"COMMON KNOWLEDGE OF A MULTIVARIATE AGGREGATE STATISTIC"

by

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Common Knowledge of a Multivariate Aggregate Statistic

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Abstract

If a stochastically monotone function of asymmetrically informed individuals' expectations of a random vector is common knowledge, then all the individuals must agree on their expectations. This result generalizes the theorem of Nielsen, Brandenburger, Geanakoplos, McKelvey and Page (1989) from random variables to random vectors. It holds for general information structures given by sigma-algebras. In the illustrative case of normal distributions and linear signals, it is a statement about linear algebra, and it can be interpreted geometrically. Applied to a version of Grossman's (1975, 1976, 1978) securities market model with asymmetric information, the result implies that the equilibrium price is common knowledge only if all investors agree on their conditional distributions of asset returns. Combined with a result about pooling of linear signals, this observation implies that the linear rational expectations equilibrium is unique.

Keywords: Asymmetric information, common knowledge, stochastic monotonicity, rational expectations equilibrium.
1 Introduction

The most basic observation in the theory of common knowledge, initiated by Aumann (1976), is that rational agents cannot "agree to disagree" on the basis of asymmetric information. Specifically, they cannot differ in their probability estimates of an event if they are commonly aware of each other's estimates.

This result has been extended in several directions. Sebenius and Geanakoplos (1983), Nielsen (1984) and Nielsen, Brandenburger, Geanakoplos, McKelvey and Page (1989) have generalized it from information structures represented by partitions to general information structures and from conditional probabilities of events to conditional expectations of random variables. Geanakoplos and Polemarchakis (1982) have transformed it from its static setting to a result about dynamic communication and convergence to consensus. The dynamic version has been generalized to general information structures and conditional expectations of random variables by Nielsen (1984).

The concept of common knowledge has been widely applied. One of the best known applications is the no-trade theorem and the study of rational expectations equilibria by Milgrom and Stokey (1982). Common-knowledge concepts reminiscent of Aumann's have been used in the study of game theory and its philosophical underpinnings, cf. Binmore and Brandenburger (1989). Related notions of "approximate common knowledge" have been studied and used by Monderer and Samet (1989), Rubinstein (1989) and others.

The most substantial new development in the theory of common knowledge was accomplished by McKelvey and Page (1986). Their theorem says that not only can the agents not agree on (be commonly aware of) their disagreement, but they cannot even agree on (be commonly aware of) a summary statistic of their disagreement. A simple proof has been provided in Nielsen, Brandenburger, Geanakoplos, McKelvey and Page (1989), and a dynamic version of the result has been given by Bergin (1989). However, few applications have appeared beyond those in the original paper.

The present paper has three related aims. The first is to generalize the McKelvey-Page theorem to the case where a multivariate summary statistic is common knowledge. The second is to throw further light on the
theorem by developing its implications and its associated geometry in the special case of normal distributions and linear signals. The third is to apply the theorem to a version of Grossman's (1975,1976,1978) securities market model, where it is used to show two facts. First, the naive equilibrium price is common knowledge only if all investors agree on their conditional distributions of asset returns. Second, the linear rational expectations equilibrium is unique.

The multivariate version of the Aumann theorem says that if a group of individuals form conditional expectations of a random vector based on their diverse information sources, and if all individuals' conditional expectations of the vector are common knowledge, then the conditional expectations are identical across individuals.

The multivariate McKelvey-Page theorem says that if a stochastically monotone function of the individuals' conditional expectations of a random vector is common knowledge, then the expectations are common knowledge, and they have to be identical across individuals. A univariate function is stochastically monotone if it is additively separable into strictly increasing components, cf. Brandenburger and Bergin (1989). The generalized McKelvey-Page theorem requires a definition of stochastic monotonicity for multivariate functions. The definition will be a natural extension of the characterization given by Brandenburger and Bergin (1989): A function is stochastically monotone if it can be separated into a sum of "strictly comonotonic functions."

Even with the simple proof provided by Nielsen et al. (1989), the (generalized) McKelvey-Page theorem continues to be somewhat surprising and non-intuitive. To further illustrate the theorem, we develop the special case where the random vector is a linear transformation of some normally distributed state variables, and where information consists in receiving signals which are linear transformations of the same state variables. In this case, both the Aumann theorem and the McKelvey-Page theorem can be reformulated as statements about linear algebra. Even though these statements follow from the general results, special proofs relying only on linear and bilinear algebra are provided for illustrative purposes.

In the case where the random vector equals the state variables, the multivariate McKelvey-Page theorem has the following curious geometric interpretation (stated in Proposition 15): Suppose a finite number of linear
subspaces of \(\mathbb{R}^m\) are given. If the sum of the orthogonal projection mappings onto the orthogonal complements of these subspaces is zero on the sum of subspaces, then all the subspaces are identical. A somewhat more general consequence of the multivariate McKelvey-Page theorem is this (stated in Proposition 16): Suppose a finite number of symmetric positive definite matrices are given, and think of them as defining linear mappings. If the sum of the kernels of these mappings is contained in the kernel of their sum, then all the kernels are identical.

The multivariate McKelvey-Page theorem is applied to a version of Grossman's (1975, 1976, 1978) securities market model. There is one riskless asset and several risky assets, whose returns are linear transformations of some normally distributed state variables. The investors have constant absolute risk aversion and observe signals which are linear transformations of the state variables. The theorem implies that in a naive equilibrium, prices are common knowledge only if all investors agree on the conditional distribution of returns (Proposition 19).

On the basis of this result and a theorem about pooled information (to be sketched below), it is possible to show that there exists exactly one linear rational expectations equilibrium. Existence and uniqueness was shown by Grossman (1975) in the special case where there is only one risky asset and where the information structures have the following special form. Each investor observes the sum of the risky return and a noise variable specific to him. The noise variables are identically, independently, normally distributed and independent of the return to the risky asset. The advantage of using the multivariate McKelvey-Page common-knowledge result is that it leads to a simple proof of uniqueness in the general case with several risky assets and with information structures given by multivariate linear signals whose covariances with each other and with the returns vector may be highly complicated.

Apart from the multivariate McKelvey-Page theorem, the other major ingredient in the proof of uniqueness of the linear rational expectations equilibrium is the following result about pooled information in the case of normal distributions and linear signals. If a number of agents all have the same conditional expectation of a random vector (which is a linear transformation of the normally distributed state variables), then that conditional expectation equals the conditional expectation given the pool of the agents’
information. This result is peculiar to the case of normal distributions and linear signals. We show by example that it does not hold for general random variables and general information structures.

Section 2 reviews the concept of common knowledge based on information structures described by sigma-algebras and restates the Aumann theorem. Section 3 defines stochastic monotonicity for multivariate functions and develops the generalized McKelvey-Page theorem. Section 4 sets up the case where the random vector and the signals are jointly normally distributed. Section 5 develops some technical results relating to linear information structures. Section 6 describes the linear signals that correspond to common knowledge, and restates the results for this particular case with proofs that rely only on algebra. This section also includes the geometric interpretations of the McKelvey-Page theorem. Section 7 describes the linear signals that correspond to pooled information and states the result that if the all individuals' conditional expectations of a random vector are identical, then they equal the conditional expectation given the pooled information. Section 8 applies the McKelvey-Page theorem to the securities market model. Section 9 works through the special case of information structures like those used in Grosmann (1975,1976).
2 Common Knowledge

The uncertain environment is described by a probability space \((\Omega, \mathcal{F}, P)\). The elements of \(\Omega\) are states of the world, and the sets in \(\mathcal{F}\) are called events. There are \(n\) individuals \(i = 1, \ldots, n\). Each individual \(i\)'s information structure is (represented by) a sigma-algebra \(\mathcal{G}_i\) contained in \(\mathcal{F}\).

An illustrative special case is where \(\Omega\) consists of finitely or countably many states \(\omega\), each with positive probability. Each individual \(i\) has a partition \(\Pi_i\) of \(\Omega\) such that the sigma-algebra \(\mathcal{G}_i\) consists of all unions of cells in \(\Pi_i\). The interpretation of the partitions is this. At the state \(\omega \in \Omega\), individual \(i\) knows the cell \(\Pi_i(\omega)\) of \(\Pi_i\) that contains \(\omega\), and he knows every event \(A\) such that \(\Pi_i(\omega) \subseteq A\).

An event \(A\) is common knowledge at a state \(\omega\) if everybody knows \(A\) at \(\omega\), everyone knows that each of the others knows \(A\) at \(\omega\), and so on. Aumann (1976) formalized this notion in the case of a finite state space and showed that it is equivalent to the following, which may for present purposes be taken as a definition (in the finite or countable case). Let \(\Pi\) denote the meet (the finest common coarsening) of the partitions \(\Pi_i\). An event \(A\) is common knowledge at \(\omega \in \Omega\) if \(\Pi(\omega) \subseteq A\).

A random vector \(X\) is said to be common knowledge at \(\omega \in \Omega\) if the event

\[\{\omega' \in \Omega : X(\omega') = X(\omega)\}\]

is common knowledge at \(\omega\). Say that \(X\) is common knowledge if \(X\) is common knowledge at \(\omega\) for all \(\omega \in \Omega\). An event \(A\) is common knowledge if its indicator function \(1_A\) is common knowledge. This will be the case if and only if \(A\) is common knowledge at \(\omega\) for all \(\omega \in A\) and the complement \(\Omega \setminus A\) is common knowledge at \(\omega\) for all \(\omega \in \Omega \setminus A\).

