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We study the dynamic pricing implications of a new, behaviorally motivated reference price mechanism, based on the peak-end memory model of Fredrickson and Kahneman (1993). This model suggests that consumers anchor on a reference price which is a weighted average of the lowest and most recent prices. Loss averse consumers are more sensitive to perceived losses than gains relative to this reference price. We find that a range of constant pricing policies is optimal for the corresponding dynamic pricing problem. This range is wider the more consumers anchor on lowest prices, and persists when buyers are loss neutral, in contrast with previous literature. In a transient regime, the optimal pricing policy is monotone, and converges to a steady state price, which is lower the more extreme and salient the low-price anchor is. Our results suggest that behavioral regularities, such as peak-end anchoring and loss aversion, limit the benefits of varying prices, and caution that the adverse effects of deep discounts on the firm's optimal prices and profits may be more enduring than previous models predict.

Key words: dynamic pricing, dynamic programming, consumer behavior, peak-end pule, prospect theory.

1. Introduction

In April 2009, Apple increased the price of its best-selling iTunes tracks from \$.99 to \$1.29, resulting in an abrupt sales decline in excess of 20% (Billboard 2009). Almost a year later, the slowdown in iTunes sales seems more enduring than expected (The Wall Street Journal 2010). Arguably, \$1.29 might just be too much to pay for a song. Yet, a plausible reason for the customer pushback may lie beyond simple (or downturn) economics: "people have been trained that a song, even a popular song, is worth \$.99" (True/Slant 2009), and are "reluctant to pay thirty cents more for something that cost a buck yesterday" (The Unofficial Apple Weblog 2010). Could it be that Apple (who had cleverly played the consumer anchoring card with the introduction of the iPhone) may have

underestimated the long-run effect of the low ($.99) iTunes price anchor on buyer behavior?

In repeat-purchase markets, consumers form price expectations, also known as reference prices. Prices are perceived as discounts or surcharges relative to these reference prices, and this perception affects demand and profitability. For example, while a price promotion may have a short-run positive impact on sales, the lowered price may result in the installation of a new lower minimum price in consumers' memory, eroding price expectations and willingness to pay, and thereby negatively affecting profitability on the long run. This suggests that it is important for a firm to understand (1) how its pricing policy affects consumers' price expectations and purchase decisions, and (2) how to set prices over time to maximize profitability in this context.

The literature on pricing with reference effects (Kopalle, Rao and Assuncao 1996, Fibich, Gavious and Lowengart 2003, Heidhues and Kőszegi 2005, Popescu and Wu 2007) assumes that consumers' price expectations follow an exponentially smoothed adaptive expectations process (Mazumdar, Raj and Sinha 2005). Under adaptive expectations, consumers anchor on a weighted average of all past prices, reflecting a fast-decaying memory process. Despite its prevalence in analytical pricing research, this model is "not very well grounded in behavioral evidence" (Rabin 1996, p.8).

We introduce a new internal (memory-based) reference price model, based on the *peak-end rule* (Fredrickson and Kahneman 1993), which is supported by extensive research in psychology, and provides a behaviorally compelling alternative to adaptive expectations. In the pricing context, this model suggests that consumers remember the lowest (salient) and most recent prices, and the reference price is a weighted average of the two. Our goal is to operationalize the peak-end rule in a dynamic pricing context, and understand how this consumer memory and anchoring process, combined with behavioral decision processes (e.g. loss aversion), influences demand and the optimal pricing strategies of the firm.

A large body of research in psychology suggests that memory, and hence remembered utility follows a "snapshot model" (or "moment based approach"), whereby the overall evaluation of past experiences is based only on a few salient moments, rather than a cumulative measure of all past experiences (Kahneman 2000). Moreover, evidence suggests that the most salient moments

are the peak and the end (i.e. the most extreme and most recent experiences), and memory-based judgment tasks are based on a combination (weighted average) of the two. This peak-end memory model, proposed by Fredrickson and Kahneman (1993), finds vast empirical support in the psychology literature, both for positive and negative experience contexts; see Fredrickson (2000) and Kahneman (2000) for reviews.

In the pricing context, we posit that the representative peak-end moments in reference price formation are associated with the lowest and the last price, i.e. the highest and the most recent transaction utility, consistent with the peak-end rule for positive experiences. A model where consumers anchor on the highest and last prices, appropriate in negative experience contexts (such as paying fines or taxes), is studied in Nasiry (2010). While an empirical investigation of the peak-end rule in the pricing context is still lacking, several studies find support for anchoring on most recent prices (e.g. Krishnamurthi, Mazumdar and Raj 1992, Chang, Siddarth and Weinberg 1999) and extreme prices (e.g. Nwokoye 1975, Niedrich, Sharma and Wedell 2001). Our single-brand model is consistent with Rajendran and Tellis (1994), who find empirical support for a reference price which is a combination of the lowest price across brands and the brand's last price. Anchoring on low prices is also motivated by perception of fairness (Xia, Monroe and Cox 2004, Mazumdar, Raj and Sinha 2005).

In a review and analysis of various reference price conceptualisations, Lowengart (2002) suggests that memory-based behavioral reference prices, such as the one proposed here, are most appropriate for frequently purchased experienced-quality goods (e.g. food, fragrance, music). In such a context, he suggests that "marketers would probably benefit from a constant price strategy, such as Every Day Low Prices, (EDLP), and should not use heavy price promotion tactics [...], to avoid establishing a low reference price in consumers minds [and] the loss effects that will appear after a price promotion" (p.163). We investigate the validity of these insights in an analytical framework.

Our paper contributes to a growing body of behavioral operations literature, reviewed e.g. by Loch and Wu (2007). Related work that incorporate consumer learning and memory models in various operational contexts include Gaur and Park (2007; consumers learn fill rates), Liu and van

Ryzin (2007; consumers learn about rationing risk), and Ovchinnikov and Milner (2005; consumers learn about the likelihood of last-minute sales). Specifically, our work builds on the literature on dynamic pricing with reference effects. Kopalle, Rao and Assuncao (1996), and Fibich, Gavious and Lowengart (2003) show monotonicity and convergence of the optimal price paths under a piecewise linear demand model. Popescu and Wu (2007) extend these findings to general demand functions and reference effects. These papers all assume that consumer learning follows a fast decaying, exponentially smoothed process. In contrast, to the best of our knowledge, our paper is the first to model a behaviorally established memory process, based on the peak-end rule, in an operational context.

From a methodological standpoint, the peak-end memory model makes the analysis of the dynamic pricing problem substantially more involved. The problem amounts to solving a dynamic program with non-smooth reward and transition functions, for which we develop a non-standard approach, inter-playing bounding and variational techniques.

Our results indicate that a constant pricing policy is optimal for a range of relatively low initial price expectations. This range of steady states is wider, the more loss averse consumers are, and the more they are sensitive to the low price anchor. Unlike with exponential smoothing, this range persists even when consumers are loss neutral (i.e. equally sensitive to discounts and surcharges), due to the asymmetric anchoring process. Overall, our results suggest that behavioral asymmetries in anchoring and decision processes make constant pricing (and EDLP) more prevalent, partially confirming Lowengart's (2002) intuition cited earlier.

