ON THE HISTORY OF YIELD APPROXIMATIONS

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1. INTRODUCTION

The problem of deriving approximate values for the unknown yield implicit in a given annuity and the related problem of determining the bond's yield to maturity for a given price, have challenged mathematicians, actuaries and financial analysts over the last four centuries. Indeed, no other problem in finance has a longer intellectual history. The purpose of this paper is to trace the historical developments of the efforts to find simple and accurate methods of approximating an annuity's implicit yield and a bond's yield to maturity. Most of the early contributions in the area of yield approximations can be attributed to British actuaries who are not well known outside their profession and country.

The problem that has preoccupied scholars for nearly 400 years can be rigorously stated as follows. Given the term (n) and the present value of an annuity of one unit (a), what is the implicit yield (i)? The problem then consists in solving the equation:

$$a = \sum_{t=1}^{n} (1+i)^{-t} = \frac{1-(1+i)^{-n}}{i} \qquad (1)$$

which does not have a general algebraic solution for orders higher than the fourth. A solution of $n > 4$ can only be obtained by approximation. Today, calculators and computers can solve this problem by a rapid series of iterations and approximate solutions to the above equation may have lost some of their original usefulness. This, however, has not diminished the importance of the problem which is still receiving the attention of scholars.

The related problem of determining the yield to maturity on a bond with a unit face value paying the fraction g of one unit per period for n periods and one unit at maturity consists in solving the equation:

$$p = g.a + v^n \quad (2)$$

where p is the bond's given price, v equals $1/(1+i)$ and a is defined in equation (1). Again, no general algebraic solution exists for orders higher than the fourth.

The next section of this paper presents a brief review of the early history of the problem and its modern developments. Section 3 is a survey of the most famous approximations of the implicit yield of an annuity. In Section 4 we review and evaluate alternative approximation formulas of the bond's yield to maturity which were suggested by scholars over the last 125 years. We show that the standard textbook approximation formula for the bond yield to maturity is the least accurate of a large family of formulas, some of which were suggested as early as 1855. The last section contains concluding remarks.

2. EARLY HISTORY AND MODERN DEVELOPMENTS

In the year 1556 the Italian physicist and arithmetician Nicolo Fontana (circa 1500-1577) better known as Tartaglia (the Stammerer) published a treatise titled *General trattato* in which he reports the following problem which was put to him by gentlemen from the city of Bari who declared that it really had happened (Sanford, 1930, p. 136):

> "A merchant gave a university 2814 ducats on the understanding that he was to be paid back 618 ducats a year for nine years, at the end of which the 2814 ducats should be considered as paid. What compound interest was he getting on his money?"

This may be the first published account of a problem involving the determination of the implicit yield on a given annuity. The original solution to this problem must have been derived through trial and error since neither logarithms nor published interest-tables were available then.

Logarithms were invented by John Napier in 1614, and the first published interest-tables appeared in 1558 in Jean Trenchant's *L'Arithmétique* followed in 1582 by Simon Steven's *Tables of Interest*.[2] Confidential interest-tables did exist as early as the late thirteenth century but they remained unpublished and available only to the large banking houses like the Baldi of Florence.

But even if interest-tables were available as early as the middle of the sixteenth century, they would have not been very helpful in solving the annuity problem posed by the gentlemen from Bari and reported by Tartaglia. The ratio of 2814 to 618 ducats is the present value of an annuity of one ducat over a nine-year term. An annuity table could have provided only a crude approximation of 16 per cent. A solution closer to the exact rate must be found with the help of approximation formulas which are powerful enough to give results that are near the true rate.

The origin of an analytical solution to the implicit yield problem can be traced back to a little known British writer by the name of Michael Dary.[3] On August 15, 1674, Dary forwarded to Dr. John Newton, the author of *Scale of Interest* (1668), an important theorem which constitutes the basis of an approximate solution for the problem. Dary's theorem is stated as follows:

- If a given function $\phi(\cdot)$ satisfies the property $b = \phi(a)$, $c = \phi(b)$, $d = \phi(c)$, etc., and if $\phi(a)$, $\phi(b)$, $\phi(c)$, etc., do not increase without limit but rather converge towards a finite or zero limit, then that limit is a solution to the equation $x = \phi(x)$.

Dary applies his theorem to the annuity problem for which:

$$i = \phi(i) \equiv \frac{1 - (1+i)^{-n}}{a}$$

where i is the unknown true yield.

Using a crude approximate rate as an initial yield one can compute the value of the ratio $[1 - (1+i)^{-n}] / a$ which will give a value that is closer to the unknown true yield than is the initial rate. Using this new rate one can obtain a still better approximation, and successive calculations will approach the unknown true rate.

