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We introduce a measure of elasticity of stochastic demand, called the elasticity of the lost sales rate, which offers a unifying perspective on the well-known newsvendor with pricing problem. This new concept provides a framework to characterize structural results for coordinated and uncoordinated pricing and inventory strategies. Concavity and submodularity of the profit function and sensitivity properties of the optimal inventory and price policies are characterized by monotonicity conditions, or bounds, on the elasticity of the lost sales rate. These elasticity conditions are satisfied by most relevant demand models in the marketing and operations literature. Our results explain, unify and complement previous work on price-setting newsvendor models, and provide a new tool for researchers modeling stochastic price-sensitive demand in other contexts.

1. Introduction

The importance and difficulty of modeling and understanding the interaction of pricing and inventory decisions in uncertain demand environments is well established, and has mativated a vast literature on stochastic price-inventory problems (see e.g. Tayur et al. 1999 and Chan et al. 2004 for surveys). The backbone of these operational models is the classical newsvendor with pricing model, which incorporates price sensitivity in the classical inventory decision problem with stochastic demand.

The existing newsvendor with pricing (NVP) literature is extensive, but has mainly focused on additive and multiplicative demand models, and provided model-specific approaches and results. Our goal in this paper is to develop a general, unified approach and provide robust, non-parametric conditions on the demand model, for the NVP problem to be well-behaved in several respects. Our results would empower future researchers to use general demand models (as opposed to specific

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scenarios, e.g. additive, multiplicative) in their analysis of stochastic price-inventory problems, and shed light on what drives differences and similarities between existing results.

We consider the classical, single product newsvendor model with lost sales and stochastic, price-sensitive demand, where the firm makes one pricing and/or inventory decision, and there is no opportunity to change price or replenish. Decisions are made before demand is observed, and excess demand is lost. In this context, we investigate properties of the objective function and the structure of coordinated and sequential price and inventory optimization policies.

In contrast with existing literature, we model demand as a general stochastic function of price. Studying general, non-parametric demand models has theoretical and methodological merit and appeal. In addition, practical considerations motivate us to go beyond typical additive-multiplicative models in the NVP context. As further detailed below, a general demand model allows us to (i) relax restrictive implicit assumptions made by additive-multiplicative models, and (ii) to capture empirically validated demand models or properties which are not consistent with additive-multiplicative forms (see Section 2.1 for details).

First, additive-multiplicative models make restrictive implicit assumptions which drive existing structural results (Petruzzi and Dada 1999). For example, such models restrict the relationship between price and demand variability, in ways which are not necessarily validated empirically, see e.g. Raz and Porteus (2006). These authors argue that using a simplified, additive or multiplicative model, instead of a general one, can result in substantial profit losses. This motivates the need to study general models, which make no upfront restriction on demand.

Second, several relevant market response models used in the marketing, operations and economics literature (as surveyed e.g. in Hanssens et al. 2001) are not covered by current NVP result, but encompassed by our general model. In particular, this includes attraction models (e.g. logit or power), which are "among the most commonly used market share models, in both empirical studies and theoretical models" (Bernstein and Federgruen 2004, p. 874).

Contribution. The main contributions of this paper can be summarized as follows: (1) we unify the existing literature on NVP models and explain what drives existing (additive-multiplicative) results; (2) we extend existing results to allow for general demand models, including practically relevant functional forms; (3) we propose a unified measure, analysis, and general set of conditions for the NVP model to be well-behaved in various respects; (4) we provide a new modeling tool and approach for researchers using price-dependent stochastic demand in other operational contexts. We elaborate on these points below.

We propose a new measure, the *lost sales rate (LSR) elasticity*, for a given price and inventory level, which captures the structural essence of the NVP model under general, stochastic price-dependent demand. Specifically, we provide a "nested" set of conditions (from necessary and sufficient, to easy to verify sufficient ones), based on the elasticity of the lost sales rate, for the NVP problem to satisfy desirable regularity and sensitivity properties. This sheds light on the restrictiveness of a well-behaved model, and on what demand properties drive structural results.

Our results characterize structural properties of the newsvendor with pricing model based on the LSR elasticity, under both coordinated and sequential decision processes. In a perfectly coordinated environment, where price and quantity decisions are made simultaneously, we show that the existence of a unique optimal price-quantity solution is guaranteed by bounds on, or monotonicity of the LSR elasticity. Specifically, if LSR elasticity exceeds 1/2 (globally, respectively pathwise), the joint pricing and newsvendor problem is concave (jointly, resp. pathwise), hence easy to solve, and admits a unique solution. The pathwise bounds, and hence uniqueness of the solution, are guaranteed by increasing LSR elasticity. Monotone LSR elasticity also ensures that the optimal NVP price increases with unit cost (the optimal inventory always decreases with cost). This result is driven by the sensitivity of optimal prices to changes in inventory.

Price and production/inventory decisions are usually not perfectly coordinated, but rather made separately, by different units of the firm (e.g sales and marketing, respectively operations or supply chain planning divisions). In such an uncoordinated setting, we investigate the optimal response of a marketing, respectively operations division to a change in production, respectively price. In this context, monotone policies are relevant, because they support managerial intuition, and set the base for structural results in more complex, coordinated, dynamic, competitive or multi-product settings. Necessary and sufficient conditions for price, respectively inventory policies to be monotone are captured by a pathwise lower bound of 1 on the LSR elasticity, along the corresponding optimal decision path. A global lower bound of 1 is equivalent to submodularity of expected profits. Monotonicity of the optimal price, respectively inventory policies is also ensured by LSR elasticity increasing in quantity, respectively price.

In sum, we find that increasing LSR elasticity is an important property, which guarantees uniqueness of the optimal coordinated NVP solution, as well as sensitivity results for coordinated and uncoordinated policies. It is therefore important to understand how restrictive this property is, and what demand functions have monotone LSR elasticity.

It is natural to assume that a stochastic demand function $\mathbf{D}(p)$ is decreasing in p in some stochastic sense. It turns out that increasing LSR elasticity (with respect to quantity) is precisely *equivalent* to $\mathbf{D}(p)$ being stochastically decreasing in price in the hazard rate order (formally defined in Section 5), a stronger order than first order stochastic dominance. For additive (multiplicative) models, this translates to increasing (general) failure rate properties of the underlying demand risk. We identify a general class of stochastic demand models with increasing LSR elasticity, and show that it includes the majority of models studied in the NVP literature, as well as other relevant market response models, including attraction models. Thus our results unify, generalize and complement the existing NVP literature.

Literature. The vast literature on coordinated pricing and inventory decisions has been reviewed by Yano and Gilbert (2003) and Chan, Shen, Simchi-Levi and Swann (2004), and specifically for the newsvendor problem by Petruzzi and Dada (1999). Some representative NVP works include Karlin and Carr (1962), Lau and Lau (1988), Mills (1959), Nevins (1966), Yao, Chen and Yan (2006), Young (1978) and Zabel (1970). These papers, and most of the literature, provide results for additive, multiplicative or additive-multiplicative demand models; our general stochastic price-dependent demand model encompasses and generalizes these. The only exception we are aware of is Raz and Porteus (2006), who use a general demand model (not comparable to ours) specified up to a finite number of empirically estimated fractiles, assumed piecewise linear in price.

Our model falls in the class of static NVP decision models with stochastic demand and lost sales. Both in terms of model and results, our paper is closest to Zabel (1970), Young (1978), Petruzzi and Dada (1999), Yao et al. (2006) and references within (as summarized in their Tables 1 and 2), who all investigate uniqueness of the optimal coordinated price-quantity solution under a lost sales model with stochastic demand. Our results based on LSR elasticity are more general, and not dominated by any of these. Furthermore, our sensitivity results for the optimal coordinated and sequential policies extend existing ones, including Ha (2001) and Yao et al. (2006) (cost sensitivity to cost), Zabel (1970) (price-inventory monotonicity under multiplicative demand). Finally, our results in Section 6.2 generalize existing comparisons of the optimal price for stochastic and deterministic problems.

This paper focuses on the classical NVP model with lost sales. However, we expect our LSR elasticity-based conditions to be useful for analyzing other types of operational models involving stochastic price-dependent demand, such as multi-product, competitive and dynamic models. For example, our results extend to richer inventory models, including multi-product, flexible manufacturing and revenue management models (see Kocabiyikoglu and Popescu 2007), and can be directly applied in more complex, dynamic settings (see de Vericourt and Lobo 2008). In that respect, our work is potentially relevant to (albeit not directly comparable with) several streams of NVP literature, including multi-product models (e.g. Netessine and Rudi, 2003), competitive models (e.g. Zhao and Atkins 2008, Bernstein and Federgruen 2005), contracting models (e.g. Ha 2001, Wang, Jiang and Shen 2004), and multi-period models (e.g. Karlin and Carr 1962, Mills 1962, Zabel 1972, Thowsen 1975, Monahan, Petruzzi and Zhao 2004, Netessine 2006). NVP models where all demand is served at the set price, and excess demand is backlogged at a price independent cost preclude lost sales as a special case, hence are not comparable to ours (e.g. Federgruen and Heching 1999, Agrawal and Seshadri 2000, and Chen and Simchi-Levi 2003).

Finally, our contributions are similar in spirit to Ziya, Ayhan and Foley (2004) and Lariviere (2006), although the setup and results are not comparable. These authors discuss and unify important demand assumptions used for pricing and/or inventory problems. Their conditions translate desirable concavity results of a deterministic revenue function into failure rate properties of the stochastic willingness to pay distribution underlying a deterministic price-demand function.

Structure. The rest of the paper is organized as follows. The model and underlying assumptions are presented in § 2. Sequential price and inventory policies are investigated in § 3, and the joint price-inventory model in § 4. Both sections translate properties of the optimal price-inventory policies into bounds and monotonicity conditions on elasticity. Equivalent characterizations of the elasticity of the lost sales rate and conditions for its monotonicity are presented in § 5, together with specific examples. A comparison with related NVP literature, including riskless price benchmarks, is provided in § 6. Finally, § 7 concludes the paper.

