"WHEN STATIONARY STRATEGIES ARE EQUILIBRIUM BIDDING STRATEGY: THE SINGLE-CROSSING PROPERTY"

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WHEN STATIONARY STRATEGIES ARE EQUILIBRIUM BIDDING STRATEGY: THE SINGLE-CROSSING PROPERTY[a]

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Abstract: We present a condition that, when verified, is sufficient to ensure that stationary strategies of an auction game are equilibrium bidding strategies. Intuitively, this condition requires the indifference curves of a higher-type bidder to cross the indifference curves of a lower-type one only once and from above. Because of this geometrical feature, this condition is called the "single-crossing property" (SCP). We show that SCP is frequently assumed, although implicitly, throughout the theoretical and empirical literature on auctions.

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1. Introduction

> *The sound procedure is to obtain first utmost precision and mastery in a limited field, and then to proceed to another, somewhat wider one, and so on ... There is no reason to assume the existence of shortcuts.*
>
> (John Von Neumann and Oskar Morgenstern)

> *... nevertheless it can happen that a certain fictive hypothesis may suffice for explaining many phenomena.*
>
> (Leonhard Euler)

The mathematical analysis of natural and social phenomena requires a sometimes harsh trade-off between tractability and accuracy. Operations research and economics are no exceptions to this dilemma: most models take the form of optimization problems that cannot be solved, given the current state of the art, unless some constraints are relaxed; but doing so often entails a serious loss of predictive power.

Many models, however, happen to verify certain conditions that safeguard a tractable relaxation. In agency theory, the task of bringing up such conditions was carried out by Rogerson (1985)[1].In a

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1 "The first-order approach to principal-agent problems involves relaxing the constraint that the agent choose an action which is utility maximizing to require instead that the agent choose an action at which his utility is at a stationary point. Although more mathematically tractable,this approach is generally invalid. This paper identifies sufficient conditions - the monotone likelihood condition and the convexity of the distribution function condition - for the first-order approach to be valid." (p. 1357)

similar spirit here, we propose **a simple** hypothesis - roughly, any two bidders with different information parameters have indifference curves that cross only once and from above - which is sufficient for a stationary point of a bidder's objective function to correspond to an equilibrium bidding strategy. We call this property the <u>single-crossing property</u>.

The single-crossing property and its consequences are stated in full generality in the next section. The relevance of the single-crossing property for auction theory is argued mostly in section 3, 4 and 5, where its links with, respectively, the design of optimal auctions, the now familiar properties of affiliation and risk aversion, and laboratory experiments are outlined.

2. <u>The single-crossing property</u>

Consider the case of a buyer (e.g. the government) to whom only one seller can propose a multi-dimensional bid $(p,q) \in \mathbb{R}^2$ (e.g. price-quality). If the buyer accepts the bid (p,q), she gets a satisfaction level V(p,q); accepting nothing gives her a satisfaction of zero. On the other hand, the seller has a privately known type (e.g. technology, accounting procedure, etc.) $t \in (t_1,\ldots,t_n) \subset R$ that determines his final utility U(p,q;t) for a sold package (p,q); a rejected proposal brings a utility U(0,0;t). To the buyer, the seller's true type is t_i with probability $\pi_i > 0$ $(i = 1,\ldots,n)$; this probability is common knowledge between the buyer and the seller.

Now assume that the buyer can commit to accepting an offer (p,q) if and only if it belongs to a chosen finite subset Ω of R^2 (e.g. there are mandatory price-quality requirements). Let V_1 and V_2, the partial derivatives of V, be negative and positive respectively; and let $U_1 > 0$, $U_2 < 0$. A rational buyer would then pick a set $\Omega = \{(p^1,q^1),\ldots,(p^n,q^n)\}$ so that her expected utility be maximized while a seller of any type makes an offer and a seller with type t_i proposes the pair (p^i,q^i) rather than (p^j,q^j), $i \neq j$. Formally, she would compute the vector $(p^1,q^1;\ldots;p^n,q^n)$ that solves the following optimization problem:

$$\text{maximize } \Sigma_{i=1}^{n} \pi_i V(p^i,q^i) \tag{H}$$

subject to:

$$\text{for all } i\text{: } U(0,0;t_i) \leq U(p^i,q^i;t_i)$$

$$\text{for all } i,j\text{: } U(p^j,q^j;t_i) \leq U(p^i,q^i;t_i).$$

Problem (H) has two sets of constraints usually called "individual rationality" and "incentive compatibility" respectively. These constraints totalize $[n+n(n-1)] = n^2$, and their number can be quite bothersome in concrete situations. But if the sellers' indifference curves were like the ones illustrated in figure 1, we claim that solving the following relaxation of (H) would also provide a solution to the buyer's problem:

$$\text{maximize } \Sigma_{i=1}^{n} \pi_i V(p^i,q^i) \tag{R}$$

subject to:

$$\text{for all } i\text{: } U(p^{i+1},q^{i+1};t_i) \leq U(p^i,q^i;t_i)$$

$$\text{for all } i: \quad U(p^{i-1},q^{i-1};t_i) \le U(p^i,q^i;t_i)$$
$$U(0,0;t_1) \le U(p^1,q^1;t_1).$$