A major drawback of Dary's approach is that the approximation converges very slowly towards the true rate and consequently the method is of little practical use. Nevertheless, Dary's theorem is probably the first genuine analytical solution to the implicit yield problem and hence represents a historical and intellectual milestone.

Other approaches for the determination of an approximate rate have been suggested since then. A review of the early literature indicates that these approaches can be classified into four distinct yet general methods of approximation:

- The first method (Method I) consists of approximations by *direct* expansion of the present value of an annuity in powers of the unknown true rate. Early writers on the subject were well aware of the shortcomings of this method, which are discussed below.
- The second method (Method II) involves approximations by expansion of the present value of an annuity *after* replacing the unknown true rate by the sum of a known trial rate near the true rate and an error term which is small relative to the true rate. It will be shown that this method circumvents the limitation of the first.
- The third method (Method III) is based on interpolation between two or more trial rates giving nearly correct values.
- The last method (Method IV) provides empirical approximations; these are formulas that were shown, through trial and error, to constitute reliable approximations of the implicit yield.

As mentioned above, approximations based on the first method will generally give poor results, particularly in the case of annuities. In the case of a bond's yield to maturity it will be seen later that Method I provides adequate approximations. The reason why reliable approximations for the implicit rate on a given annuity cannot generally be obtained by the first method is that successive terms in the expansion of the annuity in powers of the unknown rate do not diminish rapidly enough to allow the neglect of the terms in the expansion beyond the second or the third. In his textbook on interest, life annuities and assurances, Ralph Todhunter (1915) points out that

> "although the successive powers of i form a rapidly decreasing series of quantities, the coefficients by which they are multiplied may increase for a certain number of terms with equal or even greater rapidity, so that the early terms in the expansion will not necessarily exhibit rapid convergency."

Despite these limitations, several authors have derived approximation formulas for the implicit rate of an annuity based on Method I. Some of the best known formulas will be surveyed in the next section. It is worth pointing out that the shortcoming of Method I is similar to that encountered in the application of Dary's theorem to the approximation of the implicit rate on interest of an annuity.

The second method avoids the problem of slow convergency which characterizes direct expansions based on the first method. With Method II, the unknown true yield is replaced in the expansion by an approximation plus a very small unknown quantity which does not increase the coefficient of the expansion and thus produce a more rapidly convergent series than the first method. Consequently, the errors resulting from neglecting the terms beyond the second or the third in the expansion are greatly reduced. This method provides practical approximation formulas for the implicit yield of an annuity or a bond as will be shown in the following two sections.

3. APPROXIMATIONS OF THE IMPLICIT YIELD OF AN ANNUITY

In a paper published in 1859, Augustus De Morgan provides a brief but comprehensive survey of the history of the subject from the late seventeenth century up to the middle of the nineteenth century. It seems that one of the earliest published approximation of the implicit yield on an annuity is a formula suggested by Dr. John Newton in 1670. Newton's approximation which is based on Method I is:

$$\log(1+i) = \frac{6m}{3(n+1) - \frac{(n-1)m}{2(.4342945)}} \tag{3}$$

where $m = \log(n/a)$. Taking a cue from the form of expansion proposed by Newton, De Morgan (1859) and Newling (1903) suggest more accurate formulas than Newton's which are also based on the first method of approximation.

As pointed out in the previous section, better approximations can be obtained, using Method II *and* interest-tables. Two of the earliest approximations based on this technique are by Francis Baily (1808) and George Barrett (1811). The former suggests the following approximation:

$$i = i' + \frac{i'(a'-a)}{a - nv'^{n+1}} \tag{4}$$

where i' is a known trial rate and a' and v' the corresponding values of the annuity and the one-period present value of a unit, respectively. Clearly, one must have interest-tables available in order to calculate an approximate rate according to Baily's formula, or any other formula based on Method II. Baily's formula can be derived as follows:

- Consider a known trial rate $i^{o'}$ which is near the unknown true rate i. The difference ρ between this trial rate and the true rate ($\rho = i - i'$) is a very small quantity relative to either i or i'. Substituting $(i' + \rho)$ for i in the definition of the value a of an annuity gives:

$$a(i'+\rho) = 1 - [\,(1+i') + \rho]^{-n}$$
$$= 1 - [\,(v')^{-1} + \rho]^{-n} \qquad (5)$$

Expanding the bracketed term on the right-hand side of equation (5) gives:

$$a(i'+\rho) = 1 - [\,v'^{n} - nv'^{n+1}\rho + \frac{n(n+1)}{2} v'^{n+2}\rho^2 - \ldots].$$

If the terms involving the second and higher powers of ρ are ignored we get Baily's formula (4).