The sigma-algebra \(\mathcal{G} = \bigcap_i \mathcal{G}_i\) consists of all unions of events in \(\Pi\). Note that an event \(A\) is common knowledge if and only if \(A \in \mathcal{G}\), and a random vector \(X\) is common knowledge if and only if it is measurable with respect to \(\mathcal{G}\). These observations allow a direct generalization beyond the finite or countable case. In the general case, an event \(A\) is common knowledge (by definition) if it differs by a probability-zero event from an event in \(\mathcal{G}\), and a random vector \(X\) is common knowledge if it is almost surely equal to a random vector which is measurable with respect to \(\mathcal{G}\). Equivalently, \(A\) is
common knowledge if it belongs to the completion $\bar{G}$ of $G$ ($\bar{G}$ is the smallest sigma-algebra containing $G$ and all probability-zero events from $\mathcal{F}$), and $X$ is common knowledge if it is measurable with respect to $\bar{G}$.

Common knowledge thus defined is a global concept—common knowledge "at every $\omega$" (or "at almost every $\omega$") as opposed to a local concept of common knowledge "at some particular $\omega$". Local definitions for the case of general information structures are developed in Nielsen (1984) and Brandenburger and Dekel (1987), but they are not needed for our purposes.

Suppose the individuals are interested in the value of an integrable random vector $X$. Individual $i$'s conditional expectation of $X$ is

$$q^i = E(X|G_i),$$

which is also a random vector. Conditional probabilities of an event $A$ is a special case of this, namely where $X = 1_A$ and $q^i = E(1_A|G_i) = P(A|G)$. In keeping with standard practice, equations and inequalities involving conditional expectations are understood to hold only almost surely.

The following is a version of Aumann's (1976) theorem about common knowledge, restated in terms of common knowledge about a random vector:

**Theorem 1** If $q^i$ is common knowledge, then $q^i = E(X|G)$. Consequently, if $q^i$ is common knowledge for all $i$, then all $q^i$ are identical.

**Proof:** $q^i = E(X|G_i) = E(E(X|G_i)|G) = E(X|G)$. $\square$

Aumann originally stated and proved the theorem for conditional probabilities in the finite case. Also in the finite case, the result for conditional expectations of a random variable is stated in Proposition 2 of Geanakoplos and Sebenius (1983). Theorem 4.1 of Nielsen (1984) is a version of the case of conditional expectations of a random variable with general information structures.
3 Aggregation of Expectations

Again, there are \( n \) individuals, \( i = 1, \ldots, n \), all of whom agree on the probability measure \( P \). Individual \( i \)'s information structure is a sigma-algebra \( \mathcal{G}_i \) contained in \( \mathcal{F} \). Individuals are interested in the value of an integrable random \( k \)-vector \( X \). For each \( i \), let

\[
q^i = E(X|\mathcal{G}_i)
\]

be individual \( i \)'s conditional expectation of \( X \), and let

\[
q = (q^1, \ldots q^n).
\]

If \( f : (\mathbb{R}^k)^n \to \mathbb{R}^k \) is a measurable function, then \( f(q) \) is interpreted as a set of summary statistics of the individuals' expectations of \( X \). Such a function \( f \) will be said to be (strictly) stochastically monotone if it has the form

\[
f(x) = f(x^1, \ldots, x^n) = f_1(x^1) + \ldots + f_n(x^n),
\]

where \( f_1, \ldots, f_n \) are strictly comonotonic functions \( \mathbb{R}^k \to \mathbb{R}^k \). A function \( g : \mathbb{R}^k \to \mathbb{R}^k \) is strictly comonotonic if \( (y - x)'(g(y) - g(x)) > 0 \) for all \( x, y \) in \( \mathbb{R}^k \) with \( x \neq y \). This definition of stochastic monotonicity generalizes the characterization for the case \( k = 1 \) given Brandenburger and Bergin (1989). In the case \( k = 1 \), a strictly comonotonic function is the same thing as a strictly increasing function.

For example, if \( A \) is a positive definite \( k \times k \)-matrix and \( b \) is a constant \( k \)-vector, then the function

\[
g(x) = Ax + b
\]

is strictly comonotonic. If \( A^i \) is a positive definite \( k \times k \)-matrix for each \( i = 1, \ldots, k \), and \( b \) is a constant \( k \)-vector, then the function

\[
f(x) = f(x^1, \ldots, x^n) = A^1x^1 + \ldots + A^n x^n + b
\]

is stochastically monotone.

The following theorem generalizes Theorem 1 of McKelvey and Page and Theorem 3 of Nielsen et al. (1989). The latter corresponds to the case where \( X \) is a (univariate) random variable.
Theorem 2 If the function $f : (\mathbb{R}^k)^n \rightarrow \mathbb{R}^k$ is stochastically monotone and $f(q)$ is common knowledge, then for all $i$, $q^i$ is common knowledge and $q^i = E(X|G)$. Consequently, all $q^i$ are identical.

PROOF: Since $f_i$ is strictly comonotonic, 

$$(q^i - E[q^i|G])(f_i(q^i) - f_i(E[q^i|G])) > 0$$

whenever $q^i \neq E[q^i|G]$. So,

$$0 \leq E[(q^i - E[q^i|G])(f_i(q^i) - f_i(E[q^i|G]))|G]$$

$$= E[(q^i - E[q^i|G])f_i(q^i)|G]$$

$$= E[(X - E[X|G])f_i(q^i)|G],$$

and the right-hand side is zero only if $q^i = E(q^i|G)$.

Since $f(q)$ is common knowledge,

$$0 = E[X - E[X|G]|G]'f(q)$$

$$= E[(X - E[X|G])'f(q)|G]$$

$$= \sum_i E[(X - E[X|G])'f_i(q^i)|G],$$

so that $E[(X - E[X|G])'f_i(q^i)|G] = 0$ for each $i$. Hence, $q^i = E(q^i|G) = E(X|G)$ for all $i$, as desired. $\square$

Remark 1 It may appear that the argument above implicitly relies on certain further integrability assumptions. For example, the conditional expectation $E[(X - E(X|G))'f_i(q^i)|G]$ is ordinarily only defined if $(X - E(X|G))'f_i(q^i)$ is assumed to be integrable. However, as noted in connection with similar calculations in Nielsen et al. (1989), no such assumptions are necessary for the argument. The calculations in the proof can be justified by the following extension of the calculus of conditional expectations. Let $\mathcal{H}$ be a sigma-algebra (in the argument above, $\mathcal{H} = G$). If $Z$ is a non-negative random variable, then $E(Z|\mathcal{H})$ makes sense, even though it may take the value $+\infty$ with positive probability. If $Y$ is a random variable which is not necessarily non-negative, call it $\mathcal{H}$-integrable above (below) if $E(Y^+|\mathcal{H}) < \infty$ almost surely ($E(Y^-|\mathcal{H}) < \infty$ almost surely), and call it
$\mathcal{H}$-integrable if it is $\mathcal{H}$-integrable both above and below. The usual rules of manipulation of conditional expectations carry over to these more general variables. For example, the sum of two $\mathcal{H}$-integrable ($\mathcal{H}$-integrable below) random variables is $\mathcal{H}$-integrable ($\mathcal{H}$-integrable below). Also, if $Z$ is a $\mathcal{H}$-measurable random variable and $Y$ is a $\mathcal{H}$-integrable ($\mathcal{H}$-integrable below) random variable, then $ZY$ is $\mathcal{H}$-integrable ($\mathcal{H}$-integrable below), and $E(ZY|\mathcal{H}) = ZE(Y|\mathcal{H})$. \qed
4 Linear Signals and Normal Distributions

This section develops the case where the random environment is described by a vector of normally distributed state variables, \( X \) is a linear transformation of the state variables, and information is acquired by observation of signals which are also linear transformations of state variables. The case of normal distributions is a prime example where information structures given by partitions are not adequate.

Suppose \( Y \) is a random \( m \)-vector which follows a standard normal distribution in \( \mathbb{R}^m \). It is interpreted as the underlying vector of state variables.

Let \( X \) be a \( h \times m \) matrix with rank \( h \). The individuals are interested in the value of the random vector \( XY \), and they get information about it by observing a linear transformation of \( Y \). Because the joint distribution is normal, linear transformations are reasonably easy to handle.

Information channels that are "linear in \( Y \)" can be described by information matrices. An information matrix is a \( k \times m \) matrix \( S \), \( k \leq m \), with full row rank. The idea is that somebody's information consists in observing the signal \( SY \), whatever value \( Y \) takes.

If \( S \) is an information matrix, then the conditional distribution \( XY \) given \( SY \) is normal with conditional mean

\[
E(XY|SY) = XS'(SS')^{-1}SY = XHY,
\]

where

\[
H = S'(SS')^{-1}S,
\]

and with conditional covariance

\[
cov(XY|SY) = XX' - XS'(SS')SX' = X(I - H)X',
\]

where \( I \) is the \( m \times m \) identity matrix. Note that the conditional mean depends on the outcome of \( Y \) (through the signal \( SY \)), but the conditional variance does not. The conditional variance depends only on the information matrix, not on the actual signal.

In particular, taking \( X = I \) to be the \( m \times m \) identity matrix, the conditional distribution of \( Y \) given \( SY \) is normal with conditional mean

\[
E(Y|SY) = HY
\]

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and with conditional covariance

$$\text{cov}(Y|SY) = I - H.$$  

For any matrix $S$, let $\text{ker}(S)$ denote the kernel of the linear mapping $y \mapsto Sy$, i.e., the space of solutions to the equation $Sy = 0$. It is also the space of vectors orthogonal to all the rows of $S$.

**Proposition 1** The matrix $H$ has the following properties.

1. $H' = H$.
2. $HH = H$.
3. $SH = S$.
4. $\text{ker}(H) \perp \text{im}(H)$.
5. $\mathbb{R}^n = \text{ker}(H) + \text{im}(H)$.
6. $\text{ker}(H) = \text{ker}(S)$.
7. The mapping $y \mapsto Hy$ is the orthogonal projection onto $\text{im}(H) = \text{ker}(S)^\perp$.