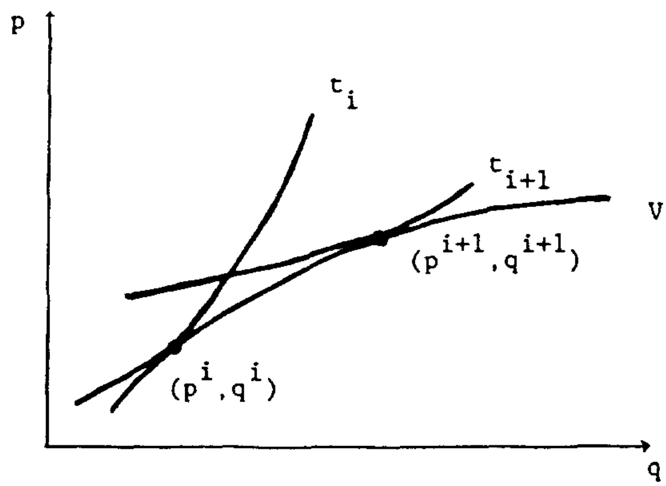

Fig. 1: A useful property of indifference curves: single-crossing

Relaxation (R) has only [2n-2+1] = 2n-1 constraints. It is therefore much less cumbersome to solve than the unrelaxed problem (H) when n is large.

The validity of (R) will now be demonstrated. We shall refer to the crucial geometrical feature figure 1 depicts as the "single-crossing property".

2.1 Not much mathematical structure is needed for the single-crossing property to be adequately stated and for some interesting theorems to be deduced from it. As we shall see, it is enough to work with order relations.

A weak ordering on a nonvoid set A is a binary relation $\le_A$

that is

i) reflexive: for all $a \in A$, $a \leq_A a$;

and ii) transitive: for all $a, b, c \in A$,

$$a \leq_A b \text{ and } b \leq_A c \text{ implies } a \leq_A c .$$

The set A with $\leq_A$ is then a weakly-ordered set.

When $a \leq_A b$ and $b \leq_A a$, it is convenient to write $a \sim_A b$. Let us also write $a <_A b$ instead of "$a \leq_A b$ and not($b \leq_A a$)".

A weak ordering that is also

iii) antisymmetric: for all $a, b \in A$,

$a \leq_A b$ and $b \leq_A a$ implies $a = b$ (a is identical to b),

is an ordering. The set A is then called an ordered set.

Finally, a weak ordering $\leq_A$ which is

iv) complete: for all $a, b \in A$,

$$a \leq_A b \text{ or } b \leq_A a,$$

is a total weak ordering. A is then a totally weakly-ordered set.

If there is a binary relation $\leq_A$ on A that satisfies properties i)-iv), it is called a total ordering and the set A with $\leq_A$ is called a chain.

Let us now consider a family $\Gamma = \{ \leq_t \mid t \in T\}$ of total weak orderings on the product $A_1 \times A_2$, where A_1 and A_2 are chains[2]. Let

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[2]The reader will later notice that it is not necessary at all to

the index set T be also a chain. We shall assume from now on that, for $\leq_t$ in Γ,

$$a,b \in A_1 \times A_2,\ a \sim_t b \text{ and } a \neq b \quad \text{imply} \quad a_1 \neq b_1. \qquad \text{(I)}$$

When $A_1 \times A_2$ is a subset of R^2 that contains the feasible vectors of quantities of two divisible goods, and the members of Γ are a consumer's possible preference relations, the last assumption just means that the indifference curves of a consumer with type t are one-dimensional with finite slope.

Now, the family Γ is said to have the <u>single-crossing property</u> (SCP) if the following assertion is true:

Take a, b $\in A_1 \times A_2$ such that $a_1 \leqslant_{A_1} b_1$. If $a \leq_t b$, then $a <_{\underline{t}} b$ for all $t <_T \underline{t}$.

2.1 In R^2, an immediate geometrical interpretation of SCP is that two indifference curves of types t and $\underline{t}$, $t \neq \underline{t}$, cross at most once. The next proposition generalizes this statement.