When consumer price expectations are outside the steady state range, firms should use skimming or penetration strategies to gradually manage price perception. Behavioral regularities lead prices to converge on the long run to a steady state. Unlike exponential smoothing, under the peak-end rule, price monotonicity is a robust result, extending for general, non-linear reference effects (see Nasiry 2010). In our case, however, the firm may benefit from alternating consumers' perception of gains and losses over time. Unlike exponential smoothing, the value of the steady state price depends on initial low price expectations, suggesting that their impact is more lasting than previously

understood. Further, the more salient the lowest price, the lower the optimal prices and profits. In the example of iTunes, our insights suggest that long term effects on prices and profits are to be expected if the \$.99 price-anchor is engrained in consumer expectations; these effects can be best mitigated by gradually increasing prices to manage expectations.

Overall, our results suggest that consumer memory and anchoring processes, in addition to decision processes, are crucial in determining how firms should manage price changes over time.

2. Model and Preliminary Results

This section describes how consumers make purchase decisions based on prices and reference prices, and how this decision affects the demand for a firm's product.

Mental accounting theory (Thaler 1985) posits that the utility from a purchase experience consists of two components: acquisition utility and transaction utility. The former reflects the monetary value of the product, whereas the latter corresponds to the psychological value of the deal, determined by the gap $x = r - p$ between the reference price, r, and the price, p.

In a deterministic context, prospect theory (Tversky and Kahneman 1991) identifies key properties of the transaction utility, which are inherited by the aggregate reference dependent demand, and validated empirically in the pricing context (Kalyanaram and Winer 1995). Specifically, demand increases in the magnitude of the gap $x = r - p$ (reference dependence), and it is more sensitive to perceived surcharges than discounts of the same magnitude (loss aversion). For simplicity of exposition, we assume that consumers' marginal sensitivity to perceived discounts, respectively surcharges, is constant, i.e. reference effects are piecewise linear (our insights extend to non-linear effects, see Nasiry 2010). This motivates the following demand model:

$$d(p,r) = d_0(p) - \lambda(p-r)^+ + \gamma(r-p)^+ = \begin{cases} d_0(p) + \lambda(r-p), & \text{if } p \geq r; \\ d_0(p) + \gamma(r-p), & \text{if } p \leq r. \end{cases} \tag{1}$$

Loss aversion is captured by $\lambda \geq \gamma > 0$. We assume that the base demand, $d_0(p)$, is non-negative, bounded, continuously differentiable and decreasing in price, and the base profit $\pi_0(p) = p\, d_0(p)$ is non-monotone and strictly concave. The firm's short term profit is denoted $\pi(p,r) = p\, d(p,r)$; all our results extend for a non-zero marginal cost c.

The peak-end rule (Kahneman et al. 1993) suggests to model consumers' reference price, r_t, at any time t as a weighted average of the minimum price, m_{t-1}, and the most recent price, p_{t-1}:

$$r_t = \theta m_{t-1} + (1-\theta)p_{t-1}, \;\; m_{t-1} = \min(m_{t-2}, p_{t-1}), \;\; t > 1, \tag{2}$$

where $\theta \in (0,1]$ captures how much consumers anchor on the lowest price.

Given initial conditions m_0 and p_0, the firm maximizes infinite horizon β-discounted revenues:

$$J(m_0, p_0) = \max_{p_t \in \mathbf{P}} \sum_{t=1}^{\infty} \beta^{t-1} \pi(p_t, \theta m_{t-1} + (1-\theta)p_{t-1}), \text{ where } m_t = \min(m_{t-1}, p_t).$$

Here $\beta \in (0,1)$, and prices are confined to a bounded interval $\mathbf{P} = [0, \overline{p}]$, where, for simplicity, $d_0(\overline{p}) = 0$ (to avoid trivial boundary solutions). The infinite horizon model implicitly assumes that lowest prices can be remembered indefinitely. While this is a reasonable approximation in a context where the frequency of transactions is high relative to the horizon length, our insights remain valid when modeling the possibility of forgetting or updating the minimum price (Nasiry 2010).

The Bellman Equation for this problem is:

$$J(m_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ \pi(p_t, r_t) + \beta J\Big(\min(p_t, m_{t-1}), p_t \Big) \Big\}, \text{ where } \; r_t = \theta m_{t-1} + (1-\theta)p_{t-1} \tag{3}$$

Intuitively, we expect that higher reference prices (i.e. memory of higher prices) should enable the firm to extract higher profits from the market. All proofs are in the Appendix.

Lemma 1. *The value function, $J(m,p)$, is increasing in both arguments.*

Defining $\pi_k(p,r) = \Big[d_0(p) + k(r-p) \Big] p = \pi_0(p) + k(r-p)p$, for $k \in \{\lambda, \gamma\}$, loss aversion ($\lambda \geq \gamma$) implies the following result, essential for our future developments.

Lemma 2. *The short-term profit, $\pi(p,r) = \min\Big(\pi_\lambda(p,r), \pi_\gamma(p,r)\Big)$, is supermodular in (p,r).*

By Topkis' Theorem (Topkis 1998, Theorem 2.8.2), Lemma 2 confirms the intuition that myopic firms, i.e. those focused on short term profits ($\beta = 0$), should charge higher prices when consumers have higher price expectations. Lemma 2 allows us to write the Bellman Equation as follows:

$$J(m_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ \min(\pi_\lambda, \pi_\gamma)(p_t, r_t) + \beta J\Big(\min(p_t, m_{t-1}), p_t \Big) \Big\}; \; r_t = \theta m_{t-1} + (1-\theta)p_{t-1}. \tag{4}$$

3. Steady States

In this section, we characterize the steady state prices (i.e. the optimal constant price policies) of the firm facing loss averse consumers with demand given by (1). Identifying the steady states of Problem (4) requires a non-standard approach, because, in this problem, both the short-term profit and the transition in the value function (memory structure) are non-smooth. Our analysis is based on a bounding technique, which identifies the steady states of Problem (4) based on those of a series of smooth problems, for which standard methods can be applied.

For $\nu \in [0,1]$, and $m \in \mathbf{P}$, consider the following smooth problem with one-dimensional state:

$$J_m^\nu(p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ (1-\nu)\pi_\lambda(p_t, \theta m + (1-\theta)p_{t-1}) + \nu \pi_\gamma(p_t, p_{t-1}) + \beta J_m^\nu(p_t) \Big\}. \tag{5}$$

We first show that the family J_m^ν, $\nu \in [0,1]$, provides upper bounds for the value function J.

Lemma 3. *For any $m \le p$, we have $J(m,p) \le J_m^\nu(p)$.*

We next argue that by approximating the value function J by a smooth upper bound J_m^ν, for an appropriate subset of values ν, the firm will charge optimal prices on the long run. Technically, this amounts to matching supergradients of the original problem with gradients of a smooth upper bound Problem (5) for an appropriate value ν.

