the envelope theorem,

$$R_C^*(C) = \frac{\partial R(p,x;C)}{\partial C}\Big|_{x=x^{**},p=p^{**}} = \bar{q}(x^{**}) + p^{**}\mathbb{E}[q(p^{**},C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}]. \tag{EC.4}$$

From (EC.3) and (EC.4) we can write

$$CR_C^*(C) - R^*(C) = \big[-\bar{q}(x)x\,(pq(p,x)-1) - \bar{q}(x)\mathbb{E}\left[\pi(p,\mathbf{Z});\mathbf{\Omega}\right]$$

$$+\mathbb{E}\left[\left(pq(p,C-\bar{\mathbf{D}})-1\right)\bar{\mathbf{D}};\bar{\mathbf{\Omega}}\right] - \mathbb{E}\left[\pi(p,\mathbf{Z});\mathbf{D}\le C-\bar{\mathbf{D}};\bar{\mathbf{\Omega}}\right]\big]\Big|_{x=x^{**},\,p=p^{**}}.$$

The first term is equal to zero, because $p^{**}q(p^{**},x^{**})=1$. Negativity of the third term follows because $q(p^{**},C-\bar{\mathbf{D}}) \le 1/p^{**}$ on $\bar{\mathbf{\Omega}} = C-\bar{\mathbf{D}} \ge x$. The second and fourth terms are negative because $\pi(p,z)\ge 0$. It follows that $CR_C^*(C) - R^*(C) \le 0$, concluding the proof. □

EC.3. Proof of Proposition 10.

(a) By definition, $p^{**}(C) = \text{argmax}_p R(p,x^*(p;C);C) = \text{argmax}_p R^*(p;C)$, so it is sufficient, by Topkis' Theorem (see Topkis 1998, Theorem 2.8.2) to show that $R^*(p;C)$ is submodular in (p,C). By the envelope theorem, $\dfrac{\partial R^*(p;C)}{\partial p} = R_p(p,x^*(p;C);C)$. From Littlewood's rule (10), the optimal protection level for a given price, $x^*(p;C)\equiv x^*(p)$ is independent of capacity. This allows us to write:

$$\frac{\partial}{\partial C}\frac{\partial R^*(p;C)}{\partial p} = \frac{\partial}{\partial C}R_p(p,x^*(p);C) = R_{pC}(p,x^*(p);C).$$

The derivative of $R_p(p,x;C) = \mathbb{E}[r_p(p,\max(x,C-\bar{\mathbf{D}}))] = \bar{q}(x)r_p(p,x) + \mathbb{E}[r_p(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}]$ with respect to C is:

$$R_{pC}(p,x;C) = \mathbb{E}[r_{pC}(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}] = \mathbb{E}[q(p,C-\bar{\mathbf{D}})(1-\mathcal{E}(p,C-\bar{\mathbf{D}}));\bar{\mathbf{\Omega}}] \tag{EC.5}$$

$$\le \big(1-\mathcal{E}(p,x)\big)\mathbb{E}[q(p,C-\bar{\mathbf{D}}));\bar{\mathbf{\Omega}}], \tag{EC.6}$$

because $\mathcal{E}(p,x)$ is increasing in x, and $\bar{\mathbf{\Omega}} = C-\bar{\mathbf{D}} \ge x$. We obtain:

$$R_{pC}(p,x^*(p);C) \le \big(1-\mathcal{E}(p,x^*(p))\big)\mathbb{E}[q(p,C-\bar{\mathbf{D}});\bar{\mathbf{\Omega}}] \le 0,$$

whenever $\mathcal{E}^*(p) = \mathcal{E}(p, x^*(p)) \geq 1$. This is implied by $\mathcal{E}(p,x)$ increasing in p implies, via Propositions 2 and 4.

(b) By definition, $x^{**}(C) = x^*(p^{**}(C); C) = x^*(p^{**}(C))$, by Littlewood's rule (10), i.e. the optimal protection level for a given price is influenced by changes in C only through $p^{**}(C)$. The derivative of $x^{**}(C)$ with respect to C is

$$\frac{\partial x^{**}(C)}{\partial C} = \frac{\partial x^*(p^{**}(C))}{\partial C} = \frac{\partial x^*(p^{**}(C))}{\partial p} \frac{\partial p^{**}(C)}{\partial C}.$$

The second term is negative from part (a), whereas the second is negative if $\mathcal{E}^*(p) \geq 1$, from Proposition 2, or in particular if $\mathcal{E}(p,x)$ increasing in p, by Proposition 4. $\square$

EC.4. Proof of Proposition 11.