<u>Proposition</u>: If Γ satisfies SCP, the sets (equivalence classes) $\sim_t^a = \{b \in A_1 \times A_2 \mid b \sim_t a\}$ and $\sim_{\underline{t}}^a = \{c \in A_1 \times A_2 \mid c \sim_{\underline{t}} a\}$, $t \neq \underline{t}$, have only one point in common, namely a.

PROOF:

Suppose that $\sim_t^a \cap \sim_{\underline{t}}^a \supset \{a,b\}$, where $a \neq b$.

We can assume, without loosing generality, that $t <_T \underline{t}$ and

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assume any weak ordering on A_2. However, taking A_2 as a chain should not be too stringent for someone who accepts the Axiom of Choice.

$a_1 <_{A_1} b_1$, the last strict inequality being justified by postulate (I) and the fact that A_1 is a chain. Then $a \leq_t b$ and SCP imply that $a <_{\underline{t}} b$. This contradiction establishes the result. □

The above proposition justifies the name "single-crossing property". The assertion $\mathcal{N}_t^a \cap \mathcal{N}_{\underline{t}}^a = (a)$, however, is not by itself sufficient for the single-crossing property to hold because SCP carries a much richer geometrical structure into Γ.

Let A_1 and A_2 be equal to $[0,\infty)$. Let us suppose that the preference relations $\leq_t$ in Γ can be <u>represented</u> by differentiable utility functions $U(\cdot;t)\colon A_1 \times A_2 \to R$, i.e.

$$(a_1,a_2) \leq_t (b_1,b_2) \text{ if and only if } U(a_1,a_2;t) \leq U(b_1,b_2;t).$$

Let U_1 (U_2) denote the first-order partial derivative of U with respect to its first (second) component. Provided SCP holds,

i) the slope $-U_1(\cdot;t)/U_2(\cdot;t)$ of the indifference curves is <u>decreasing</u> in t, when U_1 and U_2 are positive for all t; (Figure 2a)

ii) $-U_1(\cdot;t)/U_2(\cdot;t)$ <u>increases</u> with t, if U_1 is positive and U_2 is negative for all t; (Figure 2b)

iii) $-U_1(\cdot;t)/U_2(\cdot;t)$ <u>decreases</u> with t, if U_1 is negative and U_2 is positive for all t; (Figure 2c)

iv) $-U_1(\cdot;t)/U_2(\cdot;t)$ is <u>increasing</u> in t, when both U_1 and U_2 are negative. (Figure 2d)