We first identify steady states of Problem (5) which will help characterize those of Problem (4). It is useful to consider three price-memory scenarios (low, medium and high): $\mathbf{R}_1 = [0, \underline{m}]$, $\mathbf{R}_2 = [\underline{m}, \overline{m}]$, and $\mathbf{R}_3 = [\overline{m}, \overline{p}]$, where the thresholds $\underline{m} = \underline{m}(\lambda, \theta)$ and $\overline{m} = \overline{m}(\gamma)$ solve respectively:

$$\pi_0'(p) - \lambda(1-\beta(1-\theta))p = 0, \tag{6}$$
$$\pi_0'(p) - \gamma(1-\beta)p = 0. \tag{7}$$

Uniqueness of $\underline{m}$ and $\overline{m}$ follows because the above left hand sides (LHS) are strictly decreasing in p, by strict concavity of π_0. Moreover, $\underline{m} \le \overline{m}$ because $\lambda \ge \gamma > 0$ and $1-\beta(1-\theta) \ge 1-\beta$.

Lemma 4. *(a) For $\nu \in [0,1]$ and $m \in \mathbf{P}$, Problem (5) admits a unique steady state, which solves:*

$$\pi_0'(p) - \Big[\lambda(1-\nu)(2-(1-\theta)(1+\beta)) + \nu\gamma(1-\beta)\Big]p + \lambda(1-\nu)\theta m = 0. \tag{8}$$

(b) For any $m \in \mathbf{R}_2$, there is $\nu \in [0,1]$ so that m is a steady state of the corresponding Problem (5).

Denote $p_\lambda^{**}(m)$ the unique steady state of Problem (5) for $\nu = 0$. By Lemma 4(a), $p_\lambda^{**}(m)$ solves:

$$\pi_0'(p) - \lambda(2-(1-\theta)(1+\beta))p + \lambda\theta m = 0, \tag{9}$$

in particular $p_\lambda^{**}(\underline{m}) = \underline{m}$. The thresholds $\underline{m}$ and $\overline{m}$ defined above, correspond to those values m for which the steady state of J_m^ν equals m, for $\nu = 0$, respectively $\nu = 1$. It turns out that $(\underline{m}, \underline{m})$ and $(\overline{m}, \overline{m})$ are steady states for our Problem (4). The next result identifies steady states of Problem (4) based on the steady states of Problem (5), identified in Lemma 4.

Lemma 5. *(a) For $m \in \mathbf{R}_1$, $(m, p_\lambda^{**}(m))$ is a steady state of Problem (4), where $p_\lambda^{**}(m)$ solves (9). (b) For $m \in \mathbf{R}_2$, (m,m) is a steady state of Problem (4).*

The main result in this section confirms that these are the only steady states of Problem (4).

Proposition 1. *The set of steady states of Problem (4) is $\{(m, p_\lambda^{**}(m)) | m \in \mathbf{R}_1\} \cup \{(m,m) | m \in \mathbf{R}_2\}$. In particular, the value of the steady state prices is decreasing in λ, and increasing in β.*

The result implies that a more patient firm (higher β) charges higher steady state prices. Moreover, the value of the steady state is lower, the more sensitive consumers are to deviations from the reference price. Furthermore, the range $\mathbf{R}_2 = [\underline{m}(\lambda), \overline{m}(\gamma)]$ expands as consumers are more loss averse, i.e. as λ increases or γ decreases, by (6, 7). These sensitivity results are consistent with the predictions of exponentially smoothed memory models (e.g. Popescu and Wu 2007). In our model, however, the range of steady states is wider, and persists when consumers are loss neutral (i.e. for $\lambda = \gamma$, $\underline{m}(\lambda) < \overline{m}(\lambda)$), unlike with adaptive expectations, where it reduces to a single point.

Intuitively, there is relatively less opportunity value to manipulating prices under peak-end anchoring compared to exponential smoothing. On one hand, offering steep discounts can permanently erode demand in the future, as lowest prices remain salient in the memory anchoring process. On the other hand, the future benefit of increasing prices is short lived, as these high prices only affect the reference price in the next period. This is unlike exponential smoothing, where the effect of all (even extreme) past prices lingers in memory, but eventually vanishes.

A firm which ignores reference dependence and consumer anchoring processes offers in each period a static price p^0, which maximizes $\pi_0(p)$. Arguably, in this case, consumers' price expectation

is anchored at p^0. Yet, Proposition 1 implies that (p^0, p^0) is not a steady state (because $p^0 > \overline{m}$, by (7)), so the firm can actually increase profits by deviating from this constant policy. In general, consumer price expectations need not be in steady state, for a variety of contextual reasons (e.g. iTunes, or gasoline prices). The firm can be responsible for setting inadequate price expectations if it fails to estimate demand or behavioral processes correctly, or if it does not actively optimize prices in response to these effects and in anticipation of future price changes. The next section prescribes how a firm should manage prices in a transient regime, when it needs to change consumers' initial price expectations.

4. Optimal Policy and Price Paths

This section characterizes the firm's optimal transient pricing policy, and investigates convergence and monotonicity properties of the price paths of Problem (4). The optimal pricing policy $p^*(m_{t-1}, p_{t-1})$ solves (4). For any initial state (m_0, p_0), $m_0 \le p_0$, the optimal price path $\{p_t\}_t$ is given by $p_t = p^*(m_{t-1}, p_{t-1})$, with $m_t = \min(m_{t-1}, p_t), t \ge 1$; the corresponding state path is $\{(m_t, p_t)\}_t$.

Our results can be previewed in Figure 1, and suggest to divide the state space in the following regions: $\overline{\mathbf{R}}_{1a} = \{(m,p) \mid p \ge p_\lambda^{**}(m),\, m \le \underline{m}\}$, $\overline{\mathbf{R}}_{1b} = \{(m,p) \mid p \le p_\lambda^{**}(m),\, m \le \underline{m}, p\}$, $\overline{\mathbf{R}}_1 = \overline{\mathbf{R}}_{1a} \cup \overline{\mathbf{R}}_{1b}$, $\overline{\mathbf{R}}_2 = \{(m,p) \mid p \ge m,\, \underline{m} \le m \le \overline{m}\}$, and $\overline{\mathbf{R}}_3 = \{(m,p) \mid p \ge m,\, m \ge \overline{m}\}$, where $p_\lambda^{**}(m)$ is given by (9). We will show that, if (m_0, p_0) is in any of these regions, the state path remains in that region.

Proposition 2. *If $m_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$, then $p_t \ge m_0$ for all t. If $m_0 \in \mathbf{R}_3$, then $m_t \in \mathbf{R}_3$ for all t.*

The first part of this proposition shows that, if the initial minimum price m_0 is below a threshold ($m_0 \le \overline{m}$), the optimal price path stays above m_0, i.e. the minimum price does not change over time. On the other hand, if the initial minimum price m_0 is relatively high ($m_0 > \overline{m}$), the firm offers lower prices over time, but never below $\overline{m}$. So in either case, the optimal state path remains in the same region $\overline{\mathbf{R}}_i, i = 1,2,3$ as the initial state; so if it converges, it must converge to a steady state in the same region as the initial state (m_0, p_0). These steady states are identified in Proposition 1.