(a) By the implicit function theorem, $\dfrac{\partial p^*(x;C)}{\partial x} = \left.\dfrac{-R_{px}(p,x;C)}{R_{pp}(p,x;C)}\right|_{p=p^*(x;C)}$. This is negative whenever $\mathcal{E}^*(x) \geq 1$, because, from (13),

$$R_{xp}(p^*(x;C), x; C) = \bar{q}(x) q(p^*(x;C), x)(1 - \mathcal{E}^*(x)) \leq 0.$$

(b) By the same line of proof as part (a), it suffices to show that $R_{pC}(p^*(x;C), x; C) \leq 0$. This is obtained by evaluating (EC.6) at $p = p^*(x;C)$, and using $\mathcal{E}^*(x) \geq 1$:

$$R_{pC}(p^*(x;C), x; C) \leq \big(1 - \mathcal{E}^*(x)\big)\mathbb{E}[q(p^*(x;C), C - \bar{\mathbf{D}})); \bar{\mathbf{\Omega}}] \leq 0.$$

$\square$

EC.5. Proof of Proposition 12.

Proposition 1 insures the existence of a unique optimal protection level $x^*(p)$. Hence, concavity of $R^*(p)$ is sufficient for the existence of a unique optimum. Denote low fare revenue by $\bar{r}(p,x) = \mathbb{E}[\min(C - x, \bar{\mathbf{D}}(p))]$, and $\bar{\mathbf{C}}(p) = C - \bar{\mathbf{D}}(p)$, the excess capacity after all low fare demand has been served. From the envelope theorem,

$$\frac{\partial R^*(p)}{\partial p} = R^*_p(p) = \bar{r}^*_p(p) + \bar{q}^*(p) r^*_p(p) + \mathbb{E}[r_p(p, \bar{\mathbf{C}}(p)); \bar{\mathbf{\Omega}}^*]. \tag{EC.7}$$

To show that this is decreasing, we first calculate the derivative of each term with respect to p. From $\bar{r}_p^*(p) = \mathbb{E}[\bar{d}_p(p, \bar{\mathbf{Z}}); \bar{\mathbf{\Omega}}^*]$, we obtain the derivative of the first term:

$$\frac{\partial \bar{r}_p^*(p)}{\partial p} = \mathbb{E}[\bar{d}_{pp}(p, \bar{\mathbf{Z}}); \bar{\mathbf{\Omega}}^*] - \bar{d}_p^*(p)\bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}. \tag{EC.8}$$

The derivative of the second term in (EC.7) can be written as:

$$\frac{\partial}{\partial p}\bar{q}^*(p)r_p^*(p) = \bar{q}^*(p)\frac{\partial r_p^*(p)}{\partial p} + \frac{\partial \bar{q}^*(p)}{\partial p}r_p^*(p). \tag{EC.9}$$

Finally, we calculate the derivative of the third term in (EC.7):

$$\begin{aligned}
\frac{\partial}{\partial p}\mathbb{E}[r_p(p, \bar{\mathbf{C}}(p)); \bar{\mathbf{\Omega}}^*] = \; & \mathbb{E}[\pi_{pp}(p, \mathbf{Z}); \mathbf{D}(p) \le \bar{\mathbf{C}}(p); \bar{\mathbf{\Omega}}^*] \\
& -\mathbb{E}[\bar{\pi}_{pp}(p, \bar{\mathbf{Z}})q(p, \mathbf{C}(p)); \bar{\mathbf{\Omega}}^*] \\
& -\mathbb{E}[(d_p(p, z(p, \bar{\mathbf{C}}(p))) + \bar{d}_p(p, \bar{\mathbf{Z}}))^2 f(p, \bar{\mathbf{C}}(p)); \bar{\mathbf{\Omega}}^*] \\
& -\bar{f}^*(p)r_p^*(p) \\
& +\bar{f}^*(p)\bar{d}_p^*(p)pq^*(p)\frac{\partial x^*(p)}{\partial p}.
\end{aligned}$$