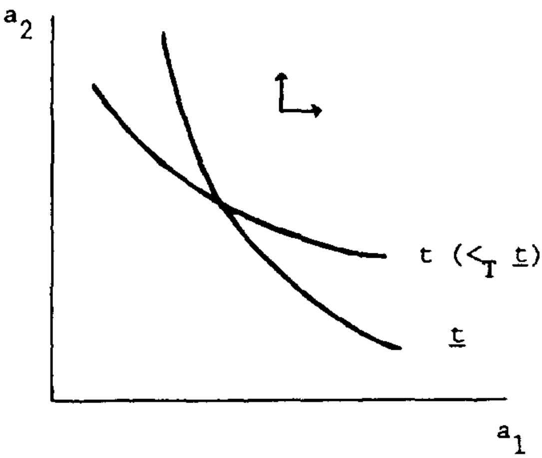

Fig. 2a: SCP; $U_1, U_2 > 0$

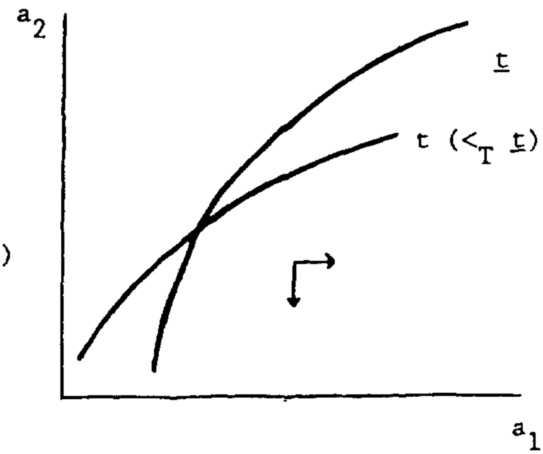

Fig 2b: SCP; $U_1 > 0$, $U_2 < 0$

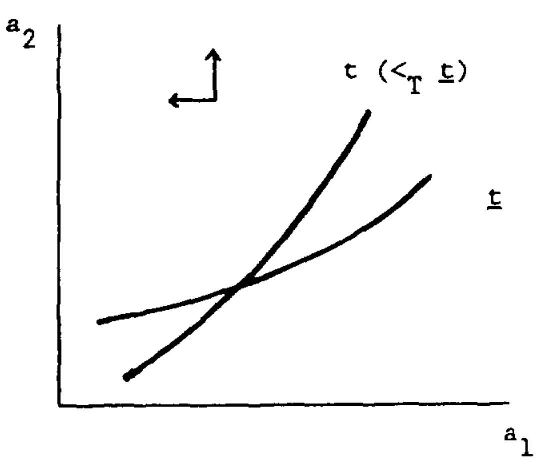

Fig. 2c: SCP; $U_1 < 0$, $U_2 > 0$

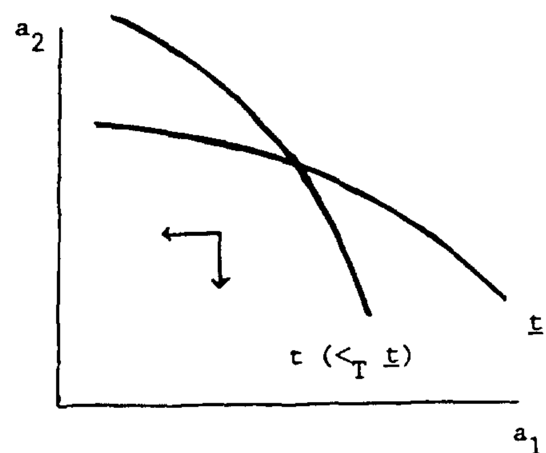

Fig. 2d: SCP; $U_1, U_2 < 0$

Briefly, in the basic framework of consumer theory, the single-crossing property implies that the indifference curves of a consumer with type $\underline{t}$ cross the indifference curves of one with type-t $(<_T \underline{t})$ at most once and from above. Although this fact might seem innocuous, it has important consequences that we shall now see.

2.2 We are interested in SCP for three main reasons: first, it greatly reduces the task of verifying global optimality to one of checking local optimality; second, it holds quite frequently in economic models; and third, it is relatively easy to check. The illustrations just brought were to make the third reason clear. We shall come back to this point in the upcoming sections, where we will also emphasize the second reason. The first reason is a central claim we will now try to support by mathematical reasoning.

Let us suppose SCP holds in the previous general context, where A_1, A_2 and T are chains,

$\Gamma = \{ \leq_t \mid t \in T\}$ is a family of total weak orderings.

Then, we can prove a general statement about the variation of the first component a_1 of $a \in A_1 \times A_2$ with respect to $t \in T$.

<u>Lemma</u>: (Monotonicity) Assume Γ satisfies SCP. Let a^i be the $\leq_{t_i}$-most prefered element in the set $A_1 \times A_2$; $i = 1,\ldots,n$ and $t_1 \leq_T t_2 \leq_T \ldots \leq_T t_n$. Then

$$a_1^1 \leq_{A_1} a_1^2 \leq_{A_1} \ldots \leq_{A_1} a_1^n.$$

PROOF:

It suffices to show that $a_1^i \leq_{A_1} a_1^{i+1}$ for each i.

Suppose not. Then $a^{i+1} \leq_{t_i} a^i$ and $a_1^{i+1} <_{A_1} a_1^i$ implies that $a^{i+1} <_{t_{i+1}} a^i$ by SCP. This is a contradiction. □

The main result - that global optimality can be checked locally - is now at hand.

<u>Proposition</u>: (Global optimality of local solutions)

Assume SCP holds in Γ. Take $t_1 <_T \dots <_T t_n$ in T. Suppose there are elements $a^1, \dots, a^n \in A_1 \times A_2$ such that

$$a_1^i \neq a_1^j \quad \text{for all } i,j = 1,\dots,n ,$$
$$a^{i+1} <_{t_i} a^i \quad \text{for } i = 1,\dots,n-1 ,$$
$$\text{and } a^i <_{t_{i+1}} a^{i+1} \quad \text{for } i = 1,\dots,n-1 .$$

Then $a^j <_{t_i} a^i$ for all $i,j = 1,\dots,n$, $i \neq j$.