We now turn to characterize the optimal price paths of Problem (4). For $m_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$, $m_t = m_0$ by Proposition 2, so Problem (4) can be written (with m_0 as a parameter) as follows:

$$J_{m_0}(p_{t-1}) = \max_{p_t \ge m_0} \Big\{ \pi(p_t, r_t) + \beta J_{m_0}(p_t) \Big\}, \tag{10}$$

where $r_t = \theta m_0 + (1-\theta)p_{t-1}$. That is, $J(m,p) = J_m(p)$ for $m \in \mathbf{R}_1 \cup \mathbf{R}_2$, and $m \le p$. Because π is supermodular (Lemma 2), the optimal policy in Problem (10) is monotone, so $p_t^*(m_0, p_{t-1})$ is increasing in p_{t-1}. Therefore, the optimal price path is monotonic in a bounded interval, and hence converges to a steady state. This must be $(m_0, p_\lambda^{**}(m_0))$, by Proposition 1.

For $m_0 \in \mathbf{R}_3$, we show in the Appendix that optimal prices decrease, approaching $\overline{m}$. This is done by observing that prices must eventually fall below m_0 (but not below $\overline{m}$, by Proposition 2), at a certain time T. Until time T, a finite horizon version of Problem (10) is solved (and the same structural results hold). After time T, we show that optimal prices $p_t = m_t$ solve the problem:

$$\tilde{J}(p_{t-1}) = \max_{p_t \in \mathbf{P}} \{\pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t)\}. \tag{11}$$

Starting at $p_T = m_T \ge \overline{m}$, the optimal path decreases to $\overline{m}$, by supermodularity of π.

The next result characterizes the optimal price paths for Problem (4), as illustrated in Figure 1.

Proposition 3. *Given (m_0, p_0), the optimal price path of Problem (4) converges monotonically to a steady state, which is: (a) $p_\lambda^{**}(m_0)$, if $m_0 \in \mathbf{R}_1$, (b) m_0, if $m_0 \in \mathbf{R}_2$, and (c) $\overline{m}$, if $m_0 \in \mathbf{R}_3$.*

Corollary 1. *The optimal pricing policy, $p^*(m,p)$, is increasing in both m and p.*

Proposition 3 shows that, starting at (m_0, p_0), the price path converges monotonically to a steady state, which typically depends on the initial low-price anchor m_0 (p_0 influences the price path, but

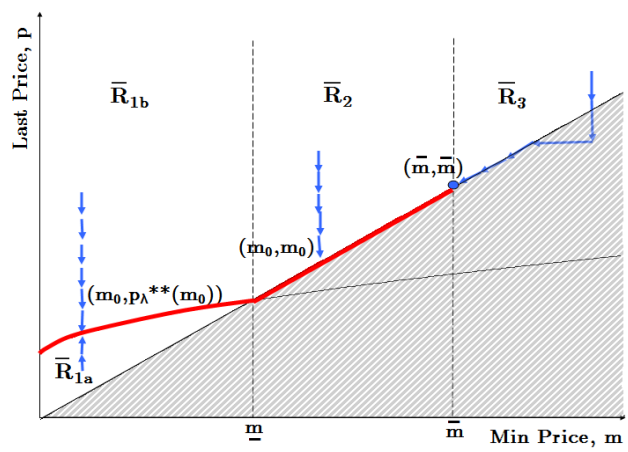

Figure 1 **Steady states and optimal price paths of Problem (4). The red bold line marks the range of steady states, and the blue arrows show the optimal price paths starting at generic points in each region.**

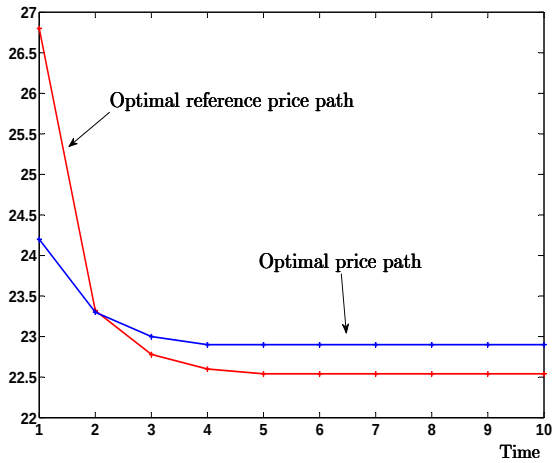

Figure 2 **Alternating price perception.** $d_0(p) = 500 - 10p, \lambda = 3, \gamma = 2, \theta = 0.4, \beta = 0.7, \underline{m} = 23, \overline{m} = 25, m_0 = 22, p_0 = 30$. **On this optimal price path, consumers initially perceive a discount** $r > p$**, and then a surcharge** $r < p$.

not the steady state). Price monotonicity is driven by supermodularity of short-term profits, $\pi(p, r)$ (Lemma 2), within the relevant regions, via Proposition 2. When initial expectations are very low $m_0 \leq p_0 < p_\lambda^{**}(m_0)$ (Region $\overline{\mathbf{R}}_{1a}$), the optimal price path is increasing and induces a consistent perception of loss. Otherwise, the optimal price path decreases, but, in contrast with exponential smoothing predictions, the gain/loss perception may alternate (see e.g. Figure 2).

Our results identify a threshold, $\overline{m}$ given by (7), such that for large enough initial low-price anchors ($m_0 \geq \overline{m}$), and independent of how much weight θ consumers put on the minimum price, the optimal price path decreases to the global steady state, $\overline{m}$. Therefore, as long as initial price perception is sufficiently high, m_0 and θ do not affect the long run optimal policy of the firm.

In contrast, an initial low price anchor $m_0 \leq \overline{m}$ affects the firm's long term strategy (steady state price). If m_0 is sufficiently low ($m_0 \leq \underline{m}$), the firm's optimal long run policy is $p_\lambda^{**}(m_0) = p_\lambda^{**}(m_0, \theta) \geq m_0$, which solves (9). As consumers anchor more on minimum prices, long run optimal prices are lower, i.e. $p_\lambda^{**}(m; \theta)$ is decreasing in the memory anchoring parameter θ. This is because the LHS in (9) is decreasing in θ for $m \leq p$. As $\theta \to 0$, i.e. as consumers anchor more on the last price, $p_\lambda^{**}(m_0; \theta)$ converges to p_λ^{**}, the unique solution of $\pi_0'(p) - \lambda(1-\beta)p = 0$. By (6), $\underline{m} = \underline{m}(\theta)$ is decreasing in θ, and $\underline{m}(\theta) \leq \underline{m}(0) = p_\lambda^{**}$. This shows that the range $[\underline{m}(\theta), \overline{m}]$ of steady states of

the type (m,m) gets wider with θ, i.e. the more consumers pay attention to the minimum price.