Summing up (EC.8), (EC.9) and (EC.10), because $\frac{\partial \bar{q}^*(p)}{\partial p} = \bar{f}^*(p)$, the last term of (EC.9) and the fourth term of (EC.10) add up to zero. Similarly, $p^*q^*(p) = 1$ implies that the last term of (EC.8) and (EC.10) sum to zero. Hence, we can write,

$$\begin{aligned}
\frac{\partial R_p^*(p)}{\partial p} = \; & \mathbb{E}[\pi_{pp}(p, \mathbf{Z}); \mathbf{D}(p) \le \bar{\mathbf{C}}(p); \bar{\mathbf{\Omega}}^*] \\
& -2\mathbb{E}[\bar{d}_p(p, \bar{\mathbf{Z}})q(p, \mathbf{C}(p)); \bar{\mathbf{\Omega}}^*] \\
& -\mathbb{E}[\bar{d}_{pp}(p, \bar{\mathbf{Z}})(pq(p, \mathbf{C}(p)) - 1); \bar{\mathbf{\Omega}}^*] \\
& -\mathbb{E}[(d_p(p, z(p, \bar{\mathbf{C}}(p))) + \bar{d}_p(p, \bar{\mathbf{Z}}))^2 f(p, \bar{\mathbf{C}}(p)); \bar{\mathbf{\Omega}}^*] \\
& +\bar{q}^*(p)\frac{\partial r_p^*(p)}{\partial p}.
\end{aligned} \tag{EC.10}$$

The first term is negative by concavity of $\pi(p, z)$, and the second because $\bar{d}_p(p, z) \ge 0$. Negativity of the third term follows because $\bar{d}_{pp}(p, \bar{z}) \le 0$ and $q(p, \bar{\mathbf{C}}(p)) \le q^*(p) = 1/p$ on $\bar{\mathbf{\Omega}}^* = (\bar{\mathbf{C}}(p) \ge x^*(p))$. The fourth term is obviously negative. From Proposition 7, $r_p^*(p)$ is decreasing in p if $\mathcal{E}^*(p) \ge 1/2$, implying the negativity of the last term, which remains the same as in Proposition 7. It follows that $R_p^*(p)$ is decreasing, so $R^*(p)$ is concave in p. $\square$

EC.6. Proof of Proposition 13.

(a) By (19), $p^P(x) = \text{ argmax}_p V(p,\bar{p},x) = \text{ argmax}_p r(p,x)$, is independent of $\bar{p}$. Optimality of $p^P(x)$ and the implicit function theorem imply that $p^P(x)$ is decreasing in x whenever $\mathcal{E}(p^P(x),x) \geq 1$, because:

$$V_{xp}(p,\bar{p},x)|_{p=p^P(x)} = r_{xp}(p,x)|_{p=p^P(x)} = q(p^P(x),x)(1-\mathcal{E}(p^P(x),x)) \leq 0.$$

It remains to show that $\mathcal{E}(p^P(x),x) \geq 1$ if $\mathcal{E}(p,x)$ is increasing in x. Write $r_p(p,x) = \int_0^x Q(p,v)dv$, where $Q(p,x) = q(p,x)(1-\mathcal{E}(p,x))$. Because $\mathcal{E}(p,x)$ is increasing in x, and $q(p,x) \geq 0, Q(p,x)$ crosses zero at most once, and from above. Therefore, the first order condition $\int_0^x Q(p^P(x),v)dv = 0$ implies $Q(p^P(x),x) \leq 0$, that is $\mathcal{E}(p^P(x),x) \geq 1$, which concludes the proof. Monotonicity of $\bar{p}^P(x)$ in x follows by a similar argument.

(b) By the envelope theorem, we have,

$$\frac{\partial^2 V(p^P(x),\bar{p}^P(x),x)}{\partial x^2} = r_{xx}(p,x) + \bar{r}_{xx}(\bar{p},x) - \frac{r_{xp}^2(p,x)}{r_{pp}(p,x)} - \frac{\bar{r}_{xp}^2(\bar{p},x)}{\bar{r}_{\bar{p}\bar{p}}(\bar{p},x)}\bigg|_{p=p^P(x),\bar{p}=\bar{p}^P(x)}.$$