PROOF:

Apply the last lemma with $T = \{t_1, t_2\}$ and $A_1 \times A_2 = \{a^1, a^2\}$ to deduce that $a_1^1 \leq_{A_1} a_1^2$. Similarly, it is easy to see that

$$a_1^k \leq_{A_1} a_1^{k+1} \quad \text{for } k = 2,\dots,n-1.$$

Assume inductively that for all i,j such that $0 < |j-i| \leq k$, $a^j <_{t_i} a^i$. Note that $a^{i-1} <_{t_i} a^i$, $a^{i-(k+1)} <_{t_{i-1}} a^{i-1}$ and $a_1^{i-(k+1)} \leq_{A_1} a_1^{i-1}$. Therefore, by SCP, $a^{i-(k+1)} <_{t_i} a^{i-1}$. Hence, $a^{i-(k+1)} <_{t_i} a^i$. Similarly, one can show that $a^{i+(k+1)} <_{t_i} a^i$. The proof now follows by induction. □

This result should have many important applications anywhere optimization is involved. The example at the beginning of section 2 was one. The above proposition might, however, give the wrong impression that SCP is fruitless when we deal with a nondenumerable set of types. This would really be unfortunate since most foreseen applications of SCP are to models with a continuum of types. We now intend to correct that impression and state a version of proposition 2.2 with a continuum of types.

2.3 Again, let A_1, A_2, T be open subintervals of **R**, and let continuously differentiable utility functions $U(\cdot;t)$ represent the preference relations $\le_t$. Consider a continuously differentiable function

$$a:T \to A_1 \times A_2,\ a(t) = (a_1(t), a_2(t))$$

that satisfies the <u>first-order condition</u>

$$U_1(a(t);t)a_1'(t) + U_2(a(t);t)a_2'(t) = 0.$$

Let's assume that $a_1'(\cdot) \neq 0$ and $U_2 \neq 0$. Suppose SCP holds. Then for $\tau < t$ ($> t$), we have by i)-iv) of 2.2.2 that

i) $\quad - U_1(a(\tau);t)/U_2(a(\tau);t) < (>)\ a_2'(\tau)/a_1'(\tau)$

when U_1 and U_2 are positive for all t;

ii) $\quad - U_1(a(\tau);t)/U_2(a(\tau);t) > (<)\ a_2'(\tau)/a_1'(\tau)$

when U_1 is positive and U_2 is negative for all t;

iii) $\quad - U_1(a(\tau);t)/U_2(a(\tau);t) < (>)\ a_2'(\tau)/a_1'(\tau)$

when U_1 is negative and U_2 is positive for all t;

and iv) $\quad - U_1(a(\tau);t)/U_2(a(\tau);t) > (<)\ a_2'(\tau)/a_1'(\tau)$

when both U_1 and U_2 are negative for all t.

If $a_1'(\cdot) > 0$ at every t, the above inequalities imply that

$$U_1(a(\tau);t)a_1'(\tau) + U_2(a(\tau);t)a_2'(\tau) > (< 0)$$

for $\tau < t$ ($> t$): i.e. a(t) is a global maximum of $U(\cdot;t)$ over the set $\{a(\tau) \mid \tau \in T\}$. We just argued that this version of proposition 2.2

was valid.

<u>Proposition</u>: Assume SCP holds in the context where A_1, A_2, T are open subintervals of **R**, and the preference relations $\leq_t$ can be represented by differentiable utility functions $U(\cdot;t)$. Suppose there is a differentiable function $a:T \to A_1 \times A_2$, $a(t) = (a_1(t), a_2(t))$, such that

i) $a_1'(\tau) > 0$ for all $\tau \in T$,

ii) $a(\cdot)$ verifies the first-order condition

$$U_1(a(t);t)a_1'(t) + U_2(a(t);t)a_2'(t) = 0$$

at all t.

Then, for all $\tau \neq t$ in T, $U(a(\tau);t) < U(a(t);t)$.

This result suggests that the single-crossing property could be a very relevant assumption in auction theory and, more generally, in information economics. It is in fact implicit in many of these models, where stronger hypothesis have already been brought in. We are going to illustrate this point in the following sections.

3. <u>The single-crossing property in optimal auction design</u>

As figure 1 may have already suggested, SCP is a crucial assumption for the design of optimal auctions. We can actually support this assertion much further, thanks to Guesnerie and Laffont's (1984) analysis of principal-agent problems.

Let us consider a class **K** of auction game forms $R = ((\Theta)^n, (H_i^r; \beta_i^r, \alpha_i^r)_{i=1,\ldots,n})$ where

- n is the number of (admitted) bidders,
- Θ is the set of allowed individual bidding policies (say a subinterval of the real line),
- $H_i^r: \Theta^n \to [0,1]$ is the probability that bidder i wins,

$$H_1^r + \ldots + H_n^r \leq 1.$$

- $\beta_i^r: \Theta^n \to (-\infty, \infty)$ is a winner's required-payment function, and
- $\alpha_i^r: \Theta^n \to (-\infty, \infty)$ tells a losing bidder what he must pay.