**Proof:** Statements 1–3 are easily proved by calculation. To show 4, let $y \in \text{ker}(H)$ and $z = xH \in \text{im}(H)$. Then $zy = xHy = x0 = 0$. To show 5, let $y \in \mathbb{R}^n$. Then $H(y - Hy) = Hy - HHy = 0$, so $y - Hy \in \text{ker}(H)$ and $y \in \text{ker}(H) + \text{im}(H)$. Statement 6 follows from 3 and the definition of $H$. To show 7, note that $y - Hy \in \text{ker}(H) = \text{ker}(S)$, as shown above. □

Because of property 7 in Proposition 1, let us call $H$ the projection matrix associated with the information matrix $S$.

The conditional distribution of $Y$ given $SY$ is concentrated on $HY + \text{ker}(S)$. The conditional covariance is regular on $\text{ker}(S) = \text{ker}(H)$, but $z'(I - H)y = 0$ for all $z$ when $y \in \text{im}(H) = \text{ker}(S)^\perp$.

If two information matrices have the same kernel, then they also have the same projection matrix, and so they will give rise to the same conditional distribution of $Y$.

Let $\sigma(S)$ denote the sigma-algebra generated by the mapping $Y \mapsto SY$. This sigma-algebra represents the information revealed by $SY$. Two
different information matrices may well reveal the same information and generate the same sigma-algebra. However, the kernel of an information matrix determines the information structure (sigma-algebra) in a manner to be explained presently.

Suppose \( S \) and \( T \) are two information matrices (possibly with different dimensions \( k \)). Then \( T \) corresponds to a finer information structure than \( S \) (formally, \( \sigma(S) \subset \sigma(T) \)) precisely if \( \ker(T) \subset \ker(S) \). This will be the case if and only if there exists a matrix \( M \) such that \( S = MT \). Moreover, \( T \) and \( S \) correspond to the same information structure (\( TY \) and \( SY \) generate the same sigma-algebra) if and only if \( \ker(T) = \ker(S) \), and if and only if there exists a regular matrix \( M \) such that \( S = MT \). To see that a regular \( M \) exists, argue as follows. If \( T \) and \( S \) correspond to the same information structure, then there exist matrices \( M \) and \( N \) such that \( S = MT \) and \( T = NS \). This implies that \( SS' = MTS' = MNSS' \) and \( MN = SS'(SS')^{-1} = I \). A similar argument shows that \( NM = I \), so that \( M \) is regular with \( M^{-1} = N \).

**Proposition 2** The following statements are equivalent:

1. \( XY \) is a function of \( SY \).
2. \( \ker(S) \subset \ker(X) \).
3. \( X = XH \)

**PROOF:** Statement 2 implies statement 3: For all \( y \),

\[
(X - XH)y = X(y - Hy) = 0
\]

because \( y - Hy \in \ker(H) = \ker(S) \subset \ker(X) \). It is obvious that 3 implies 1 and 1 implies 2. \( \square \)

The equivalence of 1 and 2 in Proposition 2 is a linear algebra version of the following result for general probability spaces: If \( \mathbf{\tilde{z}} \) and \( \mathbf{\tilde{s}} \) are random vectors, then \( \mathbf{\tilde{z}} \) is a measurable function of \( \mathbf{\tilde{s}} \) if and only if the sigma-algebra generated by \( \mathbf{\tilde{z}} \) is coarser than the sigma-algebra generated by \( \mathbf{\tilde{s}} \). See Billingsley (1986), Theorem 20.1.

Geometrically, Statement 3 in Proposition 2 says that it makes no difference to project orthogonally onto \( \ker(S)^\perp \) before applying the mapping \( y \mapsto Xy \).
When necessary in order to make clear the dependence of $H$ on $S$, write $H_S$ instead of $H$:

$$H_S = S'(SS')^{-1}S.$$ 

**Proposition 3** If $S$ and $T$ are information matrices with $\ker(S) \subset \ker(T)$, then

$$H_T = H_S H_T = H_T H_S.$$ 

**PROOF:** It follows from Proposition 2 that $H_T = H_T H_S$. But then $H_T = H_T' = H_S H_T' = H_S H_T$. □

Proposition 3 is a linear algebra version of the law of iterated expectations. Recall that the mappings $y \mapsto H_S y$ and $s \mapsto H_T z$ are the orthogonal projection mappings onto $\ker(S)\perp$ and $\ker(T)\perp$, respectively. Geometrically, Proposition 3 says that if one linear space is contained in another, then projecting first on one of them and then on the other is the same as projecting only on the smaller space.

**Proposition 4** For every information matrix $S$,

$$\ker(XH_S) = \ker(X) \cap \ker(S)\perp + \ker(S)$$

**PROOF:** Follows easily from Proposition 1. □

**Proposition 5** For every information matrix $S$,

$$XH_{XH_S} = XH_S$$

**PROOF:**

$$XH_{XH_S} = X \left[ H_SX'(XH_SH_SX')^{-1}XH_S \right] $$

$$= XH_SX'(XH_SX')^{-1}XH_S $$

$$= XH_S$$

□

Proposition 5 says that the conditional expectation of $XY$ given the conditional expectation of $XY$ given $SY$ is the same as the the conditional expectation of $XY$ given $SY$. 

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Proposition 6  If $S$ and $T$ are two information matrices with
\[ \ker(XH_S) = \ker(XH_T) \]
then
\[ XH_S = XH_T \]

**Proof:**
\[ XH_S = XH_{XH_S} = XH_{XH_T} = XH_T \]

\[ \square \]

Proposition 6 says that if the conditional expectations of $XY$ given each of two information matrices contain the same information, then the conditional expectations are identical.

Proposition 7  If $a \in \mathbb{R}^k$, then the conditional variance $a'X(I - H)X'a$ of $a'XY$ given $SY$ is zero if and only if $X'a \in \ker(S)^\perp$, and if and only if $a'XY$ is a (linear) function of $SY$. The conditional covariance matrix $X(I - H)X'$ of $XY$ given $SY$ is regular if and only if
\[ \ker(X)^\perp \cap \ker(S)^\perp = \{0\} \]

**Proof:** $a'X(I - H)X'a = 0$ if and only of $(I - H)X'a = 0$, which is equivalent to $X'a \in \ker(S)^\perp$. Also, $a'X(I - H)X'a = 0$ if and only of $a'X = a'XH$, which, by Proposition 2, is equivalent to $a'XY$ being a function of $SY$. The statement about the regularity of $X(I - H)X'$ follows from the fact that $\ker(X)^\perp = \{X'a : a \in \mathbb{R}^k\}$. \[ \square \]

In Proposition 7, there is perfect information about the linear transformation $a'XY$ if $a'XY$ is a function of $SY$. The proposition implies that there is perfect information about $a'XY$ if and only if the conditional variance of $a'XY$ is zero.
5 Linear Sigma-Algebras

Let $\sigma(S)$ denote the sigma-algebra generated by the mapping $Y \mapsto SY$, and let $\bar{\sigma}(S)$ denote the corresponding augmented sigma-algebra (the augmentation of $\sigma(S)$ by zero-probability Borel events). In this section we shall characterize the sigma-algebras of the form $\sigma(S)$ and $\bar{\sigma}(S)$ and show that if a linear mapping is almost surely equal to a mapping which is measurable with respect to $\sigma(S)$, then the linear mapping is itself measurable with respect to $\sigma(S)$.

The sigma-algebra $\sigma(S)$ consists of the sets of the form $C + \ker(S)$, where $C$ ranges over measurable subsets of $\ker(S)^\perp$.

In general, a linear sigma-algebra is a sub-sigma-algebra $\mathcal{G}$ of the Borel sigma-algebra $\mathcal{B}$ of the form

$$\mathcal{G} = \{C + K : C \in \mathcal{B}(K^\perp)\},$$

where $K$ is a linear subspace of $\mathbb{R}^m$ and where $\mathcal{B}(K^\perp)$ denotes the Borel sigma-algebra on $K^\perp$.

If $\mathcal{G}$ is a linear sigma-algebra, then the linear space $K$ is uniquely determined, because it is the largest linear space with the following invariance property: for all $D$ in $\mathcal{G}$, $D + K \subset D$. Moreover, given $K$, $\mathcal{G}$ is characterized by the invariance property in the following way:

$$\mathcal{G} = \{D \in \mathcal{B} : D + K \subset D\}.$$

This is shown in the following proposition.

**Proposition 8** Let $\mathcal{G}$ be a linear sigma-algebra.

1. $K = \ker(\mathcal{G})$ is the largest linear space such that $D + K \subset D$ for all $D$ in $\mathcal{G}$.

2. $\mathcal{G} = \{D \in \mathcal{B} : D + K \subset D\}$.

**Proof:** To prove the second statement, note that if $D \in \mathcal{B}$ and $D + K \subset D$, then $D = C + K$, where $C$ is the image of $D$ under the orthogonal projection onto $K^\perp$. To prove the first statement, note that $K \in \mathcal{G}$. If $L$ is another linear space, then $K + L \subset K$ only if $L \subset K$. $\square$

Call $K$ the kernel of $\mathcal{G}$ and write $K = \ker(\mathcal{G})$.

A linear sigma-algebra is determined by its kernel, in the following sense.
Proposition 9 If $G$ and $H$ are two linear sigma-algebras, then $G \subseteq H$ if and only if $\ker(G) \supseteq \ker(H)$, and $G = H$ if and only if $\ker(G) = \ker(H)$.

PROOF: This follows from the invariance property of the kernel. \(\square\)

Every linear sigma-algebra $G$ has the form $G = \sigma(S)$ for some information matrix $S$ (which is not unique). Simply set $k = m - \dim(\ker(S))$ and choose a $k \times m$ matrix $S$ such that $\ker(S) = \ker(G)$.

The augmentation $\tilde{G}$ of a sigma-algebra $G$ consists of the sets of the form $A \triangle N$, where $A \in G$ and $N$ is a null event. An event $B$ belongs to $\tilde{G}$ if and only if there exists an event $A \in G$ such that $B \triangle A$ is a null event.