The second order derivatives are: $r_{xx}(p,x) = -pf(p,x)$, $\bar{r}_{xx}(\bar{p},x) = -\bar{p}\bar{f}(\bar{p},x)$, and

$$\begin{aligned} r_{pp}(p,x) &= \mathbb{E}\left[\pi_{pp}\left(p,\mathbf{Z}\right);\mathbf{\Omega}\right] - pf(p,x)d_p^2(p,z(p,x)), \\ \bar{r}_{\bar{p}\bar{p}}(\bar{p},x) &= \mathbb{E}\left[\bar{\pi}_{\bar{p}\bar{p}}\left(\bar{p},\bar{\mathbf{Z}}\right);\bar{\mathbf{\Omega}}\right] - \bar{p}\bar{f}(\bar{p},x)\bar{d}_{\bar{p}}^2(\bar{p},\bar{z}(\bar{p},x)). \end{aligned}$$

Therefore, we can write:

$$r_{xx}(p,x)r_{pp}(p,x) - r_{xp}^2(p,x) = -pf(p,x)\mathbb{E}\left[\pi_{pp}\left(p,\mathbf{Z}\right);\mathbf{\Omega}\right] + q(p,x)^2(2\mathcal{E}(p,x)-1).$$

The first term is positive by concavity of π. The second, evaluated at $p = p^P(x)$, is positive when $\mathcal{E}(p^P(x),x) \geq 1/2$. The latter is guaranteed by monotonicity of elasticity in x, based on part (a). Similarly, we can show that $\bar{r}_{xx}(\bar{p},x)\bar{r}_{\bar{p}\bar{p}}(\bar{p},x) - \bar{r}_{xp}^2(\bar{p},x)\big|_{\bar{p}=\bar{p}^P(x)} \geq 0$, when $\bar{\pi}$ is concave and $\bar{\mathcal{E}}(\bar{p}^P(x),x) \geq 1/2$. Finally, because $r_{pp}, \bar{r}_{\bar{p}\bar{p}} \leq 0$, it follows that $\dfrac{\partial^2 V(p^P(x),\bar{p}^P(x),x)}{\partial x^2} \leq 0$, so the (PA) model has a unique optimal solution. $\square$

EC.7. Proof of Proposition 14.

Recall that x^* and x^P denote the optimal protection level, respectively class 1 allocation set by the (PRM) and (PA) models, respectively. Using $pq(p, x^*) = \bar{p}$ from Littlewood's rule (10), in the marginal revenue condition (21), we obtain:

$$V_x(p, \bar{p}, x)\big|_{x=x^*} = pq(p, x^*) - \bar{p}\bar{q}(\bar{p}, x^*) = \bar{p}\bar{F}(\bar{p}, x^*) \geq 0. \tag{EC.11}$$

Because $V_{xx}(p, \bar{p}, x) = -pf(p, x) - \bar{p}\bar{f}(\bar{p}, x) \leq 0$, (EC.11) implies $x^P(p, \bar{p}) \geq x^*(p, \bar{p})$. $\square$

EC.8. Proof of Proposition 15.

Recall that p^* and p^P denote the optimal prices obtained from the (PRM) and (PA) models, respectively. We can write,

$$R_p(p, x)\big|_{p=p^P} = \bar{q}(x) r_p(p^P, x) + \mathbb{E}[r_p(p^P, C - \bar{\mathbf{D}}); \bar{\mathbf{\Omega}}].$$

From the optimality of p^P, $r_p(p^P, x) = 0$. For the second term, note if $\mathcal{E}(p, x)$ is increasing in x, $r_{px}(p^P(x), x) \leq 0$ or $r_p(p^P(x), x)$ is decreasing in x (Proposition 13). Since on $\bar{\mathbf{\Omega}}$, $C - \bar{\mathbf{D}} \geq x$, this implies $\mathbb{E}[r_p(p^P, C - \bar{\mathbf{D}}); \bar{\mathbf{\Omega}}] \leq r_p(p^P, x) = 0$. It follows that $R_p(p, x)\big|_{p=p^P} \leq 0$, and $p^P(x, \bar{p}) \geq p^*(x, \bar{p})$. $\square$

Europe Campus
Boulevard de Constance,
77305 Fontainebleau Cedex, France
Tel: +33 (0)1 6072 40 00
Fax: +33 (0)1 60 74 00/01

Asia Campus
1 Ayer Rajah Avenue, Singapore 138676
Tel: +65 67 99 53 88
Fax: +65 67 99 53 99

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