Every R in that class is such that losers incur no cost ($\alpha_i^r(\cdot) = 0$ for all i), the H_i^r's, β_i^r's are piecewise continuously differentiable, a bidder's probability of winning increases with his message ($\partial_i H_i^r(\cdot) > 0$), and the functions H_i^r and β_i^r are symmetric, i.e invariant under any permutation of their arguments other that the ith and

$$H_i^r(\ldots\theta_i, \ldots, \theta_j \ldots) = H_j^r(\ldots\theta_j, \ldots, \theta_i \ldots),$$
$$\beta_i^r(\ldots\theta_i, \ldots, \theta_j \ldots) = \beta_j^r(\ldots\theta_j, \ldots, \theta_i \ldots).$$

An auctioneer now has to operate an auction game form R from class K in an environment where bidders have randomly distributed private types in Θ and expected (continuously differentiable) utility

$$\underline{U}(\underline{H}^r(\theta_i), \underline{\beta}^r(\theta_i); \theta_i) := \mathbf{P}_{\theta_i} U(H_i^r(\cdot), \beta_i^r(\cdot); \theta_i),$$

($\mathbf{P}_{\theta_i}$ being the expectation operator conditional on θ_i). She is interested in finding an auction game form R with payment functions

$\beta_i^r(\cdot)$ that yields the highest expected income nP $H_i^r(\cdot)\beta_i^r(\cdot)$. We shall call the β_i^r's <u>implementable</u> if the corresponding H_i^r's are such that

$$\underline{U}(\underline{H}^r(\theta_i),\underline{\beta}^r(\theta_i);\theta_i) \geq \underline{U}(\underline{H}^r(\sigma),\underline{\beta}^r(\sigma);\theta_i). \tag{1}$$

for all $\sigma \in \Theta$.

If the expected utility function $\underline{U}(\cdot)$ satisfies the single-crossing property, proposition 2.3 implies that the function $\underline{\beta}(\cdot)$, corresponding to an implementable payment function $\beta_i^r(\cdot)$, obeys the differential equation

$$\underline{\beta}'(\theta) = - \frac{\underline{U}_1(H(\theta),\beta(\theta);\theta)}{\underline{U}_2(H(\theta),\beta(\theta);\theta)} \underline{H}'(\theta).$$

We can say furthermore that SCP is also a necessary condition for implementability.

<u>Proposition</u>. If the piecewise continuously differentiable payment function $\beta_i^r(\cdot)$ is implementable, then

(SCP) $$\frac{\partial}{\partial\theta}\,(-\,\underline{U}_1/\underline{U}_2) \geq 0.$$

PROOF: (Guesnerie and Laffont (1984), p. 337)

Computing the first-order necessary condition for inequality (1) we get

$$\partial_1\, \varphi(\theta,\theta) \;:=\; \underline{U}_1(\underline{H}(\theta),\underline{\beta}(\theta);\theta)\underline{H}'(\theta) + \underline{U}_2(\underline{H}(\theta),\underline{\beta}(\theta);\theta)\, \beta'(\theta) \;=\; 0,$$

Since this equation must hold almost everywhere, it must be that

$$\partial_{11}\, \varphi(\theta,\theta) + \partial_{12}\, \varphi(\theta,\theta) = 0$$

also almost everywhere. But the second-order necessary condition for

yields $$\partial_{11}\, \varphi(\theta,\theta) \leq 0. \tag{1}$$

Hence, $$\partial_{12}\, \varphi(\theta,\theta) \geq 0;$$

that is

$$-\frac{(\frac{\partial}{\partial\theta}\underline{U}_2)\underline{U}_1 + (\frac{\partial}{\partial\theta}\underline{U}_1)\underline{U}_2}{\underline{U}_2}\,\underline{H}'(\theta) \geq 0,$$

which implies SCP. □

This proposition shows why the single-crossing property must be assumed, at least implicitly, in the design of optimal auctions. Optimal auctions have been proposed so far in environments that contain risk averse or risk neutral bidders with independently distributed private signals. We shall now verify that the single-crossing property is indeed present in these particular environments, and more general ones.

4. Affiliation and risk aversion imply SCP

Deriving the single-crossing property in an auction model with affiliated private valuations and risk neutral bidders was recently performed by Riley (1985). However, the omnipresence of SCP was not yet underlined. We now want to emphasize the general validity of SCP by verifying it in Milgrom and Weber's (1982) model of a first-price sealed-bid auction with affiliation and risk aversion.