If $G$ is a linear sigma-algebra, then the kernel $K = \ker(G)$ is uniquely determined not only by $G$ but by the augmentation $\tilde{G}$, through an “almost sure” version of the invariance property described above. Moreover, given $K$, $\tilde{G}$ is characterized by the same “almost sure” invariance property. This is brought out in the following proposition.

Proposition 10 Let $G$ be a linear sigma-algebra.

1. $K = \ker(G)$ is the largest linear space such that for all $D \in \tilde{G}$, there exists a null event $N$ such that $D \triangle N + K \subseteq D \triangle N$.

2. $\tilde{G}$ consists of the events $D \in B$ for which there exists a null event $N$ such that $D \triangle N + \ker(G) \subseteq D \triangle N$.

PROOF: For all $D$ in $\tilde{G}$, there exists a null event $N$ such that $D \triangle N \in G$ and hence $D \triangle N + \ker(G) \subseteq D \triangle N$. To complete the proof of (1), suppose $K$ is a linear space with the property in question, and suppose it is not contained in $\ker(G)$. Let $v \in \ker(G)^\perp$ and $w \in K$ with $v'w < 0$. Let $D = \{d : v'd \geq 0\}$. Then $D \in G \subseteq \tilde{G}$. Let $N$ be a null event such that $D \triangle N + K \subseteq D \triangle N$. Set $C = \{x : 0 > v'x \geq v'w\}$. Then $C \subseteq D + w$, so

$$C \triangle (N + w) \subseteq D \triangle N + w \subseteq D \triangle N + K \subseteq D \triangle N$$

This contradicts the fact that $C$ and $D$ have positive probability and are disjoint, while $N + w$ and $N$ are null events. Statement 2 follows directly from (2) of Proposition 8. \(\square\)

A linear information structure is represented equally well by a linear sigma-algebra or its augmentation. This is a consequence of the following Proposition.
Proposition 11 If $G$ and $H$ are two linear sigma-algebras, then $G \subset H$ if and only if $\bar{G} \subset \bar{H}$, and $G = H$ if and only if $\bar{G} = \bar{H}$.

Proof: This follows from Propositions 9 and 10. □

Because of Proposition 11, we use only linear sigma-algebras, not their augmentations. One consequence of the proposition is this: If $S$ is an information matrix, then a linear transformation of $Y$ is almost surely equal to a function which is measurable with respect to $\sigma(S)$ if and only if the transformation itself is measurable with respect to $\sigma(S)$, and if and only if it is measurable with respect to $\bar{\sigma}(S)$. Also, two linear transformations are identical if and only if they are identical almost surely.
6 Common Knowledge with Linear Signals

Suppose there are $n$ individuals $i = 1, \ldots, n$, and that individual $i$ observes the signal $S(i)Y$, where $S(i)$ is an information matrix with $k_i$ rows, $k_i \leq m$. We shall construct an information matrix $C$ which corresponds to the common information structure, i.e., a matrix $C$ with $\sigma(C) = \bigcap_i \sigma(S(i))$. Note that there has to be some degree of arbitrariness in this construction. If the information matrix $C$ corresponds to the common information structure, then so does any matrix $T$ of the form $T = MC$, where $M$ is a regular $k \times k$ matrix, i.e., any matrix which has the same kernel as $C$.

**Proposition 12** Let $C$ be an information matrix. The following statements are equivalent:

1. $C$ corresponds to the common information structure.
2. $\ker(C) = \sum_i \ker(S(i))$.
3. $C$ has these two properties:
   
   (a) For each $i$, there exists a matrix $A(i)$ such that $C = A(i)S(i)$.
   
   (b) If $T$ is a matrix such that for each $i$, there exists a matrix $B(i)$ with $T = B(i)S(i)$, then there exists a matrix $B$ with $T = BC$.

The equivalence of (2) and (3) in Proposition 12 is a simple matter of linear algebra. Property (a) says that $C$ is a (linear) transformation of $S(i)$ for each $i$, so that $CY$ is no more discriminating than any of the individual signals $S(i)Y$. Property (b) says that $CY$ is the "most discriminating" signal with this property: any other signal $TY$ with this property has to be less discriminating in the sense that it is a (linear) transformation of $CY$.

It follows from (a) that $CY$ is a transformation of $S(i)Y$, so that the information structure generated by $C$ is no finer than that generated by $S(i)$. (1) implies that any function of $Y$ which is a function of $S(i)Y$ for each $i$ is in fact (almost surely equal to) a function of $CY$. Condition (b) says only that this is true of functions that are linear transformations of $S(i)$; but it turns out to be true also of non-linear transformations.

The proof of Proposition 12 relies on the following lemma.
Lemma 1 If $G_i$ is a linear sigma-algebra for each $i$, then $\bigcap_i G_i$ is a linear sigma-algebra, and
\[ \ker(\bigcap_i G_i) = \sum_i \ker(G_i). \]

**Proof:** Follows from the invariance characterization of linear sigma-algebras in Proposition 8. \( \square \)

**Proof of Proposition 12:**

(1) equivalent to (2): This follows from Lemma 1.

(2) equivalent to (3): (a) is equivalent to $\ker(S(i)) \subseteq \ker(S)$ for all $i$; and
(b) is equivalent to the statement that if $T$ is a matrix with $\ker(S(i)) \subseteq \ker(T)$ for all $i$, then $\ker(S) \subseteq \ker(T)$. That statement is equivalent to $\ker(S) \subseteq \ker(S(i))$ for all $i$. \( \square \)

To construct a matrix $C$ with the properties in Proposition 12, one way to proceed is as follows. For each $i$, form the linear space spanned by the rows of $S(i)$. Pick a linear basis for the intersection of those spaces, treat the basis vectors as row vectors, and stack them on top of each other to form a matrix. That matrix $C$ has the required property, that $\ker(C) = \sum_i \ker(S(i))$.

An alternative construction is possible. For each $i$, let $T(i)$ be a $m \times (m - k_i)$ matrix with full column rank $m - k_i$, such that $S(i)T(i) = 0$; in other words, the columns of $T(i)$ are orthogonal to the rows of $S(i)$. Line the matrices $T(i)$ up next to each other to form a matrix whose columns are all of the columns of the $T(i)$'s. Let $l$ be the rank of the matrix thus created. Delete all but a set of $l$ linearly independent columns, and call the resulting $m \times l$ matrix $T$. Now let $C$ be a $k \times m$ matrix whose $k = m - l$ rows are linearly independent and orthogonal to all the columns in $T$. Again, $C$ has the required property, that $\ker(C) = \sum_i \ker(S(i))$.

For each $i$, let
\[ H(i) = S(i)'(S(i)S(i)')^{-1}S(i). \]
Let $C$ be an information matrix corresponding to the common information structure, and let
\[ H_C = C'(CC')^{-1}C. \]
Then
\[ \ker(H_C) = \ker(C) = \sum_i \ker(S(i)). \]
Since \( \ker(S(i)) \subset \ker(C) \), it follows from Proposition 3 that

\[
H_C = H(i)H_C = H_CH(i).
\]

The generalized Aumann result says that if \( E(XY|S(i)Y) \) is common knowledge, then it equals \( E(XY|CY) \). In the present setting, this is a statement about matrix algebra, as is apparent from the following version of the result.

**Theorem 3** The following statements are equivalent.

1. \( \ker(C) \subset \ker(XH(i)) \).
2. \( XH(i) = XH_C \).

Theorem 3 follows from Theorem 1, but it will be proved here by means of matrix algebra:

**Proof of Theorem 3**: To show that (1) implies (2), note that it follows from Proposition 2 that \( XH(i) = XH(i)H_C \). But then \( XH(i) = XH(i)H_C = XH_C \). It is clear that (2) implies (1). \( \Box \)

The geometry of Theorem 3 is brought out in the following restatement:

**Proposition 13** Let \( L, K_1, \ldots, K_n \) be linear subspaces of \( IR^n \). The following statements are equivalent.

1. \( \sum_j K_j \subset L \cap K_i^\perp + K_i \)
2. \( L \cap K_i^\perp = L \cap (\sum_j K_j)^\perp + \sum_j K_j \)

**Proof**: The equivalence follows from Theorem 3, Proposition 4 and Proposition 6 by setting \( L = \ker(X) \) and \( K_j = \ker(S(j)) \) (the equivalence can also easily be demonstrated directly). \( \Box \)

In connection with Theorem 3, we may note that if everybody has the same conditional expectation of \( XY \), then that conditional expectation equals the conditional expectation given the common information:

**Proposition 14** The following statements are equivalent:

1. \( XH(i) = XH(j) \) for all \( i, j \).
2. $XH(i) = XH_C$ for all $i$.

**Proof:** It is clear that (2) implies (1). If (1) holds, then

$$\ker(C) = \sum_i \ker(S(i)) \subset \sum_j \ker(XH(j)) = \ker(H(i)),$$

so that (2) follows from Theorem 3. ☐

The theorem of McKelvey and Page (1986), as generalized in Section 3, tells us that if $\sum_i E(XY|S(i)Y)$ is common knowledge, then $E(XY|S(i)Y) = E(XY|CY)$ for all $i$. Again, this is a statement about linear algebra, as is apparent from the following version of the theorem. For illustrative purposes, we provide a new proof which relies on linear and bilinear algebra.

**Theorem 4** The following statements are equivalent.

1. $\ker(C) \subset \ker(X \sum_i H(i))$.
2. $XH(i) = XH_C$ for all $i$.

**Proof:** It follows from Proposition 2 that (1) is equivalent to

3. $X \sum_i H(i) = X (\sum_i H(i)) H_C$.