2. The Model

This paper considers a profit maximizing firm seeking to optimize inventory x and/or price p decisions for a single product. These decisions are made either sequentially or simultaneously, before observing an uncertain, price dependent demand, $\mathbf{D}(p)$, and excess demand is lost. For simplicity, we assume a constant unit cost c; all our results extend without loss of generality to increasing and convex cost functions $c(x)$. In a coordinated setting (studied in § 4), the firm jointly optimizes price and quantity decisions in order to maximize expected profit:

$$\max_{p,x} \Pi(p,x), \text{ where } \Pi(p,x) = p\mathbb{E}\left[\min(\mathbf{D}(p),x)\right] - cx. \tag{1}$$

The constrained (or truncated) revenue is denoted by

$$R(p,x) = p\mathbb{E}\left[\min(\mathbf{D}(p),x)\right]. \tag{2}$$

In an uncoordinated environment (studied in § 3), a sales and marketing division (or a price-setting firm) optimizes the objective $\Pi(p,x)$ in (1) with respect to price p, for a given inventory x, whereas an operations division (or quantity-setting firm) optimizes $\Pi(p,x)$ with respect to x, for a given p.

2.1 Demand Model

The price dependent stochastic demand is modeled as

$$\mathbf{D}(p) = d(p,\mathbf{Z}), \tag{3}$$

where d is a deterministic demand function, and $\mathbf{Z}$ is a random variable with price-independent cdf Φ and density ϕ, capturing demand risk. In empirical estimation, $\mathbf{Z}$ can be random noise, or an independent variable in a (possibly non-linear) regression model. Conceptually, it is convenient think of $\mathbf{Z}$ as sales driver which is uncertain, or not perfectly controlled by the decision maker, or relevant division of the firm. Examples include market size, personal disposable income of the target market, product quality, advertising spend, brand awareness, a reference price or a competitor's price (see e.g. Hanssens et al. 2001).

The demand function $d(p,z)$ is decreasing in price p, strictly increasing in z, and twice differentiable in p and z. Throughout the paper the terms increasing/decreasing, positive/negative are used in their weak sense. Monotonicity of d allows us to uniquely define the stocking factor[1] $z(p,x)$ as the value of the sales driver z for which demand perfectly matches supply, $d(p,z(p,x))=x$. The riskless (or pathwise) unconstrained revenue is denoted $r(p,z)=pd(p,z)$.

The general demand model (3) encompasses the additive and multiplicative demand models commonly used in the NVP literature, as well as the additive-multiplicative one (Young, 1978):

$$\mathbf{D}(p)=d(p,\mathbf{Z})=\alpha(p)\mathbf{Z}+\beta(p), \tag{4}$$

where $\alpha(p),\beta(p)$ are decreasing functions of p. For $\alpha(p)\equiv 1$ this is the additive model (price influences the location of the demand distribution), whereas for $\beta(p)\equiv 0$ this is the multiplicative model (price influences demand scale).

Additive-multiplicative models, while easy to estimate, make restrictive implicit assumptions, which drive the nature of NVP results. For example, additive and multiplicative models restrict the nature of substitution between risk (z) and price (p), as well as the effect of price on demand variability, in ways which critically affect existing results (see Section 6.2, respectively Petruzzi and Dada 1999, p.187). Specifically, for additive models, demand variance is price independent and the coefficient of variation is increasing in price, whereas for multiplicative models, variance

[1] This is consistent with the model-specific stocking factor definitions in Petruzzi and Dada (1999), but their normalized mean-variance interpretation is limited to additive and multiplicative settings.

is decreasing in price, and coefficient of variation is not influenced by price. In general, for the additive-multiplicative model (4), demand variance is decreasing (or constant) in price. Empirical observations, however, suggest that demand variability may be non-monotone in price, in which case existing parametric approximations for the NVP problem perform poorly (see Raz and Porteus, 2006, p. 1765). Our general demand model fills this gap by making no *a priori* assumptions regarding the effect of price on demand variability.

The advantages for studying the general demand model (3), as discussed in the introduction, are theoretical and methodological, as well as practical. In particular, model (3) encompasses other practically relevant market response models, used in the marketing, economics and operations literature, which are not additive-multiplicative, including attraction models, such as logit and power choice models, further described in Section 5.3.

2.2 Lost Sales Rate (LSR) Elasticity

For a given price p and stock x, the *lost sales rate (LSR)* is denoted $q(p,x) = 1 - F(p,x)$, where $F(p,x) = P(\mathbf{D}(p) \leq x)$ is the demand cdf, or probability of no lost sale. The keystone for our developments is a new elasticity concept, the price elasticity of the rate of lost sales q, for a given price p and quantity level x. Specifically, the *lost sales rate (LSR) elasticity* is the percentage change in the rate of lost sales with respect to the percentage change in price, for a given stocking quantity. Throughout the paper partial derivatives are denoted by corresponding subscripts.

Definition 1. The LSR elasticity for a given price p and inventory level x, is defined as

$$\mathcal{E}(p,x) = -\frac{pq_p(p,x)}{q(p,x)} = \frac{pF_p(p,x)}{1-F(p,x)}. \tag{5}$$

The LSR elasticity $\mathcal{E}(p,x)$ combines the relative sensitivity of (lost) sales with respect to its underlying factors, inventory and price. Our results in this paper characterize relevant structural properties of the NVP problem in terms of (necessary and) sufficient conditions on $\mathcal{E}$, suggesting that this particular elasticity concept captures essential features of the NVP setup. A more detailed discussion on the LSR elasticity, and equivalent expressions, are provided in § 5.

Throughout the paper, price and quantity are implicitly optimized in Problem (1) over (positive) compact intervals $p \in P = [p_L, p_H]\,, x \in X = [x_L, x_H]\,,$ where p_H, x_H are arbitrary, possibly infinite, and $p_L \geq c$. We set $x_L = 0$ without loss of generality, and $p_L = \arg\max\{d(p, \Phi^{-1}(1 - \frac{c}{p})); p \geq c\}$; our results hold for any subintervals of P and X thus defined. The lower bound on price, p_L, is in a sense necessary, and not very restrictive, as argued in Section 3.2. It helps to express monotonicity of the optimal inventory level in terms of LSR elasticity (in Propositions 4 and 6).

Table 1 provides a summary of the main notation used throughout the paper.

Table 1: Summary of notation

c	unit cost
x	inventory, stock or order quantity
p	price
p_L	lower bound on price, $p \geq p_L$
$\mathbf{D}(p) = d(p, \mathbf{Z})$	stochastic price-dependent demand
$\mathbf{Z}$	random component of $\mathbf{D}(p)$
$d(p, z)$	deterministic (riskless) demand function
$z(p, x)$	stocking factor, $d(p, z(p, x)) = x$
$r(p, x) = pd(p, z)$	riskless revenue function
$R(p, x)$	expected revenue function
$\Pi(p, x) = R(p, x) - cx$	expected profit function
$\phi(z)$; $\Phi(z) = P(\mathbf{Z} \leq z)$	density; resp. cdf of $\mathbf{Z}$
$f(p, x)$; $F(p, x) = P(\mathbf{D}(p) \leq x)$	density; resp. cdf of $\mathbf{D}(p)$
$q(p, x) = 1 - F(p, x)$	lost sales rate; survival function of $\mathbf{D}(p)$
$\mathcal{E}(p, x)$	elasticity of the lost sales rate $q(p, x)$

3. Price-Inventory Interactions in a Sequential Decision Process

In most practical settings, price and inventory decisions are not managed simultaneously, but rather sequentially, by separate units of the firm (Zhao and Atkins, 2008). Sales and marketing divisions set prices based on planned inventory, whereas operations (or supply chain planning) divisions decide production or inventory levels based on pre-set prices. This amounts to optimizing $\Pi(p, x)$ in (1) with respect to p, respectively x, with the other variable as a parameter. This section investigates properties of the optimal price and inventory policies in such uncoordinated settings. Specifically, we ask how a marketing group should adjust price in response to a change in production, and how an operations division should adjust production in response to a price change. These results are used in the next section, to shed light on the coordinated price-inventory decision.

Despite the asymmetry of the price and inventory-setting problems, the structure of the obtained results is surprisingly similar. This motivates us to present the two problems in parallel.

3.1 Optimality Equations

We first provide the well known optimality equation for quantity, and express the optimality equation for price in terms of the LSR elasticity. All proofs are in the Appendix.

PROPOSITION 1. *(a) For any given price p, the optimal order quantity $x^*(p)$ solves:*

$$q(p,x) = c/p. \tag{6}$$

(b) For any given quantity x, the optimal price $p^*(x)$ solves:

$$\int_0^x q(p,v)(1-\mathcal{E}(p,v))dv = 0. \tag{7}$$

Throughout the paper, we denote the evaluation of any generic function $\mathfrak{f}$ along the optimal path $x^*(p)$ as $\mathfrak{f}^*(p) = \mathfrak{f}(p,x)|_{x=x^*(p)} = \mathfrak{f}(p,x^*(p))$. For example, at a price p, the optimal profit for a quantity-setting firm is $\Pi^*(p) = \Pi(p,x^*(p))$. Similarly, we can write $\mathfrak{f}^*(x) = \mathfrak{f}(p,x)|_{p=p^*(x)}$. This is a slight abuse of notation; however, the generic argument of $\mathfrak{f}^*$ makes the evaluation path unambiguous. We also denote $\mathfrak{f}^*_x(p) = \mathfrak{f}_x(p,x)|_{x=x^*(p)}$, the derivative of $\mathfrak{f}(p,x)$ with respect to x evaluated at the optimal quantity, and $\mathfrak{f}^*_x(x) = \mathfrak{f}_x(p,x)|_{p=p^*(x)}$, the derivative evaluated at the optimal price. In this notation, used throughout the paper, the derivative always precedes functional evaluation.

For a given price p and stock x, the stocking factor $z(p,x)$ was defined so that $d(p,z(p,x)) = x$. According to the above notation, for a quantity-setting firm, the optimal stocking factor at price p is $z^*(p) = z(p,x^*(p))$. This satisfies $P(\mathbf{Z} \geq z^*(p)) = c/p$, from (6). The following properties of the stocking factor are used throughout the paper. The proof is simple, and omitted for conciseness.

LEMMA 1. *(a) $z(p,x)$ is increasing in p and x, with $z_p(p,x) = \frac{-d_p(p,z(p,x))}{d_z(p,z(p,x))}$ and $z_x(p,x) = \frac{1}{d_z(p,z(p,x))}$.*

(b) Under the optimal inventory policy, $z^*(p) = \Phi^{-1}(1-c/p)$, which is increasing in price.

A price increase commands larger optimal stocking factors. Can we expect such a systematic effect of price on order-quantity $x^*(p)$ (and vice versa for $p^*(x)$), and if so, under what conditions? The next section investigates these issues.