In this model, a bidder's expected utility function is given by $V(X,b;z) = \mathbf{P}\,[\,U(v(X_1,Y_1)-b)\cdot I_{(Y_1<X)}\mid X_1 = z]$

where X_1 is the bidder's private **signal**, Y_1 is the highest private signal received by his competitors, $v(X_1,Y_1)$ is the bidder's <u>random</u> valuation of the good, $\mathbf{P}\ I_{(Y_1<X)}$ is his unconditional probability of winning when the competitors' bids are increasing functions of their signal, b is the amount he bids, U is his utility function and **P** is the expectation operator. Conditional on X_1, Y_1 is distributed according to a density $f_Y(\cdot|z)$. The following assumptions are made:

1) (individual rationality) $V(\cdot;z) \geq 0$;

2) $v(\cdot,\cdot)$ is increasing in its first argument;

3) the utility function is twice continuously differentiable and strictly concave: i.e. $U'(\cdot) > 0$, $U''(\cdot) < 0$;

4) $f_Y(\cdot|z) > 0$;

5) the density $f(\cdot|z)$ verifies the <u>affiliation relation</u>:

for all α, X: $\quad f_Y(\alpha|z)/f_Y(X|z)$ **is decreasing in z when $\alpha \leq X$.**

Now, we can write

$$V(X,b;z) = \mathbf{P}\ [\ \mathbf{P}\ [U(v(X_1,Y_1)-b)\cdot I_{(Y_1<X)}\ |\ X_1,Y_1]\ |\ X_1=z\ ]$$

$$= \mathbf{P}\ [\ \mathbf{P}\ [U(v(X_1,Y_1)-b)\ |\ X_1,Y_1]\cdot I_{(Y_1<X)}\ |\ X_1=z\ ]$$

$$= \int_{-\infty}^{X} \mathbf{P}\ [U(v(X_1,Y_1)-b)\ |\ X_1=z,\ Y_1=\alpha]\ f_Y(\alpha|z)\ d\alpha\ .$$

Then, the partial derivatives of $V(\cdot;z)$ are respectively

$$V_1(X,b;z) = \mathbf{P}\ [U(v(X_1,Y_1)-b)\ |\ X_1=z,\ Y_1=X]\ f_Y(X|z)$$

$$V_2(X,b;z) = -\int_{-\infty}^{X} \mathbf{P}\ [U'(v(X_1,Y_1)-b)\ |\ X_1=z,\ Y_1=\alpha]\ f_Y(\alpha|z)\ d\alpha\ .$$

To demonstrate that the (parametrized by z) family of expected utilities $V(X,b;z)$ has the single-crossing property, one must verify that the ratio

$$-V_1/V_2 = \frac{P\,[U(v(X_1,Y_1)-b)\ \mid\ X_1=z,\ Y_1=X]}{\int_{-\infty}^{X} P\,[U'(v(X_1,Y_1)-b)\ \mid\ X_1=z,\ Y_1=\alpha]\ \frac{f_Y(\alpha|z)}{f_Y(X|z)}\ d\alpha} \tag{2}$$

is <u>increasing</u> in z. This is actually true and follows directly from the previous assumptions.

PROOF:

By the first two assumptions and Milgrom and Weber's theorem 5, the numerator of (2) is positive and increasing in z.

By assumption 3 and Milgrom and Weber's theorem 5, $P\,[U'(v(X_1,Y_1)-b)\ \mid\ X_1=z,\ Y_1=\alpha]$ is positive and decreasing in z. In addition,by the affiliation relation, the ratio $\frac{f_Y(\alpha|z)}{f_Y(X|z)}$ is also decreasing in z. Therefore, the denominator is positive and decreasing in z.

This establishes SCP in our context. □

This result completes Milgrom and Weber's (1982) analysis of the first-price sealed-bid auction where bidders are risk averse and have affiliated private information: precisely, it implies that the solution of the differential equation characterizing a stationary strategy is an equilibrium. Coupled with proposition 3, it supports our claim that SCP was omnipresent in the theoretical analysis of auctions. We shall see now that the single-crossing property is important as well in experimental studies of auctions.