It is clear that (2) implies (1). To see that (3) implies (2), use Propositions 2 and 3 to verify that

$$H(i)(I - H_C)[H(i)(I - H_C)]' = H(i)(I - H_C).$$

It follows that

$$\sum_i XH(i)(I - H_C)[XH(i)(I - H_C)]' = \sum_i XH(i)(I - H_C)X'$$

$$= X \left( \sum_i H(i)(I - H_C) \right) X'$$

$$= 0.$$

Consequently, for all $i$, $XH(i)(I - H_C) = 0$, and $XH(i) = XH(i)H_C = XH_C$. ☐

The geometry of Theorem 4 in the special case $X = I$ is brought out in the following restatement.
Proposition 15 Let $K_1, \ldots, K_n$ be linear subspaces of $\mathbb{R}^m$, and for each
$i$, let $\Pi_i : \mathbb{R}^m \to K_i^\perp$ be the orthogonal projection mapping. If
\[ \sum_i K_i \subset \ker \left( \sum_i \Pi_i \right) \]
then
\[ K_1 = \ldots = K_n. \]

PROOF: For each $i$, choose an information matrix $S(i)$ such that $\ker(S(i)) = K_i$. Let $C$
be an information matrix corresponding to common knowledge. Then $\Pi_i(y) = H(i)y$ and $\sum_i \Pi_i(y) = \sum_i H(i)y$ for all $y$. The assumption is that
\[ \ker(C) \subset \ker \left( \sum_i H(i) \right). \]
It follows from Theorem 4 that $H(1) = \ldots = H(n)$, which implies that
$K_1 = \ldots = K_n$. □

Theorem 4 can be generalized to the following situation. For each $i$, let $A(i)$ be a positive
definite matrix. The generalized theorem of McKelvey and Page tells us that if $\sum_i A(i)E(XY|S(i)Y)$
is common knowledge, then $E(XY|S(i)Y) = E(XY|CY)$ for all $i$. Again, this is a statement about
linear algebra, and for illustrative purposes, we provide a new proof which relies on linear and bilinear algebra.

Theorem 5 The following statements are equivalent.

1. $\ker(C) \subset \ker(\sum_i A(i)XH(i))$.

2. $XH(i) = XHC$ for all $i$.

PROOF: It follows from Proposition 2 that (1) is equivalent to the following:

3. $\sum_i A(i)XH(i) = \sum_i A(i)XH(i)HC$.

It is clear that (2) implies (1). To see that (3) implies (2), first replace each $A(i)$ by a symmetric matrix:

\[ B(i) = \frac{1}{2}(A(i) + A(i)'), \]

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Note that
\[
\text{tr}[A(i)'XH(i)(I - H_C)X'] = \text{tr}[(H(i) - H_C)X'A(i)'X]
\]
\[
= \text{tr}[X'A(i)XH(i)(I - H_C)]
\]
\[
= \text{tr}[A(i)XH(i)(I - H_C)X'],
\]
so that \(\text{tr}[A(i)XH(i)(I - H_C)X'] = \text{tr}[B(i)XH(i)(I - H_C)X']\). Since \(B(i)\) is symmetric and positive definite, there exists a non-singular matrix \(P(i)\) such that \(B(i) = P(i)'P(i)\). The trace of the matrix

\[
P(i)XH(i)(I - H_C)(I - H_C)H(i)X'P(i)'
\]

is the sum of squared entries in the matrix \(P(i)XH(i)(I - H_C)\). Hence, the trace is non-negative, and it is zero only if \(P(i)XH(i)(I - H_C) = 0\). Now,

\[
0 \leq \text{tr}[P(i)XH(i)(I - H_C)(I - H_C)H(i)X'P(i)']
\]
\[
= \text{tr}[P(i)XH(i)(I - H_C)X'P(i)']
\]
\[
= \text{tr}[P(i)'P(i)XH(i)(I - H_C)X']
\]
\[
= \text{tr}[B(i)XH(i)(I - H_C)X']
\]
\[
= \text{tr}[A(i)XH(i)(I - H_C)X']
\]

Since

\[
\sum_i \text{tr}[A(i)XH(i)(I - H_C)X'] = \text{tr}[\sum_i A(i)XH(i)(I - H_C)X']
\]
\[
= \text{tr}[0X']
\]
\[
= 0,
\]
it follows that

\[
\text{tr}[P(i)XH(i)(I - H_C)(I - H_C)H(i)X'P(i)'] = 0
\]
for all \(i\), so that \(P(i)XH(i)(I - H_C) = 0\) and \(XH(i) = XH_C\) for all \(i\), as desired. \(\square\)

In the case where \(X = I\), Theorem 5 has the following linear algebra consequence.
Proposition 16 Let $B(1), \ldots, B(n)$ be symmetric positive semidefinite $m \times m$ matrices. If

$$\sum_i \ker(B(i)) \subset \ker(\sum_i B(i))$$

then $\ker(B(1)) = \ldots = \ker(B(n))$.

PROOF: For each $i$, choose an information matrix $S(i)$ such that $\ker(S(i)) = \ker(B(i))$. Then $B(i) = B(i)H(i)$ for all $i$. There exist positive definite matrices $A(i)$ such that $A(i)H(i) = B(i)$. To see this, let $d_i = \dim(\ker(B(i)))$ and choose a $d_i \times m$ matrix $Q(i)$ with rank $d_i$ such that $Q(i)H(i) = 0$. Then $Q(i)z = 0$ only if $z \in \text{im}(H(i)) = \ker(S(i))^\perp$. Set $A(i) = B(i) + Q(i)'Q(i)$. Then $A(i)H(i) = B(i)H(i) = B(i)$. Because $B(i)$ is symmetric, $y'A(i)y = 0$ implies $Q(i)y = 0$ and $B(i)y = 0$. Hence, $A(i)$ is positive definite. Set $A = \sum_i B(i) = \sum_i A(i)H(i)$. Let $C$ be an information matrix corresponding to common knowledge. Since $\sum_i \ker(S(i)) = \ker(C)$, the assumption is that $\ker(C) \subset \ker(A)$. It follows from Theorem 5 that $H(1) = \ldots = H(n)$, which implies that $\ker(B(1)) = \ldots = \ker(B(n))$. □
7 Pooled Information

One can also construct an information matrix $P$ which corresponds to the pooled information structure $\forall_i \sigma(S(i))$ in the sense that $\sigma(P) = \forall_i \sigma(S(i))$. Again, there is some arbitrariness in this choice. If the information matrix $P$ corresponds to the pooled information structure, then so does any matrix $T$ of the form $T = MP$, where $M$ is a regular matrix, i.e., any matrix $T$ which has the same kernel as $P$.

Proposition 17 Let $P$ be an information matrix. The following statements are equivalent:

1. $P$ corresponds to the pooled information structure.
2. $\ker(P) = \bigcap_i \ker(S(i))$.
3. $P$ has these two properties:
   (a) For each $i$, there exists a matrix $A(i)$ such that $S(i) = A(i)P$.
   (b) If $T$ is a matrix such that for each $i$, there exists a matrix $B(i)$ with $S(i) = B(i)T$, then there exists a matrix $B$ with $P = BT$.

The equivalence of (2) and (3) in Proposition 17 is a matter of linear algebra. Property (a) says that $S(i)$ is a (linear) transformation of $P$ for each $i$, so that $PY$ is no less informative than any of the individual signals $S(i)Y$. Property (b) says that $PY$ is the "least informative" signal with this property: any other signal $TY$ with this property has to be more informative in the sense that $PY$ can be derived from it as a (linear) transformation.

As in Proposition 12, it is remarkable is that (3) implies (1) in Proposition 17. According to (a), $S(i)Y$ is a transformation of $PY$, so that the information structure generated by $P$ is at least as fine as that generated by $S(i)$. However, it also has to be shown that any function of $Y$ from which $S(i)Y$ can be computed for each $i$ in fact contains enough information that $PY$ can be computed. Condition (b) says only that this is true of functions that are linear transformations of $Y$; but it turns out to be true also of non-linear transformations.

The proof of Proposition 17 relies on the following lemma.
Lemma 2 \(\text{If } G_i \text{ is a linear sigma-algebra for each } i, \text{ then } \bigvee_i G_i \text{ is a linear sigma-algebra, and} \)

\[ \ker\left(\bigvee_i G_i\right) = \bigcap_i \ker(G_i). \]

Proof: It suffices to prove the lemma for \(n = 2\). Set \(K_i = \ker(G_i)\) for each \(i = 1, 2\), set \(K = K_1 \cap K_2\), and let \(H\) be the linear sigma-algebra with kernel \(K\). It has to be shown that \(H = G_1 \vee G_2\). It is clear that \(G_1 \vee G_2 \subset H\). The hard part is to show the opposite inclusion.

Choose subspaces \(U_1, U_2, W\) such that \(K_1 = (K_1 \cap K_2) \oplus U_1, K_2 = (K_1 \cap K_2) \oplus U_2, \) and \(\mathbb{R}^m = (K_1 + K_2) \oplus W\). By a dimension argument, \(K_1 + K_2 = (K_1 \cap K_2) \oplus U \oplus V\) and

\[ \mathbb{R}^m = (K_1 \cap K_2) \oplus U_1 \oplus U_2 \oplus W. \]

The sigma-algebra \(G_1\) contains all sets of the form \((K_1 \cap K_2) + U_1 + A_2 + B\), where \(A_2 \in B(U_2)\) and \(B \in B(W)\), and \(G_2\) contains all sets of the form \((K_1 \cap K_2) + A_1 + U_2 + B\), where \(A_1 \in B(U_1)\) and \(B \in B(W)\). The intersection of such sets is \((K_1 \cap K_2) + A_1 + A_2 + B,\) and these generate \(H\). Hence, \(H \subset G_1 \vee G_2\).

Proof of Proposition 17:

(1) equivalent to (2): Follows from Lemma 2.

(2) equivalent to (3): (a) is equivalent to \(\ker(S) \subset \ker(S(i))\) for all \(i\); and (b) is equivalent to \(\ker(S(i)) \subset \ker(S)\) for all \(i\).