3.2 Sensitivity results

Economic considerations suggest that lower prices lead to higher mean demand, and consequently should drive up the preferred inventory levels. On the other hand, stochastic inventory theory predicts that lower prices lead to lower underage costs, hence lower safety stocks, driving inventory levels down. These two arguments suggest that, under demand uncertainty, the relationship between price and quantity is generally ambiguous, as previously acknowledged in the literature (Zabel 1970, Raz and Porteus 2006).

When does the economic driver of the price-quantity relationship dominate the safety stock effect due to demand uncertainty? This section presents several sets of conditions for decreasing optimal price $p^*(x)$, respectively quantity $x^*(p)$ policies. Such properties are important, on one hand because they support managerial intuition, and on the other hand because they are the base for structural results in more complex, coordinated, dynamic, competitive or multi-product settings (e.g. Kocabiyikoglu and Popescu 2007, de Vericourt and Lobo 2008). For example, these results will be useful in the next section, to characterize conditions for the coordinated problem to be well behaved in various ways (e.g. regularity and cost sensitivity). We provide both necessary and sufficient conditions, as well as sufficient conditions which are easier to verify; all are expressed in terms of the LSR elasticity, $\mathcal{E}(p, x)$.

A widely used sufficient condition for comparative statics is submodularity of the objective function, guaranteed by Topkis' theorem (see Topkis 1998, Theorem 2.8.2). A function $\mathfrak{f}(x, y)$ submodular if it has decreasing differences, that is $\mathfrak{f}(x^+, y^+) - \mathfrak{f}(x^+, y^-) \leq \mathfrak{f}(x^-, y^+) - \mathfrak{f}(x^-, y^-)$ for all $x^+ \geq x^-, y^+ \geq y^-$. For differentiable functions $\mathfrak{f}$, submodularity is equivalent to a negative cross-derivative $\mathfrak{f}_{xy}(x, y) \leq 0$. Supermodularity is defined by the opposite inequality.

Proposition 2. *The profit function $\Pi(p,x)$ is submodular if and only if $\mathcal{E}(p,x) \geq 1$. In this case, the inventory and pricing policies $x^*(p)$ and $p^*(x)$ are decreasing in their respective arguments.*

While weaker conditions are subsequently obtained for our problem, characterizing submodularity of the objective can be useful to obtain structural properties for more complex (e.g. dynamic or competitive) settings. For multiplicative demand models, the elasticity bound of 1 reduces to condition (10) in de Vericourt and Lobo (2008, p.17), who use it to obtain pathwise concavity and price monotonicity in a dynamic model of non-profit operations. These authors emphasize the importance of obtaining stronger sufficient conditions which can be propagated in a dynamic setting. Similar bounds on different elasticity measures have been used to obtain unimodality by Bernstein and Federgruen (2005), in a competitive setting (the GFR of the deterministic demand is bounded by 1 in a multiplicative model), and Ziya et al. (2006) in a queuing setting. In general, checking $\mathcal{E}(p,x) \geq 1$ amounts to bounds on p and/or x, as illustrated in Section 4.2.

The uniform lower bound of 1 on $\mathcal{E}$ is sufficient, but not necessary, for the optimal price and inventory policies to be monotone. Necessary and sufficient monotonicity conditions are captured by a lower bound of 1 on the LSR elasticity, along the optimal decision path (as opposed to everywhere).

Proposition 3. *(a) $p^*(x)$ is decreasing in x if and only if $\mathcal{E}^*(x) \geq 1$ for all x.*

(b) $x^*(p)$ is decreasing in p if and only if $\mathcal{E}^*(p) \geq 1$ for all p.

Proposition 3 requires the evaluation of $\mathcal{E}$ along the optimal policy path, a potentially difficult task. In Section 4.2 we illustrate how the conditions of Proposition 3 can be verified for some specific demand models used in the literature. The next result gives easier to verify sufficient conditions for monotone optimal policies, expressed in terms of monotonicity of $\mathcal{E}(p,x)$.

Proposition 4. *(a) If $\mathcal{E}(p,x)$ is increasing in x, then $p^*(x)$ is decreasing in x.*

(b) If $\mathcal{E}(p,x)$ is increasing in p, then $x^*(p)$ is decreasing in p.

The proof of the second part of this result relies on the lower bound p_L on price, defined in Section 2.2. We now argue that this bound is actually necessary, and not very restrictive. Consider a maximal domain $P = [p_L, p_H], p_L \geq c$, where $x^*(p)$ is monotone. Proposition 3 implies $\mathcal{E}^*(p) \geq 1$ for all $p \in P$, so by continuity, either $p_L = c$ or $\mathcal{E}^*(p_L) = 1$ (otherwise, p_L could be decreased, contradicting maximality of P). This is precisely what the definition of p_L ensures (see Lemma 4 in the Appendix). Similar lower bounds on price, essentially ensuring that demand elasticity exceeds 1, have been used for example by Ziya et al. (2006). Our results in this section generalize Zabel's (1970) results for $p^*(x)$ monotonicity under a multiplicative demand model with uniform or exponential risk, which has increasing LSR elasticity (see Section 5).

4. Simultaneous Price-Inventory Optimization

This section focuses on jointly optimized price and quantity decisions, corresponding to a setting where these decisions are centrally managed by the headquarters of the firm, and/or marketing and operations divisions are fully coordinated (Li and Atkins 2005). We present various sets of alternative conditions for the objective in (1) to have a unique solution (p^{**}, x^{**}), and investigate sensitivity properties relative to unit cost. This problem is referred to as the newsvendor with pricing (NVP) problem. Throughout this section, we assume $r(p,z) = pd(p,z)$ is strictly concave in p, i.e. $2d_p(p,z) + pd_{pp}(p,z) < 0$. This assumption, used to obtain uniqueness results in Propositions 5 and 6, is common in the literature (e.g. Zabel 1970, Young 1978).

4.1 Properties of the Objective and Optimal NVP Solution

We first show that a uniform elasticity (lower) bound of 1/2 is sufficient for joint concavity of the objective function (1) in price and quantity, and hence for the uniqueness of the optimal price-quantity solution.

PROPOSITION 5. *If $\mathcal{E} \geq 1/2$, then $\Pi(p,x)$ is jointly concave in p and x, and the (NVP) problem has a unique price-quantity solution.*

Remark that the deterministic revenue function $p\min(d(p,z(p,x)),x)$ is not jointly concave in (p,x), even when $r(p,z) = pd(p,z)$ is concave. Our result suggests that, with sufficient variability in excess demand, guaranteed by the elasticity bound, the extreme effect of the deterministic case can be smoothed out. A similar effect was observed numerically by Netessine and Rudi (2003) for a multi-product newsvendor model. Similar bounds on different elasticity measures have been used in the literature to ensure regularity, as noted in Section 3.2 (see discussion after Proposition 2).

Joint concavity of the NVP objective $\Pi(p,x)$ can be useful in more complex application settings, in particular in dynamic models, because it propagates over time (de Vericourt and Lobo 2008). Nevertheless, it is not necessary for existence of a unique optimal solution for the NVP problem. Weaker sufficient conditions include concavity of the optimal profit along one of the optimal paths, $x^*(p)$ or $p^*(x)$. This is guaranteed by increasing LSR elasticity, or a 1/2 lower bound along the optimal price or inventory policy path.

PROPOSITION 6. *The following alternative conditions are sufficient for the (NVP) problem to have a unique optimal price-quantity solution:*

(a) $\mathcal{E}^*(x) \geq 1/2$*; in this case* $\Pi^*(x)$ *is concave in* x*.*

(b) $\mathcal{E}^*(p) \geq 1/2$*; in this case* $\Pi^*(p)$ *is concave in* p*.*

(c) $\mathcal{E}(p,x)$ *is increasing in* x *or in* p*.*

This result allows us to solve the NVP problem numerically as a one dimensional concave optimization problem, and guarantees uniqueness of the optimal solution (p^{**}, x^{**}). We further argue that in some cases, the global, respectively pathwise 1/2 lower bounds are not only sufficient, but also necessary for the corresponding regularity conditions. Therefore, no weaker bound can be expected to hold for all demand functions. Indeed, the bounds are tight for linear demand models, in particular for the widely studied additive linear models (e.g. Mills 1959, Lau and Lau 1988, Ha 2001, Peruzzi and Dada 1999, Zhao and Atkins 2008).

REMARK 1. If d is linear in p, i.e. $d(p,z) = \delta(z)p + \gamma(z)$, then: (a) $\mathcal{E} \geq 1/2$ is *necessary and sufficient* for joint concavity of $\Pi(p,x)$ and (b) $\mathcal{E}^*(\cdot) \geq 1/2$ is *necessary and sufficient* for concavity of $\Pi^*(\cdot)$.

Our results in this section fully generalize those of Zabel (1970), Young (1978) and Yao et al. (2006), all of which have increasing LSR elasticity, as argued in Section 5. In terms of approach, the majority of the NVP literature, and beyond (e.g. Zabel 1970, Young 1978, Wang et al. 2006 de Vericourt and Lobo 2008) obtains the optimal price-inventory policy by optimizing the NVP problem along the optimal price path. Only Whitin (1955), for deterministic demand, and Yao et al. (2006) use the opposite sequence, optimizing the pricing problem $\Pi^*(p)$ along the optimal quantity path $x^*(p)$. Their approaches rely on the additive-multiplicative setup and specific assumptions on $\mathbf{Z}$ (e.g. uniform/exponential in Zabel 1970, logconcave densities in Young 1978), whereas our analysis is general, resting on the general properties of the LSR elasticity.

For completeness, we investigate the effect of changes in cost on the optimal solution. Let $(p^{**}(c), x^{**}(c))$ denote the lexicographically largest maximizer of $\Pi(p,x)$ (in case it is not unique).

Proposition 7. *(a) The optimal NVP order-quantity, $x^{**}(c)$ is decreasing in unit cost c.*

(b) The optimal NVP price, $p^{**}(c)$ is increasing in unit cost c if and only if $p^*(x)$ is decreasing in x, in particular if $\mathcal{E}(p,x)$ is increasing in x.

The optimal inventory always decreases in cost, a simple, well-known result (e.g. Lariviere 1999), but optimal prices are not necessarily well behaved. This was implicitly suggested by Zabel (1970), who showed that price can be non-monotone in the unit holding cost. Raz and Porteus (2006) find price to be non-monotone in unit cost in a discretized fractile approximation, and intuitively attribute this to non-monotonicity of $p^*(x)$. Proposition 7 formalizes the result, tying it to increasing LSR elasticity; this generalizes existing results, including Ha (2001) and Yao et al. (2006).