5. How SCP might make experimental auctions work

One nice feature about auction theory is that it seems relatively easy to set up laboratory or classroom experiments in order to test its predictions. These experiments, however, may be inconclusive because the subjects' bidding behavior is biased by external, theoretically-excluded considerations. As Smith (1982) points out, there is

> ...the possibility that economic agents may attach nonmonetary subjective cost (or value) to the process of making and executing individual decisions. The subjective cost of transacting, that is the cost of thinking, calculating and acting, need not be inconsequential. (p. 933)

In this case, Smith says, there is a fundamental precept of controlled economic experiments that is not fulfilled. Such precept he calls "Dominance" and states as follows:

> *Dominance*: The reward structure dominates any subjective costs (or values) associated with participation in the activities of an experiment. (p. 934)

Interestingly, the single-crossing property can be a sufficient condition in some contexts for Dominance to be satisfied. Let us consider, for example, the following reinterpretation of one of Smith's models (p. 932-4):

- bidders are identified by $i = 1,\ldots,n$;
- x^i denotes the number of units of a fictive commodity acquired

by subject i ;

- $V_i(x^i)$ is the (monetary) reward of bidder i for buying x^i. The reward schedule $V_i(\cdot)$ is set by the experimenter so that it is increasing and strictly concave, i.e. $V_i'(\cdot) > 0$ and $V_i''(\cdot) < 0$.

Let $\pi_i(x^i)$ be bidder i's <u>net</u> currency earnings if he buys x^i. When the Dominance precept holds, subject i gets a satisfaction level of, say, $U_i(\pi_i(x^i))$ from x^i. The theory stipulates bidders will ask for the quantity that

$$\text{maximizes} \quad U_i(V_i(x^i)-px^i),$$

where p is the current price of the fictive good. Hence, after assuming $U_i'(\cdot) > 0$, $U_i''(\cdot) < 0$ (bidders are never satiated but have decreasing marginal satisfaction), applying the first-order necessary and sufficient conditions for a maximum, it is predicted that each subject i will demand an amount

$$x^i = V_i'^{-1}(p).$$

The experiment can then be repeated with an adjusted price $p + \Delta p$,

$$\Delta p = [\sum_{i=1}^{n} V_i'^{-1}(p) - \omega],$$

where ω is the available amount of the fictive good.

Let us now suppose that bidders derive their satisfaction level not only from wealth but also from some "external consideration". For instance, bidder i's utility from earning π_i might be equal to $U^i(\pi_i, m_i)$, with the partial derivatives U_1^i, U_2^i being positive and negative respectively. In this case, the Dominance precept is generally violated, and even rational subjects who

$$\text{maximize} \quad U^i(V_i(h_i(m_i))-ph_i(m_i);m_i)$$

with respect to m_i will make distorted claims

$$x_-^i \equiv h_i(m_i^*) = V_i'^{-1}(p - U_2^i/(U_1^i \cdot h_i')).$$

One remedy the experimenter has is to increase the bidders' monetary rewards by a factor α. If subject i's preferences can be represented by

$$\underline{U}^i(\pi_i,m_i;\alpha) := U^i[\alpha(V_i(h_i(m_i)) - p\cdot h_i(m_i));m_i],$$

and if he is a utility-maximizer, he will ask for

$$x_*^i = V_i'^{-1}(p - U_2^i/(U_1^i \cdot h_i' \cdot \alpha)).$$

Therefore, Dominance can be restored, and x_*^i will get pretty close to x^i, provided the factor α is large enough <u>and the single-crossing property holds</u>, i.e. $-\underline{U}_1^i/\underline{U}_2^1$ is increasing in α.

This illustrates the important role SCP also plays for the success of experimental auctions.

6. Conclusion

This essay was devoted to presenting a simple mathematical hypothesis, the single-crossing property, that has many applications in auction theory and information economics in general. Roughly, such property stipulates that in a parametrized family of indifference curves, the ones with a higher parameter cross the ones with a lower parameter at most once and from above. When this single-crossing property holds, it greatly simplifies the computation of an equilibrium, because it implies that a strategy that solves the first-order necessary conditions for a bidder's optimal offer is an equilibrium bidding strategy.

In section 3, we proved SCP to be a necessary condition for the implementability of optimal auctions. Then we showed that SCP held under affiliation and risk aversion in Milgrom and Weber's (1982) general theory of auctions and competitive bidding. Finally, we illustrated how SCP lies in the background of experimental auctions.

Given the framework we provide in section 2, it is easy to check that SCP is already assumed in many other fields of information economics: in job market signalling (where the parameter of the utility function is ability, and the arguments are wage and education), in limit pricing (where the parameter is production cost, and the arguments are output and probability of entry), in warranties (where the parameter is quality of the product, and the arguments are sales and completeness of warranty), etc... The single-crossing property surely constitutes a unifying hypothesis for the mathematical treatment of the economics of information.

We were careful in this essay to state the single-crossing property in its most general fashion, for we also hope that, like constraints qualification or convexity, it can find its niche as a standard assumption of optimization theory.

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