To construct a matrix \(P\) with the properties in Proposition 17, proceed as follows. Stack the matrices \(S(i)\) on top of each other to form a matrix whose rows are all of the rows of the \(S(i)'s\). Let \(k\) be the rank of the matrix thus created, and delete all but a set of \(k\) linearly independent rows. The resulting matrix \(P\) has the required property, that \(\ker(P) = \bigcap_i \ker(S(i))\).

An alternative construction is possible. For each \(i\), let \(T(i)\) be a \(m \times (m - k_i)\) matrix with full column rank \(m - k_i\), such that \(S(i)T(i) = 0\); in other words, the columns of \(T(i)\) are orthogonal to the rows of \(S(i)\). For each \(i\), form the linear space spanned by the columns of \(T(i)\). Pick a linear basis for the intersection of those spaces, treat the basis vectors as column vectors, and line them up next to each other to form a matrix \(T\) with rank \(l\). Let \(P\) be a \(k \times m\) matrix, \(k = m - l\), whose rows are linearly independent.
and orthogonal to all the columns in \( T \). Again, \( P \) has the required property, that \( \ker(P) = \bigcap_i \ker(S(i)) \).

For each \( i \), let
\[
H(i) = S(i)'(S(i)S(i)')^{-1}S(i).
\]

Let \( P \) be an information matrix corresponding to the pooled information, and let
\[
H_P = P'(PP')^{-1}P.
\]

Then
\[
\ker(H_P) = \ker(P) = \bigcap_i \ker(S(i)).
\]

Since \( \ker(P) \subseteq \ker(S(i)) \),
\[
H(i) = H(i)H_P = H_P H(i).
\]

When the distributions are normal and the signals linear, the pooled information has the following property, which will be important in the analysis of linear rational expectations equilibria in Section 8:

**Theorem 6** The following statements are equivalent:

1. \( XH_C = XH(i) \) for all \( i \).
2. \( XH_C = XH_P \).

**Proof:**

(2) implies (1): For every \( i \),
\[
XH(i) = XH_P H(i) = XH_C H(i) = XH_C
\]

(1) implies (2):

\[
XH_C H(i) = XH_C = XH(i), \text{ all } i \implies \quad (X - XH_C)H(i) = 0, \text{ all } i \implies \quad \ker(S(i)) = \text{im}(H(i)) \subseteq \ker(X - XH_C), \text{ all } i \implies \quad \ker(P)^\perp = [\bigcap_i \ker(S(i))]^\perp = \sum_i \ker(S(i))^\perp \subseteq \ker(X - XH_C) \implies \quad (X - XH_C)H_P = 0 \implies \quad XH_P = XH_C H_P = XH_C
\]
Theorem 6 says that if everybody has the same conditional expectation of $XY$, then that conditional expectation equals the conditional expectation given the pooled information. That is particular to the case of normal distributions and linear signals, as the following example shows.

Example 1 Consider the probability space $(\Omega, \mathcal{F}, P)$ where

\[ \Omega = \{1, 2, 3, 4\}, \]

$\mathcal{F}$ is the set of all subsets of $\Omega$, and $P$ is given by

\[ P(1) = P(2) = P(3) = P(4) = 1/4 \]

Let $X$ be the random variable defined on $(\Omega, \mathcal{F})$ by

\[ X(1) = X(2) = 1 \]

\[ X(3) = X(4) = -1. \]

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the sigma-algebras (information structures) given by

\[ \mathcal{G}_1 = \{\emptyset, \Omega, \{1, 3\}, \{2, 4\}\} \]

\[ \mathcal{G}_2 = \{\emptyset, \Omega, \{1, 4\}, \{2, 3\}\} \]

Then the pooled information is represented by

\[ \mathcal{G}_1 \lor \mathcal{G}_2 = \mathcal{F} \]

but

\[ E(X|\mathcal{G}_1) = E(X|\mathcal{G}_2) = 0 \]

while

\[ E(X|\mathcal{G}_1 \lor \mathcal{G}_2) = E(X|\mathcal{F}) = X \neq 0. \]
Equilibria in Asset Markets

This section presents an example of a securities market with asymmetric information where the vector of market prices is a stochastically monotone function of agents' expectations. In an equilibrium where the market price is observed and taken into account by agents when they form their expectations, all agents must have the same expectations according to the multivariate McKelvey-Page theorem, and the market price reveals the relevant part of each investor's information. In combination with Theorem 6, this result implies that given the investors' exogenous signals, there is only one linear rational expectations equilibrium.

There are two types of asset trading models that have equilibria where prices are so informative that all agents must have the same conditional distribution of returns. Some models rely on complete markets contingent on payoff-relevant states (Grossman, 1981). McKelvey and Page (1986) show that their theorem can be used to establish the equality of relevant information in a model of this type when agents have logarithmic utility. Other models do not involve complete markets but assume normal distributions (Grossman, 1975, 1976, 1978). Here, I shall apply the multivariate McKelvey-Page theorem to a version of Grossman's model.

As in Section 4, the random environment is based on a random m-vector $Y$, which follows a standard normal distribution in $\mathbb{R}^m$.

There are a riskless asset with total return $R_1$ per share and $h$ risky assets with a vector of random total return per share given by

$$R = \bar{R} + XY,$$

where $X$ is an $h \times m$ matrix with rank $h$.

There are $n$ investors $i = 1, \ldots, n$, with utility functions

$$u_i(c) = -\exp(-a_ic).$$

Investor $i$ observes the signal $S(i)Y$, where $S(i)$ is a $k_i \times m$-matrix with rank $k_i \leq m$. The conditional distribution of $R$ is normal with mean

$$q^i = E(R|S(i)Y) = \bar{R} + XH(i)Y,$$

where

$$H(i) = S(i)'(S(i)S(i)')^{-1}S(i),$$
and covariance matrix
\[ \Omega_i = X(I - H(i))X', \]
which is assumed to be positive definite. According to Proposition 7, this is equivalent to \( \ker(X) \perp \cap \ker(S(i)) \perp = \{0\} \).

Investor \( i \) has an initial endowment of \( \phi_i \) shares of the riskless asset and a vector \( \psi^i \) of shares of the risky assets. The total supply of shares of the risky assets is
\[ \psi = \sum_i \psi^i, \]
which is assumed to be non-random. In the present model, there is no “noise” in the sense of uncertainty about parameters in addition to uncertainty about returns. In particular, there is no noise in the total supply \( \psi \) of risky shares.

The price per share of the riskless asset is normalized at one, and the vector of prices per share of the risky assets is \( p \). The investor’s wealth is
\[ w_i = \phi_i + p'\psi^i. \]

We denote by \( v_i \) the number of shares of the riskless asset that he demands, and by \( z^i \) his vector of demanded numbers of shares of the risky assets. If he buys a vector \( z^i \) of shares of the risky assets, then his random total return is
\[ (w_i - p'z^i)R_f + R'z^i = w_iR_f + (R - R_f p)'z^i. \]

A (naive) equilibrium relative to the information matrices \( S(i), i = 1, \ldots, n \), is a family \( (p, (v_i), (z^i)) \) of random vectors (functions of \( Y \)), such that

1. \( \sum_i v_i = \sum_i \phi_i \) almost surely.
2. \( \sum_i z^i = \psi \) almost surely.
3. \( v_i + p'z^i = \phi_i + p'\psi^i \) almost surely, for all \( i \).
4. For almost all \( Y \), \( (\hat{v}_i, \hat{z}^i) = (v_i(Y), (z^i(Y)) \) maximizes
\[ \int u_i(\hat{v}_iR_f + \hat{z}_iR)P(dR|S(i)Y) \]

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subject to
\[ \hat{v}_i + p(Y)z^i = \phi_i + p(Y)\psi^i \]

(where \( P(.)|S(i)Y) \) is a regular conditional probability distribution: \( P(.)|S(i)Y) \) is the normal distribution with mean \( q^i(Y) \) and covariance matrix \( \Omega_i \).

Requirement (1) is redundant according to Walras' law.
Set
\[ \Gamma = \left( \sum_i a_i^{-1} \Omega_i^{-1} \right)^{-1}. \]

Note that \( \sum_i a_i^{-1} \Omega_i^{-1} \) is positive definite, being a sum of positive definite matrices.

**Proposition 18** There exists one unique naive equilibrium. It is given by
\[ p = R_f^{-1} \Gamma \sum_i a_i^{-1} \Omega_i^{-1} \left( \bar{R} + XH(i)Y \right) - R_f^{-1} \Gamma \psi, \]
\[ z^i = a_i^{-1} \Omega_i^{-1} (\bar{R} + XH(i)Y - R_fp), \]
\[ v_i = \phi_i + p' \psi^i - p' z^i. \]

**Proof:** Maximizing conditional expected utility as in (4) corresponds to maximizing the function \( e - a_is^2/2 \), where \( e \) and \( s^2 \) are the conditional mean and variance of total portfolio return. So, investor \( i \) chooses \( z^i \) to maximize
\[ w_iR_f + (q^i - pR_f)'z^i - a_i(z^i\Omega_i z^i)/2. \]

The first-order condition
\[ q_i - R_fp - a_i \Omega_i z^i = 0 \]
implies
\[ z^i = a_i^{-1} \Omega_i^{-1} (q^i - R_fp). \]

The expression for \( v_i \) comes from (3). Substituting \( z^i \) into (2) gives
\[ \psi = \sum_i a_i^{-1} \Omega_i^{-1} (q^i - R_fp) = \sum_i a_i^{-1} \Omega_i^{-1} q^i - R_f \left( \sum_i a_i^{-1} \Omega_i^{-1} \right) p, \]
which yields the expression for $p$. \qed

The random vector $p$ specified in Proposition 18 will be called the **naive equilibrium price vector** relative to the information matrices $(S(i))$. Note that $p$ is a stochastically monotone function of the individual expectations $q^i = \bar{R} + XH(i)Y$, since the matrices

$$\Gamma a_i^{-1} \Omega_i^{-1}$$

are positive definite.