4.2 Summary of Results and Illustration of Our Unified Approach

Table 2 provides a summary of the alternative LSR elasticity conditions presented in Sections 3 and 4, which guarantee relevant properties of the NVP objective function and optimal price and inventory policies in coordinated and sequential environments. Together, these results suggest that the LSR elasticity concept introduced in this paper captures essential features of the newsvendor setup. In Section 5 we show how the conditions provided in Table 2 can be verified in terms of properties of the riskless demand d and the risk distribution $\mathbf{Z}$, and characterize a general class of demand models with increasing LSR elasticity.

Table 2: LSR elasticity conditions for the NVP solution and objective function

unique (p^{**}, x^{**})	$p^{**}(c) \downarrow c \Leftrightarrow p^{*}(x) \downarrow x$	$x^{*}(p) \downarrow p$	$\Pi(p,x)$	
$\mathcal{E}(p,x) \uparrow p$ or x	$\mathcal{E}(p,x) \uparrow x$	$\mathcal{E}(p,x) \uparrow p$	joint concave	$\mathcal{E}(p,x) \geq 1/2$
$\mathcal{E}(p,x) \geq 1/2$	$\mathcal{E}(p,x) \geq 1$	$\mathcal{E}(p,x) \geq 1$	submodular	$\mathcal{E}(p,x) \geq 1(^{\ddagger})$
$\mathcal{E}^{*}(\cdot) \geq 1/2$	$\mathcal{E}^{*}(x) \geq 1(^{\ddagger})$	$\mathcal{E}^{*}(p) \geq 1(^{\ddagger})$	$\Pi^{*}(\cdot)$ concave	$\mathcal{E}^{*}(\cdot) \geq 1/2$

(‡) these conditions are necessary and sufficient

To illustrate our unified approach, consider four demand models commonly used in the NVP literature, with additive-linear and multiplicative-isoelastic forms (see Petruzzi and Dada, 1999), where the distribution of $\mathbf{Z}$ is taken to be either uniform between 0 and 1, denoted $U(0,1)$, or exponential with $\lambda = 1$, denoted $Exp(1)$, as summarized in Table 3. The literature so far has deployed separate, model specific (e.g. additive vs. multiplicative, or distribution specific) arguments to understand the NVP solution for these models.

By contrast, the general conditions in Table 2 provide a common solution approach for these models. Indeed, the expressions for $\mathcal{E}(p,x)$ in Table 3 indicate that all these demand models have monotone LSR elasticity, so they satisfy the regularity and monotonicity properties summarized in Table 2. In general, it is not necessary to compute $\mathcal{E}$ in order to verify its monotonicity; sufficient conditions based on the properties of d and $\mathbf{Z}$ are provided in the next section.

Table 3: Evaluation of $\mathcal{E}(p,x)$, $\mathcal{E}^{*}(p)$ and p_L for special forms of demand $d(p,\mathbf{Z})$

	$\mathcal{E}(p,x)$		$\mathcal{E}^{*}(p)$		p_L	
$\mathbf{Z}$	$a-bp+\mathbf{Z}$	$ap^{-b}\mathbf{Z}$	$a-bp+\mathbf{Z}$	$ap^{-b}\mathbf{Z}$	$a-bp+\mathbf{Z}$	$ap^{-b}\mathbf{Z}$
$U(0,1)$	$\frac{bp}{1-x+a-bp}$	$\frac{bx}{ap^{-b}-x}$	$\frac{bp^2}{c}$	$\frac{b(p-c)}{c}$	$\max\left(c, \sqrt{\frac{c}{b}}\right)$	$c\left(1+\frac{1}{b}\right)$
$Exp(1)$	bp	$\frac{bx}{ap^{-b}}$	bp	$b\ln\frac{p}{c}$	$\max\left(c, \frac{1}{b}\right)$	$e^{1/b}+c$

Table 3 also allows us to check weaker (necessary and) sufficient conditions for sensitivity and regularity results, formulated in terms of bounds on $\mathcal{E}(p,x)$ and $\mathcal{E}^{*}(p)$. In general, uniform bounds on $\mathcal{E}(p,x)$ amount to bounds on p and/or x. For example, for $\mathbf{D}(p) = a - bp + \mathbf{Z}, \mathbf{Z} \sim Exp(1)$, $\mathcal{E}(p,x) \geq 1$ as long as $p > 1/b$. For all four models, $\mathcal{E}^{*}(p)$ is increasing in p, so bounds on $\mathcal{E}^{*}(p)$ amount to bounds on price. Specifically, $\mathcal{E}^{*}(p) \geq 1$, translates precisely to $p \geq p_L$, defined in Section 2.2 (by Lemma 4). Table 3 suggests that these bounds are not too restrictive, if demand is sufficiently price-sensitive, i.e. b is relatively large. In particular, for both additive models, $p_L = c$ whenever $b \geq 1/c$, effectively imposing no additional restriction on the permissible price range.

5. Conditions for Increasing LSR Elasticity

Our findings so far lead us to conclude that the relevant results concerning the NVP problem are driven by properties, in particular monotonicity, of LSR elasticity. This sections provides equivalent characterizations of LSR elasticity, and identifies general classes of stochastic price-sensitive demand models with increasing LSR elasticity. In particular, we show that increasing LSR elasticity coincides with demand stochastically decreasing in price in the hazard rate order, a stronger condition than first order dominance. Examples of relevant demand models satisfying increasing LSR elasticity are provided in the next section.

5.1 Alternative Elasticity Expressions

We begin by providing several alternative characterizations of the LSR elasticity $\mathcal{E}(p,x)$, in terms of other known elasticity measures, such as the riskless elasticity of demand and the failure rate of the demand distribution. Table 4 gives the notation and relationship between the distributions of $\mathbf{Z}$ and $\mathbf{D}(p)=d(p,\mathbf{Z})$, as well as their failure (or hazard) rates (FR) and generalized failure rates (GFR). A distribution is IFR, respectively IGFR if it has increasing FR, respectively GFR.

Table 4: Notation and relationships for $\mathbf{Z}$ and $\mathbf{D}(p)$.

	$\mathbf{Z}$	$\mathbf{D}(p)=d(p,\mathbf{Z})$	Relationship for $z=z(p,x)$
Density	$\phi(z)$	$f(p,x)$	$=\phi(z)/d_z(p,z)$
CDF	$\Phi(z)$	$F(p,x)=1-q(p,x)$	$=\Phi(z)$
FR	$h_Z(z)=\frac{\phi(z)}{1-\Phi(z)}$	$h_D(p,x)=\frac{f(p,x)}{1-F(p,x)}$	$=h_Z(z)/d_z(p,z)$
GFR	$g_Z(z)=zh_Z(z)$	$g_D(p,x)=xh_D(p,x)$	$=g_Z(z)/\tilde{\epsilon}_Z(p,z)$

Denote $\tilde{\epsilon}_P(p,z)=-\dfrac{pd_p(p,z)}{d(p,z)}$, $\tilde{\epsilon}_Z(p,z)=\dfrac{zd_z(p,z)}{d(p,z)}$ the (absolute) riskless demand elasticities and $\tilde{\epsilon}_{PZ}(p,z)=\dfrac{\tilde{\epsilon}_P(p,z)}{\tilde{\epsilon}_Z(p,z)}=\dfrac{-pd_p}{zd_z}$, the corresponding cross elasticity. The following expressions allow us to provide various sufficient conditions for monotone LSR elasticity.

LEMMA 2. *(a)* $\mathcal{E}(p,x)=\dfrac{-pf(p,x)d_p(p,z(p,x))}{q(p,x)}$.

(b) $\mathcal{E}(p,x)=g_D(p,x)\epsilon_P(p,x),$ *where* $\epsilon_P(p,x)=\tilde{\epsilon}_P(p,z(p,x))$.

(c) $\tilde{\mathcal{E}}(p,z)=h_Z(z)\dfrac{-pd_p(p,z)}{d_z(p,z)}=g_Z(z)\tilde{\epsilon}_{PZ}(p,z),$ *where* $\tilde{\mathcal{E}}(p,z)=\mathcal{E}(p,d(p,z))$.

In general, there is a natural correspondence between elasticity and GFR. The riskless price elasticity is the percentage change in riskless demand with respect to price, which is precisely the GFR of customers' willingness to pay distribution (see Lariviere 2006). In our case, the GFR of demand, $g_D(p,x)$, is the percentage change in the excess demand rate with respect to stocking quantity for a given price p, which can be interpreted as the elasticity of the lost sales rate with respect to *quantity* (as opposed to *price*, which would be $\mathcal{E}$).

5.2 Conditions for Monotone LSR Elasticity

We next present conditions for monotonicity of $\mathcal{E}(p,x)$ in quantity x and price p, respectively, which are the key drivers of concavity and sensitivity results obtained in the previous sections.

It is natural to assume that a stochastic demand function $\mathbf{D}(p)$ is decreasing in p, in some stochastic sense. It turns out that $\mathcal{E}(p,x)$ increasing in x is precisely *equivalent* to $\mathbf{D}(p)$ being stochastically decreasing in price in the hazard rate order,[2] or equivalently, the demand FR being increasing in price. The hazard rate order is theoretically stronger than first order dominance, and weaker than likelihood ratio order; however, the three are actually equivalent for most parametric families with the natural parameter order (see Müller and Stoyan, 2002, Table 1.1).

Proposition 8. *$\mathcal{E}(p,x)$ is increasing in x if and only if the failure rate of $\mathbf{D}(p)$, $h_D(p,x)$, is increasing in p, that is, whenever $\mathbf{D}(p)$ is decreasing in p in the hazard rate order. If moreover, $\epsilon_P(p,x)$ is increasing in p, then $\mathcal{E}(p,x)$ is also increasing in p.*

The result for LSR elasticity increasing in p follows from Lemma 2(b), and relies on the increasing (riskless) price-elasticity property (IPE) commonly used in the NVP literature, e.g. by Ziya et al. (2006), Netessine (2006) and Yao et al. (2006) (who provide a list of functions satisfying IPE).

We next provide a collection of conditions separating the riskless demand d and the distribution $\mathbf{Z}$. A preliminary result relates the monotonicity of $\mathcal{E}(p,x)$ and $\tilde{\mathcal{E}}(p,z)$, defined in Lemma 2:

[2] By definition, $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the hazard rate order ($\mathbf{Y} \succeq_{FR} \mathbf{X}$) if and only if their respective hazard rates satisfy $h_{\mathbf{X}}(z) \geq h_{\mathbf{Y}}(z)$ (Müller and Stoyan, 2002, Theorem 1.3.3).