Barret, in a letter to Baily, suggests an approximation that is more accurate and slightly different from that of Baily's (De Morgan, 1854). Barrett's formula is:

$$i = i' + i' \frac{a' - a}{a' - nv'^{n+1}} . \qquad (6)$$

In Barrett's formula the known value a is replaced in the denominator by the value a' of the annuity evaluated at the trial rate i'. The most elegant derivation of Barrett's formula is given in a note written in 1855 and signed M. Consider the function:

$$\phi(x) = \frac{1 - (1+x)^{-n}}{x} - a.$$

This implies $\phi(i) = 0$, that is, $\phi(i'+\rho) = 0$. Expanding the left-hand side of $\phi(i'+\rho) = 0$, and neglecting the terms involving powers of ρ beyond the first (recall that ρ is very small) yields:

$$\phi(i') + \phi'(i')\rho + \ldots\ldots = 0$$

or

$$\rho = -\frac{\phi(i')}{\phi'(i')} \quad . \tag{7}$$

An approximate value of ρ may be found by substituting the known trial rate i' for the unknown rate i in the right-hand side of equation (7). Following these steps will give:

$$\phi'(i') = \frac{nv'^{n+1} - a'}{i'}$$

and

$$\rho = i' - i = \frac{i'(a'-a)}{a'-nv'^{n+1}}$$

which is the expression for Barrett's formula (6).

In a paper written in 1904, John Spencer points out that the accuracy of Barrett's formula relative to that of Bailey's depends on the term of the annuity. One formula is not consistently superior to the other. In his contribution, Spencer examines this important point and determines the conditions under which one formula yields more accurate approximations than the other. A passage in Todhunter's classical textbook (1915, p. 108) should also be mentioned. In it he points out that

> "closer approximations can be obtained by working with the reciprocal of the annuity-value instead of with the annuity-value itself."

Expanding $\frac{1}{a}$ instead of a yields:

$$\frac{1}{a} = \frac{i}{1-v^n} = \frac{i'+\rho}{1-(1+i'+\rho)^{-n}} = \frac{i'+\rho}{1-v'^n+n\rho v'^{n+1} - \ldots}$$

$$= \frac{1}{a'}\,(1+\frac{\rho}{i'})\,(1-\frac{nv'^{n+1}}{a'}\cdot\frac{\rho}{i'}+\ldots)$$

$$= \frac{1}{a'}\left[1+\frac{\rho}{i'}\,(1-\frac{nv'^{n+1}}{a'}) - \ldots\right]$$

and as a first approximation:

$$i = i' + i' \frac{\frac{1}{a} - \frac{1}{a'}}{\frac{1}{a'} - \frac{nv'^{n+1}}{a'^2}}$$

which is shown to be more accurate than either Baily's or Barrett's formula.

One of the earliest approximation formulas for the implicit yield of an annuity based on Method IV (these are empirical approximations) is suggested in the note written in 1855 and signed M. This short but important contribution was mentioned earlier. The formula suggested by M. is:

$$i = \frac{8\ (n-a)}{(n+1)\ (3a+n)} \quad .$$

Another empirical approximation formula was provided by R. Henderson in 1907. He suggests the following:

$$i = h \left[\frac{n+1}{2n} + \frac{(n-1)h}{6 + 2(n-1)h} \right]^{-1}$$

where $h = (1/a) - (1/n)$.

Other approximation formulas for the implicit yield of an annuity, based on various approximation methods, were later suggested by J.F. Steffensen (1916), R. Montague Webb (1930), A.W. Evans (1944-46), M.T.L. Bizley (1962), H. Karpin (1967), H.O. Worger (1967), and Mahoney (1980) to cite only some of the noteworthy contributions.

4. APPROXIMATIONS OF THE BOND YIELD TO MATURITY

We have shown in the introduction that the determination of the yield to maturity in a bond with a unit face-value paying the fraction g of one unit per period for n periods and one unit at maturity consists in solving the equation

$$p = g.a + v^n$$

where p is the bond's given price, and $v = 1/(1+i)$.

Introducing k to denote the bond's premium $(k > 0)$ or discount $(k < 0)$ from face value, the above equation can be rewritten as:

$$k = p - 1 = g.a + v^n - 1$$

$$k = (g-i)a. \qquad (8)$$

Equation (8) provides the starting point for the examination of all the approximation formulas for the determination of the unknown yield to maturity i.