In a naive equilibrium as defined above, the random price vector $p$ must be linear in $Y$, according to the formula. However, $p$ is not necessarily a function of $S(i)Y$, i.e., it is not necessarily observable by the investor. But according to (4), in each state of the world the investor optimizes subject to a budget constraint which depends on the price in that state. So he may have to optimize subject to a constraint that he does not know. This makes the (naive) equilibrium concept unsatisfactory and leads to the idea of rational expectations.

First, consider a naive equilibrium $(p, (v_i), (z^i))$ relative to the information matrices $S(i)$, where prices happen to be common knowledge. Formally, in addition to (1) – (4), the equilibrium satisfies

5. For each $i$, $p$ is a measurable function of $S(i)Y$.

In the presence of (1) – (4), the latter implies:

6. For each $i$, $(v_i, z^i)$ is a measurable function of $S(i)Y$.

In the presence of the latter two statements, a condition equivalent to (4) is

7. $(v_i, z^i)$ maximizes $Eu_i(v_i R_f + z^i R)$ subject to $(v_i, z_i)$ being a measurable function of $S(i)Y$ and subject to $v_i + pz^i = \phi_i + p'\psi^i$ almost surely.

Let $C$ be an information matrix corresponding to the common information of the matrices $(S(i))$, and let $H_C$ be the corresponding projection matrix. Also, let

$$\Omega_C = X(I - H_C)X'$$

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be the conditional covariance matrix of $XY$ given the common information. Since $\Omega_i$ is positive definite (for every $i$), so is $\Omega_C$.

Set

$$\gamma = \left( \sum_i a_i^{-1} \right)^{-1}.$$ 

**Proposition 19** The naive equilibrium price vector $p$ relative to the information matrices $(S(i))$ is common knowledge if and only if $XH(i) = XHC$ for all $i$. If so, then all investors have the same conditional expectations of the returns vector $R$, and

$$p = \pi Y + \xi$$

with

$$\pi = R_j^{-1}XHC$$

and

$$\xi = R_j^{-1}(\bar{R} - \gamma \Omega_C \psi).$$

**Proof:** As noted above, the naive equilibrium price $p$, given the information matrices $(S(i))$, is a stochastically monotone function of the individual conditional expectations $q^i = \bar{R} + XH(i)Y$. Hence, it follows from Theorem 2 that if the naive equilibrium price $p$, given the information matrices $(S(i))$, happens to be common knowledge, then all $q^i$ must be identical and all the matrices $XH(i)$ must be identical and equal to $XHC$. Conversely, if all $q^i$ are identical in the equilibrium, then it is clear from the equilibrium equation that $p$ is common information. The expressions for $\pi$ and $\xi$ follows from the expression for $p$ in Proposition 18 by some simple calculations. \(\square\)

In a naive equilibrium as in Proposition 19, where the random price vector $p$ is common knowledge,

$$\sigma(p) = \sigma(\pi) = \sigma(XHC) = \sigma(XH(i)) = \sigma(XHP)$$

for all $i$. The last equality follows from Theorem 6. The price vector is *revealing*: it reveals the relevant aspect of each investor's information as well as the relevant aspect of the pooled information. In other words, each investor's conditional expectation of $R = \bar{R} + XY$ is a function of $p$, and the conditional expectation of $R$ given the pooled information is a function of $p$. In terms of sigma-algebras, $\sigma(XH(i)) \subset \sigma(p)$ for all $i$, and $\sigma(XHP) \subset \sigma(p)$. 33
The price vector in Proposition 19 is an $R$-sufficient transformation of each investor’s signal: it is a function of the investor’s signal, while the investor’s conditional expectation of the returns vector $R$ is a function of the price vector. In terms of sigma-algebras, $\sigma(XH(i)) \subset \sigma(p) \subset \sigma(S(i))$. The price vector is also an $R$-sufficient transformation of the pooled signal: it is a function of the pooled signal, while the conditional expectation of the returns vector $R$ given the pooled signal $PY$ is a function of the price vector. In terms of sigma-algebras, $\sigma(XH_P) \subset \sigma(p) \subset \sigma(P)$. Not only is the price vector an $R$-sufficient transformation of each investor’s signal, it is in fact a minimal $R$-sufficient transformation in the sense that it is a function of any other $R$-sufficient transformation. In terms of sigma-algebras, $\sigma(XH(i)) = \sigma(p)$. The price vector in Proposition 19 is also a minimal $R$-sufficient transformation of the pooled signal: $\sigma(XH_P) = \sigma(p)$.

In a rational expectations equilibrium, the investors calculate the conditional distribution of returns given the information contained in the equilibrium price as well as in the signals $S(i)Y$. To formalize this notion, let $p$ be a random price vector which has the linear form

$$p = \pi Y + \xi,$$

where $\pi$ is a $h \times m$ matrix with rank $h$. We may think of $\pi$ as an information matrix. Then $\sigma(\pi)$ is the information revealed by $p$. If an investor observes $p$ and derives information from it in addition to the information gained from observing $S(i)Y$, then his total information will be $\sigma(S(i)) \vee \sigma(\pi)$. An information matrix $S^*(i)$ which corresponds to this pooled information, in the sense that $\sigma(S^*(i)) = \sigma(S(i)) \vee \sigma(\pi)$, will be called an augmented information matrix. An augmented information matrix can be constructed in the following way. Stack the rows of $\pi$ on top of $S(i)$ to form a matrix, and let $\tilde{k}_i$ be its rank. Delete all but $\tilde{k}_i$ linearly independent rows, and let $S^*(i)$ be the resulting matrix. To distinguish $S(i)$ from the augmented information matrix $S^*(i)$, we will refer to $S(i)$ as investor $i$’s raw information matrix.

Let $P$ be a matrix corresponding to the pool of the information in the individual agents’ raw information matrices $S(i)$. We will require that a rational expectations equilibrium price contain no more information than the information in $P$.

By definition, a linear rational expectations equilibrium price (LREE price) relative to the information matrices $S(i)$ is a pair $(\pi, \xi)$, consisting
of an \( h \times m \) matrix \( \pi \) and a vector \( \xi \) such that

- the random price vector
  \[
  p = \pi Y + \xi
  \]
  is the naive equilibrium price vector relative to the augmented information matrices \( (S^\pi(i)) \) formed from \( \pi \) and the basic information matrices \( (S(i)) \).

- \( \pi Y \) is a function of \( PY \).

First, we construct the “artificial economy” equilibrium. Consider an artificial economy where every investor has information \( P \). The conditional distribution of the returns vector \( R \) is normal with mean \( \tilde{R} + XH_PY \) and covariance matrix
\[
\Omega_P = X(I - H_P)X'.
\]

**Assumption 1** \( \Omega_P \) is positive definite.

According to Proposition 7, Assumption 1 is equivalent to
\[
\ker(X) \cap \left( \bigcap_i \ker(S(i)) \right)^\perp = \{0\}.
\]

It implies that
\[
\ker(X) \cap \ker(S(i))^\perp = \{0\},
\]
so that, again by Proposition 7, \( \Omega_i \) is positive definite for each \( i \).

The common information in the artificial economy is given by \( P \). Hence, by Proposition 19, the naive equilibrium price vector in the artificial economy is
\[
p = \pi_P Y + \xi_P,
\]
where
\[
\pi_P = R^-1_f X H_P
\]
and
\[
\xi_P = R^-1_f (\tilde{R} - \gamma \Omega_P \psi).
\]

Since this price vector is common knowledge in the artificial economy, it is not only a naive equilibrium price but also a LREE price in the artificial economy. Moreover, according to the following proposition and theorem, it is the unique LREE price in the original economy.
Proposition 20 \((\pi_P, \xi_P)\) is a \textit{LREE} price relative to the information matrices \((S(i))\).

\textbf{Proof:} For each investor \(i\), let \(S^*(i)\) be an augmented information matrix constructed from \(S(i)\) and \(\pi_P\). It must be shown that \(p = \pi_P Y + \xi_P\) is the naive equilibrium price vector with respect to \((S^*(i))\). Let \(H^*(i)\) be the projection matrix corresponding to \(S^*(i)\). Let \(C^*\) be an information matrix corresponding to the common information of the augmented matrices \((S^*(i))\), and let \(H_{C^*}\) be the corresponding projection matrix. Since \(\sigma(S^*(i)) = \sigma(S(i)) \lor \sigma(XH_P)\), it follows that

\[\sigma(XH_P) \subset \sigma(S^*(i)) \subset \sigma(P).\]

This implies that

\[XH^*(i)Y = E(XY|S^*(i)) = E(XY|PY) = XH_PY,\]

so that \(XH^*(i) = XH_P\). Since this is true of all \(i\), \(XH^*(i) = XH_{C^*}\) and \(XH_{C^*} = XH_P\). By Proposition 19, the naive equilibrium price vector with respect to \((S^*(i))\) is \(p = \pi_P Y + \xi_P\). \(\blacksquare\)

In the proof of Proposition 20, we might alternatively use an algebraic argument to show that \(XH^*(i) = XH_P\): Since \(S^*(i)Y\) is a function of \(PY\) and \(XH_PY\) is a function of \(S^*(i)Y\), it follows that \(S^*(i) = S^*(i)H_P\) and \(XH_P = XH_P H^*(i)\). The former implies \(H^*(i) = H^*(i)H_P\). By transposing this, we get \(H^*(i) = H_P H^*(i)\). So, \(XH^*(i) = XH_P H^*(i) = XH_P\).

\textbf{Theorem 7} If \((\pi, \xi)\) is a \textit{LREE} price relative to the information matrices \((S(i))\), then \((\pi, \xi) = (\pi_P, \xi_P)\).