Proposition 4 summarizes the above results, and shows that optimal prices and profits decrease the more consumers anchor on lowest prices, as measured by θ.

Proposition 4. *(a) The optimal prices $p^*(m,p;\theta)$ and profits $J(m,p;\theta)$ in Problem (4) decrease in θ, $\theta \in (0,1]$. (b) $\underline{m}(\theta)$ and $p^{**}_\lambda(m;\theta)$ decrease in θ, and $\overline{m}(\theta) \equiv \overline{m}$ is independent of θ.*

Overall, our results indicate when and how behavioral parameters (θ,λ,γ) impact the firm's prices and profits, depending on initial consumer expectations. In order to decide which behavioral effects and parameters are relevant to assess, we conclude by proposing the following sequential estimation procedure for determining the optimal long run policy of the firm:

1. Compute the global threshold $\overline{m} = \overline{m}(\gamma)$ based on (7). Among behavioral parameters, only consumers' sensitivity to discounts (gains), γ, needs to be estimated for this.

2. Assess if prices lower than $\overline{m}$ were charged in the past (or recalled by consumers). If not, $\overline{m}$ is the optimal long term price.

3. If the lowest historic price is $m \in [p^{**}_\lambda, \overline{m}]$, where p^{**}_λ is given by (9), then m is the optimal long term price. Consumers' sensitivity to surcharges, λ, needs to be estimated for this.

4. If $m_0 < p^{**}_\lambda$, it is necessary to assess the anchoring parameter θ, and calculate $\underline{m} = \underline{m}(\theta)$, based on (6). The equilibrium price is determined by comparing m_0 with $\underline{m}$, via Proposition 3.

We conclude that the peak-end model, and its specific parameters, are particularly relevant when initial price expectations are low. Back to the example of iTunes, our results in this section suggest that Apple cannot profitably increase prices, unless $m_0 = .99 < \underline{m} = \underline{m}(\lambda,\theta)$. The latter provides an upper bound on the optimal long run price, $p^{**}_\lambda(.99;\theta) \geq .99$, which depends on the initial anchor (\$.99), and on how salient it is (θ). This is in contrast with exponential smoothing predictions, which suggest that the long term price This is in contrast with exponential smoothing predictions, which suggest that the long term price does not depend on initial expectations (Popescu and Wu, 2007).

5. Conclusions

This paper motivated a reference price model based on the peak-end rule (Fredrickson and Kahneman 1993), as a psychologically compelling alternative to the exponential smoothing memory models used in the literature. In the pricing context, the peak-end rule suggests that consumers' price expectations are based on the lowest and the most recent prices, and the reference price is formed as a weighted average of these two prices. We embedded the peak-end rule in a repeat purchase context, where loss averse consumers anchor on the lowest and most recent prices, and characterized the optimal dynamic pricing policy of the firm.

Our results showed how peak-end anchoring processes interact with loss aversion to affect the structure of optimal pricing strategies. Under the peak-end rule, we found that a range of constant pricing strategies (steady states) is optimal, supporting EDLP; the result persists even with loss neutral buyers, unlike existing literature. In our case, this range is due to the asymmetry in memory structure, in addition to loss aversion. The range of constant price policies is wider the more loss averse consumers are, and the more they anchor on the lowest price. Overall, this suggests that behavioral regularities limit the benefits of varying prices. Both profits and prices are lower the more consumers anchor on the lowest price, and the lower the initial expectations. If expectations are misaligned, the firm's optimal policy follows a traditional skimming or penetration strategy. Unlike exponential smoothing (Popescu and Wu 2007), in our model, price monotonicity is a robust result, extending under non-linear reference effects, as shown in Nasiry (2010). The latter extends our insights for heterogeneous buyers, as well as alternative memory models which update the minimum price anchor, or combine it with adaptive expectations.

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Appendix: Proofs

Proof of Lemma 1: Because $d(p_t, r_t)$ is increasing in $r_t = \theta m_{t-1} + (1-\theta)p_{t-1}$, $\pi(p_t, r_t) = p_t\, d(p_t, r_t)$ increases in p_{t-1} and m_{t-1}. Moreover, the transition in the Bellman Equation (3) is increasing in m_{t-1} (and independent of p_{t-1}). So the value function is increasing (Stokey et al. 1989, Thm. 4.7).

Proof of Lemma 2: For $p^l \le p^h$ and $r^l \le r^h$, we need to show that:

$$\pi(p^h, r^h) - \pi(p^l, r^h) \ge \pi(p^h, r^l) - \pi(p^l, r^l). \tag{12}$$

We consider all possible cases: (1) $p^l \le p^h \le r^l \le r^h$, (2) $p^l \le r^l \le p^h \le r^h$, (3) $p^l \le r^l \le r^h \le p^h$, (4) $r^l \le p^l \le p^h \le r^h$, (5) $r^l \le p^l \le r^h \le p^h$, (6) $r^l \le r^h \le p^l \le p^h$. Cases 1 and 6 follow because π_γ or π_λ, are supermodular in (p, r). After rearranging terms, (12) simplifies in each case as follows:

Case 2: $\gamma(r^h - p^h)p^h + \lambda(p^h - r^l)p^h \ge \gamma(r^h - r^l)p^l$. This holds because $\lambda > \gamma$ and $p^h \ge r^l$.

Case 3: $-\lambda(p^h - r^h)p^h - \gamma(r^h - p^l)p^l \ge -\lambda(p^h - r^l)p^h - \gamma(r^l - p^l)p^l$, or, $\lambda p^h(r^h - r^l) \ge \gamma p^l(r^h - r^l)$.

Case 4: $p^h\Big(\gamma(r^h - p^h) + \lambda(p^h - r^l)\Big) \ge p^l\Big(\lambda(p^l - r^l) + \gamma(r^h - p^l)\Big)$. It is enough to show that $\gamma(r^h - p^h) + \lambda(p^h - r^l) \ge \lambda(p^l - r^l) + \gamma(r^h - p^l)$, which holds because $\lambda(p^h - p^l) \ge \gamma(p^h - p^l)$.

Case 5: $\lambda p^h(r^h - r^l) \ge \lambda p^l(p^l - r^l) + \gamma(r^h - p^l)p^l$. Because $\lambda \ge \gamma$ and $r^h \ge p^l$, this is implied by $p^h(r^h - r^l) \ge p^l(p^l - r^l) + (r^h - p^l)p^l = p^l(r^h - r^l)$.

Proof of Lemma 3: For all $p \ge m$, the function $J^\nu_m(p)$ uniquely solves $TJ^\nu_m(p) = J^\nu_m(p)$, where the operator T is defined for any continuous function f over $[m, \overline{p}]$ by:

$$Tf(p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{(1-\nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, p_{t-1}) + \beta f(p_t)\Big\}. \tag{13}$$

Moreover, $\lim_{n\to\infty} T^n f(p) = J^\nu_m(p)$; see Stokey and Lucas (1989, Theorem 4.6).