Lemma 3. *(a) $\mathcal{E}(p,x)$ is increasing in x if and only if $\tilde{\mathcal{E}}(p,z)$ is increasing in z.*

(b) $\mathcal{E}(p,x)$ is increasing in p if $\tilde{\mathcal{E}}(p,z)$ is increasing in both z and p.

From Lemma 3 and Lemma 2(c) we obtain:

Proposition 9. *(a) Suppose that $\mathbf{Z}$ is IFR. If d_p/d_z is decreasing in z, then $\mathcal{E}(p,x)$ is increasing in x. If moreover pd_p/d_z is decreasing in p, then $\mathcal{E}(p,x)$ increases in p.*

(b) Suppose $\mathbf{Z}$ is IGFR. If the riskless cross elasticity $\tilde{\epsilon}_{PZ}(p,z) = \frac{-pd_p(p,z)}{zd_z(p,z)}$ is increasing in z, then $\mathcal{E}(p,x)$ is increasing in x. If moreover $\tilde{\epsilon}_{PZ}(p,z)$ increases in p, then $\mathcal{E}(p,x)$ increases in p.

The IFR and IGFR assumptions on $\mathbf{Z}$ are commonly used in NVP models (see e.g. Yao et al. 2006 for a summary) and impose "very mild restrictions on the demand distribution" (Wang et al. 2005, p.37). The weaker IGFR assumption "captures most common distributions a modeler would choose to employ" (Tayur et al. 1999, p. 241, also Lariviere and Porteus, 2001). IGFR means that the elasticity of the survival function is increasing, while IFR means that the percentage change in the survival function is increasing. IFR distributions are those with log-concave survival functions, and include distributions with log-concave density (also known as PF2); examples include Uniform, Exponential, Normal, truncated Normal, and lognormal distributions (see Barlow and Proschan 1996, and Bagnoli and Bergstrom 2005). All IFR distributions are IGFR, but the reverse is not true; for example Gamma and Weibull distributions are IFR over a restricted set of parameters, but IGFR for all. For more on IFR and IGFR distributions, see Barlow and Proschan (1996), Lariviere (2006) or Lariviere and Porteus (2001).

We argue that the generic conditions on d and $\mathbf{Z}$ in Proposition 9 are, relative to each other, as general as possible. They are "conditionally necessary" for $\mathcal{E}$ to be monotone in x, in the following weak sense: For an arbitrary, given function $d(p,z)$, if the corresponding LSR elasticity is increasing in x, for *all* IFR distributions $\mathbf{Z}$, then one can show that d_p/d_z must be decreasing in z. And conversely, for an arbitrary, given $\mathbf{Z}$, if $\mathcal{E}(p,x)$ corresponding to $d(p,\mathbf{Z})$ is increasing in x, for *all* demand functions d satisfying the given conditions, then $\mathbf{Z}$ must be IFR. Similar results hold for part (b) of the proposition. So, for example, if all we want to assume about $\mathbf{Z}$ is IGFR, then it is necessary that $\tilde{\epsilon}_{PZ}(p,z)$ be increasing in z, for $\mathcal{E}$ to be increasing in x.

The conditions on d in Proposition 9 say that higher prices, respectively z, increase the relative responsiveness, or sensitivity of demand to changes in price relative to the sales driver z, as measured by $\frac{-d_p}{d_z}$, respectively $\tilde{\epsilon}_{PZ}(p,z) = \frac{-pd_p}{zd_z}$. The next result provides simpler and more intuitive (albeit stronger) sufficient conditions on the riskless demand d, and is a direct consequence of Proposition 9(a).

Corollary 1. *Suppose that $\mathbf{Z}$ is IFR and $d_{zz} \le 0$ and $d_{pz} \le 0$. Then $\mathcal{E}(p,x)$ is increasing in x. If moreover $\epsilon_P(p,x)$ is increasing in p, in particular if $d_p + pd_{pp} \le 0$, then $\mathcal{E}(p,x)$ increases in p.*

Diminishing demand sensitivity in response to a sales driver, $d_{zz} \le 0$, is a common, empirically validated assumption (see e.g. Hanssens et al. 2001, p.95). The condition $d_{pz} \le 0$ states that p and z are strategic demand substitutes, which holds for additive-multiplicative models, and in general for a variety of sales drivers (such as advertising or disposable income), but not all (competitors' prices may be strategic complements). Our results in Section 6.2 suggest that $d_{pz} \le 0$ is necessary for the NVP problem to be well-behaved in other important respects. Proposition 9 relaxes this assumption to include strategic complements, as long as $log(d_p/d_z)$ is decreasing in z. Finally, the increasing (riskless) price-elasticity property (IPE), common in the NVP literature, holds for example if d is concave in p; in particular, $d_p + pd_{pp} \le 0$ implies concavity of the riskless revenue r.

The NVP literature so far has focused almost exclusively on additive and multiplicative demand models. The next result gives necessary and sufficient conditions for the LSR elasticity to be increasing in x in such models, as well as sufficient conditions for monotonicity in p.

Corollary 2. *Consider the additive-multiplicative model $d(p,z) = \alpha(p)z + \beta(p)$, as in (4).*

(a) If $\mathcal{E}(p,x)$ is increasing in x then $\mathbf{Z}$ is IGFR. If $\mathbf{Z}$ is IFR then $\mathcal{E}(p,x)$ is increasing in x; if moreover $p\alpha'(p)$ and $p\beta'(p)$ are decreasing in p then $\mathcal{E}(p,x)$ is also increasing in p.

(b) For multiplicative models ($\beta(p) = 0$), $\mathcal{E}(p,x)$ is increasing in x if and only if $\mathbf{Z}$ is IGFR. If moreover $\alpha(p)$ has increasing elasticity then $\mathcal{E}(p,x)$ is also increasing in p.

(c) For additive models ($\alpha(p)=1$), $\mathcal{E}(p,x)$ is increasing in x if and only if **Z** *is IFR. If moreover $p\beta'(p)$ is decreasing in p then $\mathcal{E}(p,x)$ is also increasing in p.*

5.3 Examples of Demand Models with Monotone LSR Elasticity

For illustration purposes, we consider some relevant demand models and characterize conditions on **Z** for these to have increasing LSR elasticity. We find that increasing LSR elasticity is usually equivalent to IFR or IGFR of **Z**. The results, presented in Table 5, are obtained based on the results of the previous section and/or directly based on the corresponding LSR elasticity expressions (see Appendix). This approach can be used to characterize monotone LSR elasticity conditions for virtually any demand model; for a review of market response models see e.g. Hanssens et al. 2001.

Table 5: Necessary and sufficient conditions on **Z** for increasing LSR elasticity
($*$ conditions are only sufficient)

Demand Model	$d(p,z)$	$\mathcal{E}(p,x)^{\ddagger}$	$\mathcal{E}(p,x)\uparrow x$	$\mathcal{E}(p,x)\uparrow p$
additive linear	$z-bp$	bph_Z	IFR	IGFR
mult. iso-elastic	$ap^{-b}z$	bg_Z	IGFR	IGFR
power	$a\frac{z}{z+p^{b}}$	bg_Z	IGFR	IGFR
logit	$a\frac{e^{z-bp}}{1+e^{z-bp}}$	bph_Z	IFR	IGFR
exponential	e^{z-bp}	bh_Z	IFR	IFR
log	$log_a(z-bp)$	bph_Z	IFR	IGFR*
economic wtp	$zP(\mathbf{W}\geq p)$	$g_W(p)g_Z$	IGFR	IGFR &**W** IGFR*

‡ the (generalized) hazard rate functions are evaluated at $z=z(p,x)$

Attraction models, such as the logit and power models, are among the most commonly used market share models, both in empirical studies and theoretical models, and emerge naturally from intuitive axioms (Bernstein and Federgruen 2004, p.873-874). Attraction models assume a fixed market size from which the firm acquires a market share that is proportional to (an attraction value given by) a function of its price, and possibly other sales drivers such as brand awareness, service level, advertising efforts etc. (Hanssens et al. 2001).

The most prevalent market share model is the logit model ($a\frac{e^{z-bp}}{1+e^{z-bp}}$), used for example in multi-product, competitive and customized pricing settings (Phillips 2005); one of its main advantages is the ease of capturing non-price factors, including order size, product or service quality, competitor's prices and other marketing instruments. An alternative market share model used in the literature is the power model of customer choice $a\dfrac{z}{z+p^b}$ (Phillips 2005, Agrawal and Ferguson 2007). In both models, a is the market size, b is a measure of price sensitivity, and z captures non-price factors affecting demand. These models are not amenable o additive-multiplicative forms, so, albeit widely used in both theory and practice, they are not covered by previous NVP results.

A model that is ubiquitous in the economic and operations literature is the standard heterogeneous willingness-to-pay model, $d(p,z) = zP(\mathbf{W} \geq p)$ (see e.g. Phillips 2005), were $\mathbf{W}$ denotes consumers' willingness-to-pay distribution, and z is the market size. For this example, we assume that demand uncertainty comes from market size, a common assumption e.g. for learning models (contrasting attraction models, where market size is assumed fixed). This leads to a multiplicative demand model. IGFR of the willingness-to-pay distribution is a common assumption in the literature, ensuring regularity of the deterministic objective when market size is known (e.g. Lariviere 2006, Ziya et al. 2004).

We also present conditions for additive linear and multiplicative iso-elastic models, commonly used in the NVP literature (Petruzzi and Dada 1999), as well as exponential and log demand models (see Hanssens et al. 2001). The exponential model, $d(p,z) = e^{z-bp}$, can be transformed into a multiplicative model by a change of variables $v = e^z$; the IFR condition on $\mathbf{Z}$ is equivalent to IGFR of $e^{\mathbf{Z}}$. The log-model is not amenable to additive-multiplicative form.

6. Relation with the Literature and Riskless Price Benchmarks

This section discusses how our results and approach relate to the existing NVP literature. In line with the literature, we briefly investigate the effect of uncertainty on optimal decisions and comparisons with a riskless price benchmark.

6.1 Relation with the Literature

All demand models used in the NVP literature have monotone LSR elasticity. The only exception is Petruzzi and Dada (1999), whose hazard rate condition is not comparable to our increasing LSR elasticity condition, as further explained below.