\textbf{Proof:} For each investor \(i\), let \(S^*(i)\) be an augmented information matrix constructed from \(S(i)\) and \(\pi\). Let \(H^*(i)\) be the projection matrix associated with \(S^*(i)\). Let \(C^*\) be an information matrix corresponding to the common information of the augmented matrices \((S^*(i))\), and let \(H_{C^*}\) be the corresponding projection matrix. The pooled information from \((S^*(i))\) is the same as the pooled information from \((S(i))\), and it is represented by the matrix \(P\). Since the price \(p\) is common knowledge with respect to
the augmented information matrices \( S^*(i) \), it follows from Proposition 19 and Theorem 6 that

\[
\pi = R_f^{-1}XH_{C^*} = R_f^{-1}XH_{P} = \pi_P
\]

and

\[
\xi = R_f^{-1}(\tilde{R} - \gamma\Omega_{C^*}\psi) = R_f^{-1}(\tilde{R} - \gamma\Omega_{P}\psi) = \xi_P
\]

\( \square \)
9 An Example

Suppose \( m = n + 1 \) and suppose \( S(i) \) is the row vector with entries \( i \) and \( m \) equal to \( \hat{\sigma} > 0 \) and \( \sigma > 0 \), respectively, and the other entries equal to zero. Then individual \( i \) observes \( \hat{\sigma}Y_i + \sigma Y_m \). If we set \( \epsilon_i = \hat{\sigma}Y_i \) for \( i = 1, \ldots, n \) and \( Z = \sigma Y_m \), then each individual \( i \) observes \( Z + \epsilon_i \), where \( Z \) and \( \epsilon_i \) are normally distributed with mean zero and standard deviation \( \sigma \) and \( \hat{\sigma} \), respectively, and all the \( \epsilon_i \)'s are mutually uncorrelated and uncorrelated with \( Z \).

Let \( X \) be a row \( m \)-vector with all entries equal to zero except the last, which is equal to \( \sigma \). Then \( XY = Z \).

The matrix \( H(i) \) has the following non-zero entries:

\[
\begin{pmatrix}
H(i)_{ii} & H(i)_{im} \\
H(i)_{mi} & H(i)_{mm}
\end{pmatrix}
= \frac{1}{\sigma^2 + \hat{\sigma}^2}
\begin{pmatrix}
\hat{\sigma}^2 & \hat{\sigma}\sigma \\
\sigma\hat{\sigma} & \sigma^2
\end{pmatrix}.
\]

The conditional expectation of \( Y \) given \( i \)'s information is given by

\[
E(Y_i|S(i)Y) = \frac{\hat{\sigma}^2}{\sigma^2 + \hat{\sigma}^2} (\hat{\sigma}Y_i + \sigma Y_m),
\]

\[
E(Y_j|S(i)Y) = 0
\]

for \( i \neq j < m \), and

\[
E(Y_m|S(i)Y) = \frac{\sigma^{2}}{\sigma^2 + \hat{\sigma}^2} (\hat{\sigma}Y_i + \sigma Y_m).
\]

It follows that

\[
E(\epsilon_i|Z + \epsilon_i) = \frac{\hat{\sigma}^2}{\sigma^2 + \hat{\sigma}^2} (Z + \epsilon_i),
\]

\[
E(\epsilon_j|Z + \epsilon_i) = 0
\]

for \( i \neq j < m \), and

\[
E(XY|S(i)Y) = E(Z|Z + \epsilon_i) = \frac{\sigma^{2}}{\sigma^2 + \hat{\sigma}^2} (\hat{\sigma}Y_i + \sigma Y_m) = \frac{\sigma^{2}}{\sigma^2 + \hat{\sigma}^2} (Z + \epsilon_i).
\]

Consequently, \( S(i)Y \) is a function of \( XH(i)Y \), a feature particular to this example:

\[
XH(i)Y = \frac{\sigma^{2}}{\sigma^2 + \hat{\sigma}^2} S(i)Y
\]
\[
S(i)Y = \frac{\sigma^2 + \hat{\sigma}^2}{\sigma^2} X H(i)Y
\]

The conditional variance of \(XY\) is

\[
\text{var}(XY|S(i)Y) = \text{var}(Z|Z + \varepsilon_i) = \sigma^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \hat{\sigma}^2}\right) = \frac{\sigma^2 \hat{\sigma}^2}{\sigma^2 + \hat{\sigma}^2}.
\]

The kernel of \(S(i)\) is \(\{Y : \hat{\sigma}Y_i + \sigma Y_m = 0\}\), which has dimension \(m - 1\). Hence, \(\sum_i \ker(S(i)) = \mathbb{R}^m\). This means that there is no common information (the common information matrix is 0).

Let \(P\) be the information matrix formed by stacking the rows \(S(i)\) on top of each other:

\[
P = \begin{pmatrix} \hat{\sigma} I & \sigma \iota \end{pmatrix}
\]

Then \(P\) corresponds to the pooled information, and

\[
\ker(P) = \bigcap_i \ker(S(i)) = \{Y : \hat{\sigma}Y_i + \sigma Y_m = 0 \text{ for all } i < m\},
\]

which has dimension one.

The matrix \(PP'\) is an \(n \times n\) matrix with diagonal entries \(\sigma^2 + \hat{\sigma}^2\) and other entries equal to \(\sigma^2\):

\[
PP' = \begin{pmatrix} \hat{\sigma} I & \sigma \iota \end{pmatrix} \begin{pmatrix} \hat{\sigma} I \\ \sigma \iota' \end{pmatrix} = \hat{\sigma}^2 I + \sigma^2 \iota\iota'
\]

where \(I\) is the \(n \times n\) identity matrix and \(\iota\) is an \(n\)-vector all of whose entries are equal to one. It is easily verified that its inverse has a similar form:

\[
(PP')^{-1} = \frac{1}{\hat{\sigma}^2} I + a\iota\iota'
\]

with

\[
a = -\frac{\sigma^2}{(n\sigma^2 + \hat{\sigma}^2)\hat{\sigma}^2}
\]

Note that

\[
\iota'(PP')^{-1} = \iota' \left(\frac{1}{\hat{\sigma}^2} I + a\iota\iota'\right) = \left(\frac{1}{\hat{\sigma}^2} + na\right) \iota' = \frac{1}{n\sigma^2 + \hat{\sigma}^2} \iota'
\]
Consequently,

\[
P'(PP')^{-1} = \left( \begin{array}{c} \hat{\sigma}I \\ \sigma \lambda' 
\end{array} \right) (PP')^{-1} = \left( \begin{array}{c} \hat{\sigma}(PP')^{-1} \\ \frac{\sigma}{n\sigma^2 + \hat{\sigma}^2} \lambda' \n\end{array} \right)
\]

It follows that \( XH_PY \) is a function of \( \lambda'PY = \sum_i(Z + \epsilon_i) \):

\[
XH_PY = XP'(PP')^{-1}PY = \left( \frac{\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \right) \lambda'PY = \left( \frac{\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \right) \sum_i(Z + \epsilon_i)
\]

The matrix \( H_P \) is computed next:

\[
H_P = P'(PP')^{-1}P
\]

\[
= \left( \begin{array}{c} \hat{\sigma}(PP')^{-1} \\ \frac{\sigma}{n\sigma^2 + \hat{\sigma}^2} \lambda' \n\end{array} \right) \left( \begin{array}{c} \hat{\sigma}I \\ \sigma \lambda' \n\end{array} \right)
\]

\[
= \left( \begin{array}{c} \hat{\sigma}^2(PP')^{-1} \\ \frac{\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \lambda' \n\end{array} \right)
\]

The conditional expectation of \( Y \) given the pooled information is given by

\[
E(Y_i|PY) = Y_i - \frac{\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \sum_{j=1}^{n} Y_j + \frac{\sigma\hat{\sigma}}{n\sigma^2 + \hat{\sigma}^2} Y_m
\]

\[
= \frac{1}{\hat{\sigma}}(Z + \epsilon_i) - \frac{\sigma^2}{(n\sigma^2 + \hat{\sigma}^2)\hat{\sigma}} \sum_{j=1}^{n}(Z + \epsilon_j)
\]

for \( i < m \), and

\[
E(Y_m|PY) = \frac{1}{\sigma}XH_PY = \frac{1}{n\sigma^2 + \hat{\sigma}^2} \sum_{j=1}^{n}(Z + \epsilon_j) = \frac{1}{n\sigma^2 + \hat{\sigma}^2} \lambda'PY.
\]

The conditional variance of \( XY \) given \( PY \) is

\[
\Omega_P = \text{var}(XY|PY) = \sigma^2 \left( 1 - \frac{n\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \right) = \frac{\sigma^2\hat{\sigma}^2}{n\sigma^2 + \hat{\sigma}^2}
\]

Now suppose there is a riskless asset with total return \( R_f \) per share and a single risky asset with total return \( R = \bar{R} + XY \) per share. The pooled-information equilibrium price \((\pi_P, \xi_P)\) is given by

\[
\pi_P = R_f^{-1}XH_P = \frac{1}{R_f} \frac{\sigma^2}{n\sigma^2 + \hat{\sigma}^2} \lambda'P
\]

40
\[ \xi_P = R_f^{-1}(\bar{R} - \gamma\Omega_P \psi) = R_f^{-1}(\bar{R} - \gamma \frac{\sigma^2 \hat{\sigma}^2}{n\sigma^2 + \hat{\sigma}^2} \psi) \]

Set
\[ y = \bar{R} + PY \]
and
\[ \bar{y} = \frac{1}{n} y. \]

Manipulations of the equations above yield
\[ \pi_P Y + \xi_P = \alpha_1 \bar{y} + \alpha_0 \]
where
\[ \alpha_1 = \frac{n\sigma^2}{R_f(n\sigma^2 + \hat{\sigma}^2)} \]
and
\[ \alpha_0 = \frac{\hat{\sigma}^2(\bar{R} \sum_i a_i^{-1} - \sigma^2 \psi)}{R_f(n\sigma^2 + \hat{\sigma}^2) \sum_i a_i^{-1}} \]

When \( \hat{\sigma} = 1 \), this reduces to the formula exhibited by Grosmann (1975,1976).
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