We argue below that, for all $p \ge m$, $J(m,p) \le TJ(m,p)$. This further implies $J(m,p) \le T^n J(m,p) \le \lim_{n\to\infty} T^n J(m,p) = J^\nu_m(p)$, concluding the proof. Indeed, for $m_{t-1} \le p_{t-1}$, we have:

$$\begin{aligned}
J(m_{t-1}, p_{t-1}) &= \max_{p_t \in \mathbf{P}} \Big\{\pi(p_t, r_t) + \beta J(\min(m_{t-1}, p_t), p_t)\Big\} \\
&\le \max_{p_t \in \mathbf{P}} \Big\{(1-\nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, r_t) + \beta J(m_{t-1}, p_t)\Big\} \\
&\le \max_{p_t \in \mathbf{P}} \Big\{(1-\nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, p_{t-1}) + \beta J(m_{t-1}, p_t)\Big\} \\
&= TJ(m_{t-1}, p_{t-1}).
\end{aligned}$$

The first inequality above holds because $\pi = \min(\pi_\lambda, \pi_\gamma) \leq (1-\nu)\pi_\lambda + \nu\pi_\gamma$, and the value function is increasing (Lemma 1). The second inequality holds because $p_{t-1} \geq r_t = \theta m_{t-1} + (1-\theta)p_{t-1}$, for $p_{t-1} \geq m_{t-1}$. Finally, the last equality follows by the definition of T in (13) applied to $f(p) = J(m,p)$.

Proof of Lemma 4: (a) It is easy to check that, if Problem (5) admits an interior steady state, this solves the Euler Equation (8). Moreover, this equation admits a unique solution, because $\pi_0'(p) < 0$ and the coefficient of p in (8) can be written as $-(1-\beta)((1-\nu)\lambda(1+\theta) + \nu\gamma) \leq 0$. It remains to verify that a steady state exists, and must be interior. Existence of a steady state follows from supermodularity of the objective function, because π_λ and π_γ are supermodular in (p,r). By Topkis Theorem (Topkis 1998, Theorem 2.8.2), this implies that the pricing paths of Problem (5) are monotonic on the bounded domain $\mathbf{P}$, hence converge to a steady state p^{**}.

We further argue that a steady state must be interior. First, $p^{**} = 0$ cannot be a steady state because any non-zero pricing strategy achieves positive profits. Second, $p^{**} < \overline{p}$ for any steady state of Problem (5). This is because $\pi_0(p)$ is non-monotone, hence, its largest maximizer $\hat{p}$ is interior, i.e. $\hat{p} < \overline{p}$. Moreover, concavity of π_0 implies: $J^\nu(\hat{p}) \geq \frac{\pi(\hat{p})}{1-\beta} \geq \frac{\pi(p^{**})}{1-\beta} = J^\nu(p^{**})$. Finally, because J^ν is increasing, we conclude that $p^{**} \leq \hat{p} < \overline{p}$, so p^{**} is interior and solves the Euler Equation (8).

Finally, by definition, $p_\lambda^{**}(m)$ solves (8) for $\nu = 0$. This has a unique solution because the LHS is strictly decreasing in p, positive at $p = 0$ and negative at $p = \overline{p}$.

(b) Substituting $p = m$ in (8), we have $L(m,\nu) = \pi_0'(m) - \lambda[(1-\nu)(1-\beta(1-\theta)) + \nu(1-\beta)]m = 0$; (6) and (7) translate to $L(\underline{m}, 0) = 0$ and $L(\overline{m}, 1) = 0$. Because $L(m,\nu)$ is decreasing in m, for all $m \in [\underline{m}, \overline{m}]$, $L(m,0) \leq 0$ and $L(m,1) \geq 0$. The result follows because $L(m,\nu)$ is continuous in ν.

Proof of Lemma 5: We first show that $p_\lambda^{**}(m)$, as defined by (9), is feasible, i.e. $p_\lambda^{**}(m) \geq m$ for $m \in [0, \underline{m}]$. Note that $p_\lambda^{**}(m)$ is increasing in m and single crosses the identity line from above at $\underline{m}$, defined by (6). Feasibility follows because, at $m = 0$, (9) has a unique positive solution, $p_\lambda^{**}(0)$.

For $m \in [0, \underline{m}]$, the constant pricing policy $p_t \equiv p_\lambda^{**}(m)$ is optimal for Problem (5) with $\nu = 0$, and feasible for Problem (4). Because $m \leq \underline{m}$, $\min(m, p_\lambda^{**}(m)) = m$, and $r = \theta m + (1-\theta)p_\lambda^{**}(m) \leq p_\lambda^{**}(m)$, which implies $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. This constant pricing policy yields the same value in both problems, so it is also optimal for Problem (4), and $(m, p_\lambda^{**}(m))$ is a steady state of (4).

For $m \in [\underline{m}, \overline{m}]$, the constant pricing policy $p_t \equiv m$ is optimal for Problem (5), feasible for Problem (4) ($\pi_\lambda = \pi_\gamma$ along this path), and yields the same value in both problems. Therefore (m, m) is a steady state of Problem (4).

Proof of Proposition 1: For any steady state (m, p), two cases are possible, either $m = p$, or $m < p$. In the second case, $r < p$ and starting at (m, p), the price path gives a consistent perception of loss. Therefore, the steady state price must be the same as the steady state of Problem (5), with $\nu = 0$, i.e. $p = p_\lambda^{**}(m)$. Thus Problem (4) has only two types of steady states. It remains to identify the regions where each type of steady state is relevant.

First assume $m < \underline{m}$. We show by contradiction that (m, m) cannot be a steady state of Problem (4). If (m, m) is a steady state, the profit from charging a constant price $p_t \equiv m$ exceeds the profit on the path $p_t = m + \delta$, for all t. Denoting $r = \theta m + (1-\theta)(m+\delta)$, this implies:

$$\frac{\pi_0(m)}{1-\beta} \geq \pi_0(m+\delta) - \lambda\delta(m+\delta) + \frac{\beta}{1-\beta}\Big(\pi_0(m+\delta) - \lambda(m+\delta-r)(m+\delta)\Big),$$

This reduces to: $\pi_0(m+\delta) - \pi_0(m) \leq \lambda\delta(m+\delta)(1-\beta(1-\theta))$. Dividing both sides by δ and letting δ go to zero, gives: $\pi_0'(m) \leq \lambda(1-\beta(1-\theta))m$, which holds with equality for $m = \underline{m}$ (see (6)). Because π_0' is strictly decreasing, it follows that $m \geq \underline{m}$, a contradiction. Hence, for $m < \underline{m}$, the only possible steady state for Problem (4) is $(m, p_\lambda^{**}(m))$.