Yao et al. (2006) obtain uniqueness of the optimal coordinated solution for general multiplicative, respectively additive models assuming that $\mathbf{Z}$ is IGFR, respectively IFR, and price elasticity of $\alpha(p)$, respectively $\beta(p)$, is increasing. Under these assumptions, Corollary 2 ensures that $\mathcal{E}(p,x)$ is monotone in both p and x. Their setup already encompasses most of the existing models in the literature, as summarized in their Tables 1 and 2. Additive models usually assume a linear price dependence and IFR (in particular uniform) risk (e.g. Mills 1959, Lau and Lau 1988, Ha 2001, Zhao and Atkins 2008). Multiplicative models are either iso-elastic with IGFR risk (Wang et al. 2004, Monahan et al. 2004), or general with exponential or uniform (hence IFR) risk (Zabel 1970). All these, as well as the additive-multiplicative model of Young (1978, with logconcave or lognormal risk) satisfy monotone LSR elasticity, in light of Corollary 2.

Petruzzi and Dada (1999) study additive linear and multiplicative iso-elastic demand models. Our results indicate that IGFR of $\mathbf{Z}$ is necessary and sufficient for monotone elasticity (in x for the multiplicative iso-elastic, and in p for the additive linear, see Table 5), so it ensures regularity in both models. Petruzzi and Dada (1999) assume that $\mathbf{Z}$ satisfies $2h_Z^2 + h_Z' \geq 0$, which holds for all IFR distributions, but is not comparable to IGFR. For example, the IGFR Weibull(λ, k) distribution does not satisfy this condition for $k < 1$. Therefore, their results are generally not comparable to our monotone LSR elasticity conditions.

6.2 The Effect of Uncertainty and Riskless Benchmark

One of the issues frequently discussed in the NVP literature is the relationship between the optimal NVP price and a riskless price benchmark (see Petruzzi and Dada 1999). We address this here for completeness, and characterize the effect of changes in uncertainty on optimal prices, under our general demand model. Our results suggest that complementarity properties of riskless demand and revenue/profit drive these results, and explain the effect of uncertainty on optimal prices.

Consider two price benchmarks commonly used in the literature:

- the *riskless price*: $p^0 = \arg\max (p-c)\mathbb{E}[\mathbf{D}(p)]$, maximizes profit for average demand;

- the *base price*: $p_B(z) = \arg\max_p (p-c)\mathbb{E}[d(p, \min(\mathbf{Z}, z))] = \mathbb{E}[\pi(p, \min(\mathbf{Z}, z))]$, maximizes profit from expected sales for a given stocking factor z; here $\pi(p,z) = (p-c)d(p,z)$ denotes the riskless profit function. These extend Petruzzi and Dada's (1999) definitions in our general setting.

Define $p^*(z) = \arg\max \Pi(p, d(p,z)) = \arg\max \mathbb{E}[r(p, \min(\mathbf{Z}, z))] - cd(p,z)$, the optimal price for a given stocking factor z.

PROPOSITION 10. *(a) If $d_{pz} \geq 0$, then $p^*(z) \leq p_B(z)$ for all z, and $p^{**} \leq p^0$.*

(b) If $d_{pz} \leq 0$, then $p^*(z) \geq p_B(z)$ for all z. If moreover $\pi_{pz} \leq 0$, or $d(p,z) = \alpha(p)z$, then $p^{**} \geq p^0$.

The relationship between the optimal NVP price and the riskless price, as identified in the NVP literature, is mixed; Mills (1959) shows that $p^{**} \geq p^0$ for additive models, whereas Karlin and Carr (1962) find the opposite for multiplicative models. Petruzzi and Dada (1999) introduce the base price $p_B^*(z)$ in order to resolve this inconsistency, and show that it is a lower bound on the optimal price (for linear-additive and multiplicative-isoelastic models). Proposition 10(a) indicates that their result extends to general demand models, as long as $d_{pz} \leq 0$. This condition, discussed in Section 5, assumes that p and z are strategic demand substitutes, and holds for all additive-multiplicative demand models. Proposition 10(b) further suggests that Petruzzi and Dada' (1999) base price is not a robust lower bound, and can become an upper bound if p and z are strategic complements for demand (e.g. for $d(p,z) = \log(z - bp)$).

We recover Petruzzi and Dada's (1999) result for additive linear models ($p_B(z) = p^*(z), p^0 \geq p^{**}$) and show that it extends for *all* additive models ($d_{pz} = 0$). We also recover their result for multiplicative iso-elastic models ($p_B(z) \leq p^*(z), p^0 \leq p^{**}$), and show that it extends for *all* multiplicative models. It also extends for additive-multiplicative models with elastic riskless demand (i.e. elasticity of $\alpha(p)$ exceeds 1), and more generally whenever $\pi_{pz} \leq 0$.[3]

We further investigate the sensitivity of optimal prices to changes in uncertainty, as captured by first order dominance and the convex order: $\mathbf{Z}_L \succeq_{FSD} \mathbf{Z}$, is stochastically *L*arger than $\mathbf{Z}$ and $\mathbf{Z}_V \succeq_{CX}$

[3] If $d_{pz} \leq 0$ and $\pi_{pz} \geq 0$, we can show that $p_B(z) \leq p^0$, but the relation between p^0 and p^{**} remains ambiguous.

$\mathbf{Z}$ is more Variable than $\mathbf{Z}$.[4] Let $p_i^*(z) = \arg\max \mathbb{E}[r(p, \min(\mathbf{Z}_i, z))] - cd(p,z)$, $i = L, V$. These results enable the comparison with another riskless benchmark, $p^0(z) = \arg\max pd(p, \min(\mu, z)) - cd(p,z)$, the price which maximizes profit when $\mathbf{Z}$ equals it mean, $\mu = \mathbb{E}[\mathbf{Z}]$.

Proposition 11. *Suppose that $\mathbf{Z}_L \succeq_{FSD} \mathbf{Z}$, and $\mathbf{Z}_V \succeq_{CX} \mathbf{Z}$.*

(a) If $r_{pz} \geq 0$ then $p_L^*(z) \geq p^*(z)$. If in addition $r_{pzz} \leq 0$, then $p_V^*(z) \leq p^*(z) \leq p^0(z)$.

(b) If $r_{pz} \leq 0$ then $p_L^*(z) \leq p^*(z)$. If in addition $r_{pzz} \geq 0$, then $p_V^*(z) \geq p^*(z) \geq p^0(z)$.

We find that higher demand, triggered by a first order increase in the sales driver, leads to higher optimal prices if p and z are strategic revenue-complements (i.e. r is supermodular, $r_{pz} \geq 0$); the relation is reversed, however, for substitutes. The effect of variability, as captured by the convex order, also depends on revenue complementarity between p and z (and a third order condition, which always holds for additive-multiplicative models $r_{pzz} = 0$). Supermodularity of r means that marginal revenue increases with z, i.e. d_z is price inelastic. This holds whenever $d_{pz} \geq 0$, and in particular for all additive models ($r_{pz} = 1$). For multiplicative and additive-multiplicative models, $r_{pz} \geq 0$ whenever (riskless) demand is price inelastic.[5] In particular, we find that $p^*(z) \leq p^0(z)$ for all additive models, and for additive-multiplicative models with inelastic $\alpha(p)$, whereas $p^*(z) \geq p^0(z)$ if $\alpha(p)$ is price elastic.

In summary, the results in this section suggest that the relationship with riskless price benchmarks may not be driven by the effect of price on variance and coefficient of variation of demand, as speculated in the literature (Petruzzi and Dada 1999, p.186-187), but primarily by the nature of substitution between p and z relative to (riskless) demand and profit. In particular, for additive-multiplicative models, the results depend on whether the multiplicative demand component is price elastic or not.

[4] By definition, (1) $\mathbf{X}$ first-order dominates $\mathbf{Y}$ ($\mathbf{X} \succeq_{FSD} \mathbf{Y}$) if $\mathbb{E}[u(\mathbf{X})] \geq \mathbb{E}[u(\mathbf{Y})]$ for all increasing functions u, or equivalently, if $P(\mathbf{X} > t) \geq P(\mathbf{Y} > t)$ for all t; (2) $\mathbf{X}$ dominates $\mathbf{Y}$ in the convex order ($\mathbf{X} \succeq_{CX} \mathbf{Y}$) if $\mathbb{E}[u(\mathbf{X})] \geq \mathbb{E}[u(\mathbf{Y})]$ for all convex functions u. If $\mathbf{X} \succeq_{CX} \mathbf{Y}$, then $\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{Y}]$, and $Var(\mathbf{X}) \geq Var(\mathbf{Y})$ (see Müller and Stoyan 2002).

[5] $r_{pz} \geq 0$ if the elasticity of $\alpha(p)$ is less than 1 ($\alpha(p) + p\alpha'(p) \geq 0$), and $r_{pz} \leq 0$ if the elasticity of $\alpha(p)$ exceeds 1.

7. Conclusion

We introduce a new elasticity concept which provides a framework to study interdependent pricing and inventory decisions with stochatic demand. Focusing on the single product price setting newsvendor model, our key contribution is to characterize general models of stochastic price dependent demand, that guarantee relevant structural properties of the optimal price and inventory policies. These include uniqueness and sensitivity properties of the joint price-inventory solution in a coordinated system, and of the optimal pricing and quantity policies in a sequential decision framework. Our approach is novel, and consists in identifying a new measure of demand elasticity, the elasticity of the lost sales rate, or LSR elasticity, whose properties, in particular monotonicity, drive the desired results. We further characterize general classes of demand models that satisfy these LSR elasticity conditions. These conditions unify, generalize and complement assumptions commonly made in the NVP literature, such as additive-multiplicative models and failure rate conditions. We expect these results to be useful in modeling stochastic, price-dependent demand and solving other types of inventory-pricing problems. For example, Kocabiyikoglu and Popescu (2007) extend the current results in a more complex operational context, including flexible manufacturing and revenue management.

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Appendix

Proof of Proposition 1. The first order condition with respect to x gives the well known equation (6). Equation (7) states the first order condition with respect to p. To see this, write (2) as

$R(p,x) = p\int_0^x q(p,v)dv$, and hence

$$R_p(p,x) = \int_0^x (q(p,v) + pq_p(p,v))\, dv = \int_0^x q(p,v)(1-\mathcal{E}(p,v))dv, \tag{8}$$

where the second equality is obtained using (5). $\square$

Proof of Proposition 2. Marginal revenue is $R_x(p,x) = pq(p,x)$. Taking its derivative with respect to p, we obtain:

$$R_{xp} = q(p,x) + pq_p(p,x) = q(p,x)(1-\mathcal{E}(p,x)), \tag{9}$$

where the second equality is obtained using (5). It follows that $R_{xp} \le 0$ whenever $\mathcal{E} \ge 1$, and $x^*(p)$ and $p^*(x)$ are decreasing in their respective arguments by Topkis' Theorem. $\square$

Proof of Proposition 3. The implicit function theorem implies $\dfrac{\partial p^*(x)}{\partial x} = \dfrac{-\Pi^*_{xp}(x)}{\Pi^*_{pp}(x)}$. The denominator is negative by second order condition. It remains to show that the numerator is negative under the conditions of the proposition. Using (9), we write,

$$\Pi^*_{xp}(x) = q^*(x)(1-\mathcal{E}^*(x)) \le 0. \tag{10}$$

The inequality is implied by the assumption of the proposition. This implies $\dfrac{-\Pi^*_{xp}(x)}{\Pi^*_{pp}(x)} \le 0$, i.e. $p^*(x)$ is decreasing in x, whenever $\mathcal{E}^*(x) \ge 1$. The second part is analogous. $\square$

Proof of Proposition 4(a). Write $R_p(p,x) = \int_0^x Q(p,v)dv$, where $Q(p,x) = q(p,x)(1-\mathcal{E}(p,x))$. Because $\mathcal{E}(p,x)$ is increasing in x and $q(p,x) \ge 0$, for any p, $Q(p,x)$ crosses zero at most once, and from above. Therefore the first order condition $\int_0^x Q(p^*(x),v)dv = 0$, implies that $Q^*(x) = Q(p^*(x),x) \le 0$, that is $\mathcal{E}^*(x) \ge 1$. The result follows by Proposition 3. $\square$

The proof of part (b) of Proposition 4 relies on the following lemma:

LEMMA 4. *For all $p \ge p_L$, $p_L q(p_L, x^*(p)) \ge c$. Moreover, $\mathcal{E}^*(p_L) = 1$ or $p_L = c$.*

Proof. By definition, $p_L = \arg\max\{d(p,\bar{\Phi}^{-1}(c/p)); p \ge c\} = \arg\max\{d(p,z^*(p)); p \ge c\} = \arg\max\{x^*(p); p \ge c\}$. Hence, $x^*(p) \le x^*(p_L)$ for all $p \ge p_L$. It follows that for all $p \ge p_L$, $q(p_L, x^*(p)) \ge q(p_L, x^*(p_L)) = \dfrac{c}{p_L}$. For the second part, if $p_L > c$ then p_L is an interior maximizer of $x^*(p)$. So, by the implicit function theorem, $\Pi^*_{xp}(p_L) = 0$, which amounts to $\mathcal{E}^*(p_L) = 1$. $\square$

Proof of Proposition 4(b). The marginal revenue condition (6) states that the following expression, evaluated at $x^*(p)$, equals zero:

$$\begin{aligned} pq(p,x)-c &= \int_{p_L}^{p} \frac{\partial}{\partial v}(vq(v,x)-c)dv + p_L q(p_L,x)-c \\ &= \int_{p_L}^{p} q(v,x)(1-\mathcal{E}(v,x))dv + (p_L q(p_L,x)-c)\,. \end{aligned}$$

By the first part of Lemma 4, the second term, evaluated at $x^*(p)$, is non-negative. So the first must be non-positive at $x^*(p)$, i.e. $\int_{p_L}^{p} Q(v,x^*(p))dv \le 0$, where $Q(p,x) = q(p,x)(1-\mathcal{E}(p,x))$. Because $\mathcal{E}(p,x)$ increases in p, and $q(p,x)\ge 0$, $Q(p,x)$ crosses zero at most once, and from above, as p increases. Therefore $Q(p,x^*(p)) \le 0$, i.e. $\mathcal{E}^*(p)\ge 1$. The rest follows from Proposition 3. $\square$

Proof of Proposition 5. We show that the Hessian matrix of $\Pi(p,x)$ is negative semi-definite. The second order derivatives are:

$$\Pi_{xx}(p,x) = -pf(p,x), \tag{11}$$

$$\Pi_{xp}(p,x) = q(p,x) + pf(p,x)d_p(p,z), \tag{12}$$

$$\Pi_{pp}(p,x) = \mathbb{E}\left[r_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] - pf(p,x)d_p^2(p,z(p,x)), \tag{13}$$

where $\mathbf{\Omega} = \mathbf{\Omega}(p,x) = (\mathbf{D}(p)\le x)$ defines the event of no lost sales (arguments are omitted for notational convenience), and we use the standard notation $\mathbb{E}[\mathbf{A};\mathbf{B}] = \mathbb{E}[\mathbf{A}|\mathbf{B}]P(\mathbf{B})$. Equation (13) is obtained by differentiating twice with respect to p the objective in (1), written as:

$$\Pi(p,x) = pxq(p,x) + \mathbb{E}\left[r(p,\mathbf{Z});\mathbf{\Omega}\right] - cx, \tag{14}$$

Because $\Pi_{xx}\le 0$, and $\Pi_{pp}\le 0$ (from concavity of r), it remains to check that the determinant of the Hessian, $\Delta(p,x)$, is positive. Pairing up terms, we obtain:

$$\Delta(p,x) = \Pi_{xx}(p,x)\Pi_{pp}(p,x) - \Pi_{xp}^2(p,x) \tag{15}$$

$$= -pf(p,x)\mathbb{E}\left[r_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] - q(p,x)\left[q(p,x) + 2pf(p,x)d_p(p,z(p,x))\right]. \tag{16}$$

Using the equivalent expression for LSR elasticity obtained in Lemma 2(a), we can write (16) as:

$$\Delta(p,x) = -pf(p,x)\mathbb{E}\left[r_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] + q(p,x)^2(2\mathcal{E}(p,x)-1). \tag{17}$$

The first term is positive by concavity of r and the second because $\mathcal{E}\ge 1/2$. $\square$

Proof of Proposition 6. By the envelope theorem, we have $\frac{\partial^2}{\partial x^2}\Pi^*(x) = \Pi_{xx}(p,x) - \left.\frac{\Pi_{xp}^2(p,x)}{\Pi_{pp}(p,x)}\right|_{p=p^*(x)}$. From the second order condition, $\Pi_{pp}^*(x) < 0$. Using (17), we write:

$$\Delta^*(x) = \Pi_{xx}^*(x)\Pi_{pp}^*(x) - \Pi_{xp}^*(x)^2 = -pf^*(x)\mathbb{E}\left[r_{pp}^*;\mathbf{\Omega}^*\right] - q^*(x)^2(2\mathcal{E}^*(x)-1),$$

which is positive, whenever $\mathcal{E}^*(x) \geq 1/2$ and $r_{pp} \leq 0$. It follows that $\frac{\partial^2}{\partial x^2}\Pi^*(x) \leq 0$, and $\Pi^*(x)$ is concave in x. Part (b) of the proposition is proved analogously. By Propositions 3 and 4, $\mathcal{E}(p,x)$ increasing in p or x guarantees the LSR elasticity bound required by parts (a) and (b) of the proposition, which settles part (c). □

Proof of Remark 1. The results follows from the proofs of Propositions 5 and 6 because $r_{pp} = 0$. □

Proof of Proposition 7. (a) By definition, $x^{**}(c) = \arg\max_x \Pi(p^*(x,c),x,c) = \arg\max_x \Pi^*(x,c)$. So it is sufficient, by Topkis' Theorem, to show $\Pi^*(x,c)$ is submodular in (x,c). By the envelope theorem, $\frac{\partial \Pi^*(x,c)}{\partial x} = \Pi_x(p^*(x,c),x,c)$. From (7), $p^*(x,c) = p^*(x)$ is independent of cost, and so:

$$\frac{\partial}{\partial c}\frac{\partial \Pi^*(x,c)}{\partial x} = \frac{\partial}{\partial c}\Pi_x(p^*(x,c),x,c) = \Pi_{xc}(p^*(x),x,c) = -1,$$

where the last part follows by taking the derivative of $\Pi_x(p,x) = pq(p,x) - c$ with respect to c. It follows that $x^{**}(c)$ is decreasing in c.

(b) From (7), $p^*(x,c) = p^*(x)$ is independent of cost, so $p^{**}(c) = p^*(x^{**}(c))$, i.e. the optimal price is influenced by changes in c only through $x^{**}(c)$. We obtain: $\frac{\partial p^{**}(c)}{\partial c} = \frac{\partial p^*(x^{**}(c))}{\partial x}\frac{\partial x^{**}(c)}{\partial c}$. The second term is negative from part (a), so monotonicity of $p^{**}(c)$ in c is driven solely by monotonicity of $p^*(x)$ in x, for which $\mathcal{E}(p,x)$ increasing in x is sufficient, by Proposition 4. □

Proof of Lemma 2. Part (a) is obtained by writing $F_p(p,x) = \frac{\partial}{\partial p}P(d(p,\mathbf{Z}) \geq x) = -f(p,x)d_p(p,z(p,x))$ in the definition (5). The other two parts follow from (a) using the definitions and relationships in Table 4, and the elasticity definitions. □

Proof of Proposition 8. Because $\mathcal{E}(p,x) = p\frac{-q_p(p,x)}{q(p,x)} = -p\frac{\partial}{\partial p}\log q(p,x)$, we obtain

$$\frac{\partial}{\partial x}\frac{\mathcal{E}(p,x)}{p} = -\frac{\partial}{\partial p}\frac{\partial}{\partial x}\log q(p,x) = \frac{\partial}{\partial p}\frac{f(p,x)}{q(p,x)} = \frac{\partial}{\partial p}h_D(p,x).$$

Therefore $\mathcal{E}$ increasing in x is equivalent to h_D increasing in p. The second part follows from Lemma 2(b). $\square$

Proof of Lemma 3. Differentiating $\tilde{\mathcal{E}}(p, z(p,x)) = \mathcal{E}(p,x)$ on both sides with respect to x, respectively p we have: $\mathcal{E}_x(p,x) = \tilde{\mathcal{E}}_z(p, z(p,x))z_x(p,x)$, respectively $\tilde{\mathcal{E}}_z(p, z(p,x))z_p(p,x) + \tilde{\mathcal{E}}_p(p, z(p,x)) = \mathcal{E}_p(p,x)$. The result follows because $z_x(p,x) \geq 0$, and $z_p(p,x) \geq 0$, by Lemma 1(a). $\square$

Proof of Proposition 9. (a) From Lemma 2(c),

$$\frac{\partial \tilde{\mathcal{E}}(p,z)}{\partial z} = -ph'_Z(z)\frac{d_p}{d_z} - ph_Z(z)\frac{\partial}{\partial z}\frac{d_p}{d_z}.$$