Moreover, because $p_\lambda^{**}(m) < m$ for $m > \underline{m}$, it follows that, for $m \geq \underline{m}$, the only possible steady state is (m, m). We prove by contradiction that (m, m) cannot be a steady state for $m > \overline{m}$. If (m, m) is a steady state, the profit from charging a constant price $p_t \equiv m$ exceeds the profit along the alternative path $p_t = m - \delta$, for all t, i.e.:

$$\frac{\pi_0(m)}{1-\beta} \geq \pi_0(m-\delta) + \gamma\delta(m-\delta) + \frac{\beta}{1-\beta}\pi_0(m-\delta), \quad \text{or} \quad \pi_0(m) - \pi_0(m-\delta) \geq \gamma\delta(1-\beta)(m-\delta).$$

Dividing by δ and letting δ go to zero, we have: $\pi_0'(m) \geq \gamma(1-\beta)m$, which holds with equality for $m = \overline{m}$ (see (7)). Because $\pi_0'(m)$ is strictly decreasing in m, it follows that $m \leq \overline{m}$, a contradiction. We conclude that steady states of the form (m, m) can only be relevant when $\underline{m} \leq m \leq \overline{m}$.

Finally, $p_\lambda^{**}(m)$ is decreasing in λ. This is because $p_\lambda^{**}(m) \geq m$ (for $m \leq \underline{m}$) solves equation (9) the LHS of which is decreasing in p, and λ (for $p \geq m$).

Proof of Proposition 2(a): We prove this in two parts, depending if $m_0 \in \mathbf{R}_1$, or $m_0 \in \mathbf{R}_2$. For $m_0 \in \mathbf{R}_1$, we consider two cases: $(m_0, m_0) \in \overline{\mathbf{R}}_{1a}$ and $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$ (see Figure 1).

Claim 1. *Given $(m_0, p_0) \in \overline{\mathbf{R}}_{1a}$, then $p_t \geq m_0$ for any t.*

Proof: Denote $J^{\nu=0}$ the objective function in Problem (5), with $\nu = 0$. $J^{\nu=0}$ is supermodular in (p, r), and thus the price path converges monotonically to the steady state price, $p_\lambda^{**}(m_0)$. Because $p_0 < p_\lambda^{**}(m_0)$, the optimal price path for $J^{\nu=0}$, increases to this steady state, and $p_t^*(r_t) \leq p^{**}(m_0)$ for all t. This implies that $m_t = m_0$ along this path (the minimum price does not change over time), and thus $r_t = \theta m_0 + (1-\theta)p_{t-1} \leq p_{t-1} \leq p_t$. Therefore, $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$, and this path is feasible for (4), and yields the same value which leads us to conclude that the same path is also optimal for Problem (4). This result is stronger than stated in the claim, because it guarantees also the existence of the steady state, and the monotonicity of the price path.

Claim 2. *Given $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$, then $p_t \geq p^{**}(m_0) \geq m_0$ for any t.*

Proof: For $m_0 \leq \underline{m}$, $(m_0, p_\lambda^{**}(m_0))$ is a steady state of Problem (4) (Proposition 1). We show that if at any time it is optimal for the price to be below $p_\lambda^{**}(m_0)$, then $(m_0, p_\lambda^{**}(m_0))$ cannot be a steady state of Problem (4) which is a contradiction.

Let $r_1^* = \theta m_0 + (1-\theta)p_\lambda^{**}(m_0)$. We show that $p_1 = p^*(m_0, p_0) \geq p_\lambda^{**}(m_0)$, and then by induction we conclude that $p_t \geq p_\lambda^{**}(m_0)$. Assume by contradiction, $p_1 < p_\lambda^{**}(m_0)$. Then:

$$\pi(p_1, r_1) + \beta J(\min(m_0, p_1), p_1) > \pi(p_\lambda^{**}(m_0), r_1) + \beta J(m_0, p_\lambda^{**}(m_0)),$$

or equivalently by defining $\Delta J = J(m_0, p_\lambda^{**}(m_0)) - J(\min(p_1, m_0), p_1)$,

$$\pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1) > \beta \Delta J. \tag{14}$$

Because $p_0 > p_\lambda^{**}(m_0) > p_1$, we have $r_1 > r_1^*$. Supermodularity of $\pi(p, r)$ (Lemma 2), then implies:

$$\pi(p_1, r_1^*) - \pi(p_\lambda^{**}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1). \tag{15}$$

Because $p_\lambda^{**}(m_0) > r_1^*$, it follows that $\pi_\lambda(p_\lambda^{**}(m_0), r_1^*) \leq \pi_\gamma(p_\lambda^{**}(m_0), r_1^*)$, and $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. Hence (15) can be written as: $\pi(p_1, r_1^*) - \pi_\lambda(p_\lambda^{**}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1)$. Combining with equation (14), we have: $\pi(p_1, r_1^*) - \pi_\lambda(p_\lambda^{**}(m_0), r_1^*) > \beta \Delta J$. Or equivalently:

$$\pi_\lambda(p_\lambda^{**}(m_0), r_1^*) + \beta J(m_0, p_\lambda^{**}(m_0)) < \pi(p_1, r_1^*) + \beta J(\min(p_1, m_0), p_1).$$

This contradicts $(m_0, p_\lambda^{**}(m_0))$ being a steady state of (4). We conclude that $p_t \geq p_\lambda^{**}(m_0)$ for all t.

Claim 3. *Given $m_0 \in \mathbf{R}_2$, then $p_t \geq m_0$ for any t.*

Proof: We show that $p_1 = p^*(m_0, p_0) \geq m_0$. By induction, this shows that $p_t^* \geq m_0$, $\forall t$. Suppose by contradiction, $p_1 < m_0$. Then, because $p_1 \leq \theta m_0 + (1-\theta)p_0 = r_1$, we have $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. Now, (4) can be written as:

$$J(m_0, p_0) = \max_{p_1 < m_0} \Big\{ \pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) \Big\}. \tag{16}$$

We show that, in this case, (m_0, m_0) cannot be a steady state of Problem (4), a contradiction. Because $p_1 < m_0$, we have $\pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) > \pi_\gamma(m_0, r_1) + \beta J(m_0, m_0)$. Equivalently,

$$\pi(p_1, r_1) - \pi(m_0, r_1) > \beta(J(m_0, m_0) - J(p_1, p_1)) \triangleq \beta \Delta J. \tag{17}$$

Because $\pi_\gamma(p, r)$ is supermodular, and $r_1 > m_0$, it follows that:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(m_0, r_1) \leq \pi_\gamma(p_1, m_0) - \pi_\gamma(m_0, m_0). \tag{18}$$

Combining (18) and (17), we have: $\pi_\gamma(p_1, m_0) - \pi_0(m_0) > \beta \Delta J$, or equivalently: $\pi_0(m_0) - \beta J(m_0, m_0) < \pi_\gamma(p_1, m_0) + \beta J(p_1, p_1)$. This contradicts the fact that (m_0, m_0) is a steady state of Problem (4). We conclude that if the initial state is such that $\underline{m} \leq m_0 \leq \overline{m}$, then $p_t \geq m_0$ for all t.

Proof of Proposition 2(b): Consider two cases: $p_0 = m_0$ and $p_0 > m_0$.