This, and Lemma 3(a) imply $\mathcal{E}(p,x)$ is increasing in x when $h_Z(z)$ is increasing and $\frac{d_p}{d_z}$ is decreasing in z. The second part follows from $\tilde{\mathcal{E}}_p(p,z) = h_Z(z)\frac{\partial}{\partial p}\frac{pd_p}{d_z}$ via Lemma 3(b) because $\frac{pd_p}{d_z}$ is decreasing in p. Part (b) follows from Lemma 2(c) and Lemma 3(a) by writing

$$\frac{\partial \tilde{\mathcal{E}}(p,z)}{\partial z} = pg'_Z(z)\tilde{\epsilon}_{PZ}(p,z) + pg_Z(z)\frac{\partial \tilde{\epsilon}_{PZ}(p,z)}{\partial z}. \quad \square$$

Proof of Corollary 2. We focus on $\mathcal{E}(p,x)$ increasing in x, which, by Proposition 8, is equivalent to $h_D(p,x)$ is increasing in p. Throughout the proof we use the general fact that $z_p \geq 0$, by Lemma 1(a). For additive-multiplicative models, $z(p,x) = \frac{x-\beta(p)}{\alpha(p)}$, and from Table 4:

$$h_D(p,x) = \frac{h_Z(z(p,x))}{d_z(z(p,x))} = \frac{h_Z(z(p,x))}{\alpha(p)} = \frac{g_Z(z(p,x))}{x-\beta(p)}. \tag{18}$$

(a) If $\mathbf{Z}$ is IFR, because $z_p \geq 0$, and α is decreasing, we obtain

$$\frac{\partial}{\partial p}h_D(p,x) = h'_Z(z(p,x))\frac{z_p(p,x)}{\alpha(p)} - h_Z(z(p,x))\frac{\alpha'(p)}{\alpha^2(p)} \geq 0.$$

On the other hand, because $z_p \geq 0$ and β is decreasing, if

$$\frac{\partial}{\partial p}h_D(p,x) = \frac{\partial}{\partial p}\frac{g_Z(z(p,x))}{x-\beta(p)} = g'_Z(z(p,x))\frac{z_p(p,x)}{x-\beta(p)} + g_Z(z(p,x))\frac{\beta'(p)}{(x-\beta(p))^2} \geq 0,$$

then g_Z must be increasing, i.e. $\mathbf{Z}$ is IGFR.

(b) For the multiplicative case, $\beta(p) = 0$, so $h_D(p,x) = \frac{1}{\alpha(p)}h_Z(\frac{x}{\alpha(p)}) = \frac{1}{x}g_Z(z(p,x))$. Again, because $z_p \geq 0$, we have that h_D is increasing in p if and only if g_Z is increasing, i.e. $\mathbf{Z}$ is IGFR.

(c) For the additive case, $\alpha(p) = 1$, so $\frac{\partial}{\partial p} h_D(p,x) = h_Z'(z(p,x)) z_p(p,x)$. Because $z_p \geq 0$, h_D is increasing in p if and only if h_Z is increasing, i.e. $\mathbf{Z}$ is IFR.

The conditions for $\mathcal{E}(p,x)$ increasing in p follow directly from Corollary 1 for parts (a) and (c), whereas part (b) follows from Proposition 9(b). $\square$

Proof of the Results in Table 5. For additive-linear models, $z(p,x) = x + bp$, and $\mathcal{E}(p,x) = bph_Z(bp+x) = \frac{bp}{bp+x} g_Z(x+bp)$. The first expression shows that this is increasing in x iff h_Z is increasing (as also implied by Corollary 2). The last expression indicates that g_Z increasing implies $\mathcal{E}$ increasing in p, because $\frac{bp}{bp+x}$ is increasing in p. Setting $x = 0$, we see that IGFR is also necessary for $\mathcal{E}$ increasing in p.

For the multiplicative iso-elastic model, $z(p,x) = \frac{xp^b}{a}$ and $\mathcal{E}(p,x) = bg_Z(\frac{xp^b}{a})$. This is increasing in x, respectively p if and only if g_Z is increasing.

For the power model, $z(p,x) = \frac{xp^b}{a-x}$ and $\mathcal{E}(p,x) = bg_Z(\frac{xp^b}{a-x})$. This is increasing in x, respectively p, if and only if g_Z is increasing (because its argument is increasing in both x and p).

For the logit model, $z(p,x) = bp + ln\frac{x}{a-x}, x \leq a$, and $\mathcal{E}(p,x) = bph_Z(bp + ln\frac{x}{a-x}) = \frac{bp}{bp+ln\frac{x}{a-x}} g_Z(x+bp)$, where $ln = log_e$. The first expression shows that this is increasing in x iff h_Z is increasing, because the argument of h_Z is increasing in x. The second indicates that g_Z increasing implies $\mathcal{E}$ increasing in p, because the fraction preceding g_Z is increasing in p. Setting $x = a/2$, we see that IGFR is also necessary for $\mathcal{E}$ increasing in p.

For the exponential, $z(p,x) = bp + lnx$ and $\mathcal{E}(p,x) = bh_Z(bp + lnx)$. This is increasing in x, respectively p iff h_Z is increasing.

For the base-a *log* model, $z(p,x) = bp + a^x$ and $\mathcal{E}(p,x) = bph_Z(bp + a^x) = \frac{bp}{bp+a^x} g_Z(bp + a^x)$. The first expression shows that this is increasing in x iff h_Z is increasing. The second indicates that g_Z increasing implies $\mathcal{E}$ increasing in p, because the fraction in front of g_Z is increasing in p.

For the economic wtp model, the result follows from Corollary 2(b); the elasticity of the riskless demand $\alpha(p) = P(\mathbf{W} \geq p)$, is precisely the GFR of the willingness to pay distribution $\mathbf{W}$. $\square$

The proof of Proposition 10 relies on parts (a) and (b) of the following additional lemma; part (c) is provided for completeness.

LEMMA 5. *(a) If $\pi_{pz} \leq 0$, the optimal base price $p_B(z)$ is decreasing in z.*

(b) If $\pi_{pz} \geq 0$, the optimal base price $p_B(z)$ is increasing in z. If moreover $d_{pz} \leq 0$, then the optimal price $p^*(z)$ is increasing in z.

Proof. For $p_B(z)$, by definition, $p_B(z)$ optimizes $\tilde{\Pi}^B(p,z) = (p-c)\mathbb{E}[d(p,\min(\mathbf{Z},z))] = \mathbb{E}[\pi(p,\min(\mathbf{Z},z))]$. So $\tilde{\Pi}^B_{pz}(p,z) = \pi_{pz}(p,z)\bar{\Phi}(z)$, and the sign of π_{pz} dictates monotonicity of $p_B(z)$, by Topkis' Theorem.

For $p^*(z)$, letting $\tilde{\Pi}(p,z) = \Pi(p,d(p,z)) = \mathbb{E}[r(p,\min(\mathbf{Z},z)] - cd(p,z)$, we have $\tilde{\Pi}_{pz}(p,z) = r_{pz}(p,z)\bar{\Phi}(z) - cd_{pz}(p,z) = \pi_{pz}(p,z)\bar{\Phi}(z) - cd_{pz}(p,z)\Phi(z) \geq 0$, when $\pi_{pz} \geq 0$ and $d_{pz} \leq 0$. So $p^*(z)$ is increasing in z by Topkis' Theorem. $\square$

Proof of Proposition 10. For part (a), by definition, $p_B(z)$ solves $\mathbb{E}[\pi_p(p,\min(\mathbf{Z},z))] = 0$, so

$$\tilde{\Pi}_p(p,z)\Big|_{p=p_B(z)} = c\mathbb{E}[d_p(p_B,\min(\mathbf{Z},z)) - d_p(p_B,z)] \leq 0,$$

because $d_p(p,z)$ is increasing in z. It follows that $p^*(z) \leq p_B(z)$. For the second part, we write $p_B(\infty) = \arg\max_p (p-c)\mathbb{E}[d(p,\min(\mathbf{Z},\infty))] = \arg\max_p (p-c)\mathbb{E}[\mathbf{D}(p)] = p^0$, by abuse of notation. Therefore, for the optimal $z^{**} = \arg\max \tilde{\Pi}(p^*(z),z)$, we have

$$p^0 = p_B(\infty) \geq p_B(z^{**}) \geq p^*(z^{**}) = p^{**}, \tag{19}$$

where the first inequality follows from Lemma 5(b) ($d_{pz} \geq 0$ implies $\pi_{pz} = (p-c)d_{pz} + d_z \geq 0$ since $d_z \geq 0$), and the second inequality from the first part of the proposition.

The proof of part (b) is analogous to (a), yielding the opposite inequalities than in (19), based on Lemma 5(a), which relies on submodularity of π. The result for multiplicative demand follows directly from the first order conditions. Indeed, $p^*(z)$ solves $\pi'(p)\mathbb{E}[\min(\mathbf{Z},z)] - c\alpha'(p)\mathbb{E}[z-\mathbf{Z}]^+ = 0$. The left hand side evaluated at p^0 is positive because α is decreasing, and $\pi'(p^0) = 0$ (by optimality of p^0) so $p^0 \leq p^*(z)$, in particular $p^0 \leq p^{**}$. $\square$

Proof of Proposition 11. From Topkis' Theorem, for the first part, it is sufficient to show that $\mathbb{E}[r(p_2,\min(\mathbf{Z}_L,z)) - r(p_1,\min(\mathbf{Z}_L,z))] \geq \mathbb{E}[r(p_2,\min(\mathbf{Z},z)) - r(p_1,\min(\mathbf{Z},z))]$ for $p_2 \geq p_1$. If r_p is increasing in z, then for arbitrarily fixed z, $u(y) = r(p_2,\min(y,z)) - r(p_1,\min(y,z))$ is increasing in y. Because $\mathbf{Z}_L \succeq_{FSD} \mathbf{Z}$, it follows that $\mathbb{E}[u(\mathbf{Z}_L)] \geq \mathbb{E}[u(\mathbf{Z})]$, which completes the proof. For the second part, if r_p is increasing and concave in z, then for arbitrarily fixed z, $u(y) = r(p_2, \min(y,z)) - r(p_1, \min(y,z))$ is concave in y. Because $\mathbf{Z}_V \succeq_{CX} \mathbf{Z}$, it follows that $\mathbb{E}[u(\mathbf{Z}_V)] \leq \mathbb{E}[u(\mathbf{Z})]$. Because $\mathbf{Z} \succeq_{CX} \mu$, these results enable the comparison of $p^*(z)$ and $p^0(z)$. Part (b) is proved analogously. $\square$

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