Case 1: Assume $m_0 = p_0$. Consider the problem:

$$J^\phi(p_{t-1}, p_{t-1}) = \max_{p_t} \Big\{ \pi(p_t, p_{t-1}) + \beta J^\phi(p_t, p_t) \Big\}. \tag{19}$$

Because the value function in Problem (4) is increasing in its arguments (Lemma 1), it follows that: $J(m_{t-1}, p_{t-1}) \leq J^\phi(p_{t-1}, p_{t-1})$. Equality occurs if $m_{t-1} = p_{t-1}$ for all t, i.e. starting from $m_0 = p_0$, the price path is decreasing. By construction, the steady state of Problem (19) is the same as the steady state of the following problem:

$$\tilde{J}(p_{t-1}) = \max_{p_t} \Big\{ \pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t) \Big\}, \tag{20}$$

which is $\overline{m}$. Therefore $(\overline{m}, \overline{m})$ is the unique steady state of Problem (19). Because $\pi(p_t, p_{t-1})$ is supermodular, starting at $p_0 > \overline{m}$, the optimal price path of Problem (19) is decreasing and converges to $\overline{m}$. Starting at an initial state (m_0, p_0) such that $p_0 = m_0 > \overline{m}$, the optimal path of

Problem (19) is feasible for Problem (4) and yields the same value. This is because at each stage $m_{t-1} = p_{t-1}$, implying $\min(p_{t-1}, p_t) = p_t$ and $r_t = p_{t-1}$. Therefore for such initial states, this price path is optimal for Problem (4) and converges to $\overline{m}$. This also implies $m_{t-1} \geq \overline{m}$, as desired.

Case 2: Now assume that $p_0 > m_0$. The following claim proves the desired result.

Claim 4. *For $p_0 > m_0 > \overline{m}$, if the optimal price p_t is such that $p_t \leq m_0$, then $p_t \geq \overline{m}$.*

Proof: We show that if p_1 is such that $p_1 \leq m_0$, then $p_1 > \overline{m}$. By induction, this implies $p_t \geq \overline{m}$.

Suppose by contradiction that $p_1 = p^*(m_0, p_0) < \overline{m}$. Then $p_1 \leq \theta m_0 + (1-\theta)p_0 = r_1$, and hence $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. This allows to write (4) as:

$$J(m_0, p_0) = \max_{p_1} \Big\{ \pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) \Big\}.$$

Because $p_1 < \overline{m}$, we have $\pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) > \pi_\gamma(\overline{m}, r_1) + \beta J(\overline{m}, \overline{m})$. Equivalently,

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(\overline{m}, r_1) > \beta(J(\overline{m}, \overline{m}) - J(p_1, p_1)) \triangleq \beta \Delta J. \tag{21}$$

Because $\theta m_0 + (1-\theta)p_0 = r_1 > \overline{m}$ and $p_1 < \overline{m}$, and $\pi_\gamma(p, r)$ is supermodular, it follows that:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(\overline{m}, r_1) \leq \pi_\gamma(p_1, \overline{m}) - \pi_\gamma(\overline{m}, \overline{m}). \tag{22}$$

Combining (22) and (21), we have $\pi_\gamma(p_1, \overline{m}) - \pi_\gamma(\overline{m}, \overline{m}) > \beta \Delta J$, or equivalently, $\pi_0(\overline{m}) + \beta J(\overline{m}, \overline{m}) < \pi_\gamma(p_1, \overline{m}) + \beta J(p_1, p_1)$. This implies that $(\overline{m}, \overline{m})$ cannot be a steady state of Problem (4), a contradiction. We conclude that, if (p_0, m_0) is such that $m_0 > \overline{m}$, we have $p_t \geq \overline{m}$.

Proof of Proposition 3: (a) Consider two possible cases: 1) $(m_0, p_0) \in \overline{\mathbf{R}}_{1a}$, and 2) $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$. The first case is proved in Proposition 2. For the second case, the argument in Section 4, shows that the price path is monotonic. Moreover, because $p_0 \geq p_\lambda^{**}(m_0)$, it follows that, in this case, the price path is decreasing and converges to $p_\lambda^{**}(m_0)$.

(b) The argument in Section 4 shows that the price path in this region is monotonic. Because $p_0 \geq m_0$, the price path is decreasing to its steady state m_0.

(c) In the proof of Proposition 2b, we showed that for initial states $p_0 = m_0 \geq \overline{m}$, the price path decreases monotonically to the steady state $(\overline{m}, \overline{m})$. Now, we focus on the case where the initial state is such that $p_0 > m_0$. The next claim ensures the price path eventually falls below m_0.

Claim 5. *Starting at (m_0, p_0), where $p_0 > m_0 \geq \overline{m}$, at some point in time, T, the optimal price falls below m_0, i.e. $p_T \leq m_0$.*

Proof: Suppose by contradiction that $p_t > m_0$, for all t. Thus the value function in Problem (4), with m_0 as a parameter, can be written as:

$$J_{m_0}(p_0) = \max_{p_1} \Big\{ \pi(p,r) + \beta J_{m_0}(p_1) \Big\}.$$

The objective function is supermodular in (p,r) (Lemma 2), so the price path is monotonic, and converges to a steady state. By Proposition 1, this must be $(\overline{m},\overline{m})$, which contradicts $p_t > m_0 \geq \overline{m}$. We conclude that at some point in time, T, the optimal price is such that $p_T \leq m_0$. $\square$

Let T be the first time that the optimal price falls below m_0. Thus at time T, the value function in Problem (4) can be written as:

$$J(m_0, p_{T-1}) = \max_{p_T \leq m_0} \Big\{ \pi(p,r) + \beta J(p_T, p_T) \Big\}.$$

Claim 4 implies $p_T > \overline{m}$. For an initial state (m_0, p_0) such that $p_0 = m_0 > \overline{m}$, the price path decreases monotonically to $\overline{m}$, and $(\overline{m},\overline{m})$ is the corresponding steady state. The value function for $t < T$ is given by the finite horizon model:

$$J_{t-1}(p_{t-1}) = \max_{p_t} \Big\{ \pi(p,r) + \beta J_t(p_t) \Big\}, \quad t < T,$$

where $J_T(p_T) = J(m_0, p_T)$. Because the objective function is supermodular in (p,r), the price path is monotonic, and decreases to p_{T-1}. (It cannot be increasing, because then it would have to converge to a steady state above m_0, contradicting Proposition 1.) In summary, starting at an initial state (m_0, p_0) such that $p_0 > m_0 > \overline{m}$, the price path decreases until it falls below m_0 and then converges decreasingly to $\overline{m}$.

Proof of Proposition 4: $\pi(p,r)$ is supermodular in (p,r) by Lemma 2, and $r = r(\theta) = p + \theta(m - p)$ is decreasing in θ for $m \leq p$. Therefore π is submodular in (p,θ) and $p^*(p,m;\theta)$ is decreasing in θ. Moreover, because π is increasing in r, and r is decreasing in θ, we conclude that the value function, $J(m,p;\theta)$, is decreasing in θ (Stokey and Lucas, 1989, Theorem 4.7).

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