One of the earliest published accounts of an approximate-bond-yield formula dates back to 1855. The scarcity of earlier recorded contributions on the subject of bond yields - and security valuation in general - is not surprising. After all, the issuance of any substantial volume of securities and their organized trading in stock exchanges did not take place before the end of the eighteenth century in Europe and the middle of the nineteenth century in the United States. The approximation formula presented in the above-mentioned note is based on Method II and hence requires the aid of interest-tables for its application. The method was discussed earlier and applied to the case of an annuity in the previous section. In the case of a bond, we can write: $\phi(i') = (g-i').a - k = 0$.

Noting that the derivation of the function $\phi(i')$ is:

$$\phi'(i') = \frac{1}{i'}(nv'^{n+1} - a')(g-i') - a'$$

and applying this result to equation (7) the following approximation formula can be easily derived:

$$i = i' + i' \frac{a'(g-i') - k}{a'g - nv'^{n+1}\ (g-i')} .$$

It is worth pointing out that for the case where the trial rate i' is taken to equal g, the coupon rate, then the above approximation formula reduces to:

$$i = g - k/a_{(g)}$$

where $a_{(g)}$ is the n-period annuity evaluated at the trial rate g.

We mentioned earlier that direct approximations based on Method I do not generally yield reliable values owing to the fact that the successive terms in the expansion do not diminish rapidly enough to allow the neglect of the terms following the third or the fourth term in the expansion. In a contribution published in 1897, Todhunter was the first to recognize that this shortcoming does not exist for approximations involving the bond yield to maturity. Todhunter suggests the following approach. Using the expression for the value of an annuity, equation (8) can be rewritten as:

$$g - i = ki\ [\ 1 - (1+i)^{-n}]^{-1} .$$

Expanding the expression $(1+i)^{-n}$ according to the binomial theorem:

$$g - i = ki\ [\ 1 - ni + \frac{n(n+1)}{2}\ i^2 - \ldots)]^{-1}$$

$$= ki(ni)^{-1}\ [\ 1 - (\frac{n+1}{2})i + \ldots]^{-1} .$$

The expansion of the expression in brackets gives:

$$g - i = \frac{k}{n}[\ 1 + (\frac{n+1}{2})i - \ldots]$$

and the neglect of all terms involving powers of i above the first yields:

$$i = \frac{g - k/n}{1 + [\frac{n+1}{2n}]k} \qquad (9)$$

which is Todhunter's approximate bond-yield formula.

In 1927, James I. Craig extended Todhunter's approximation and suggested a formula which is more accurate than (9). Craig's formula is:

$$i_2 = i_1 - \frac{nk\, i_1^2 \,(1-i_1/2)}{12\,[1 + (\frac{n-1}{2n})k]} \qquad (10)$$

where i_1 is determined by Todhunter's formula. Craig derived his approximation by neglecting all terms in the expansion involving powers of i above the third instead of just the first.

Karpin (1967) and Worger (1967) present the most recent contributions to the long list of approximation formulas for the bond's yield to maturity. Worger uses an approach similar to that of Craig and derived comparable approximation formulas which are easier to apply than Craig's and require little arithmetical work compared to any of the other formulas. One of Worger's formulas is:

$$i = \frac{g - k/n}{1 + [\frac{n+1}{2n}(1 + \frac{(n-1)i_1}{6})]k} \qquad (11)$$

where i_1 is determined by Todhunter's formula (9). In his 1967 contribution, Worger claims that his approximation formulas "leave little room for any further improvements in the accuracy of such methods."

Henderson (1907) has suggested various empirical approximation formulas (Method IV) for the determination of the bond yield to maturity. The simpliest of those is:

$$i = \frac{g - k/n}{1 + .6k} \qquad (12)$$

Another popular approximation of the bond's yield to maturity, which can be found in most modern textbooks of finance and accounting is calculated by taking the ratio of the average periodic return on the bond to the average value of the bond, that is: [4]

$$i = \frac{g - k/n}{1/2\,[\,1 + (1+k)]} = \frac{g - k/n}{1 + .5k} \qquad (13)$$

We will shortly see that this standard approximation of the yield to maturity, which has found a respectable place in some of the most popular textbooks of finance, is actually the least accurate of the formulas we have just surveyed. It is indeed surprising that the rich history of approximation formulas has remained unnoticed and that it never reached the textbook audience.

The examination of the relative merit of alternative formulas for the determination of the bond yield to maturity can be understood better if the terms of equation (8) are rearranged and expressed as a function of the *exact* yield to maturity. That is,

$$i + \frac{k}{a} = g$$

or

$$i + \frac{k}{a} - \frac{k}{n} = g - \frac{k}{n}$$

$$i + \left(\frac{1}{a} - \frac{1}{n}\right) k = g - \frac{k}{n}$$

$$i\,\left[\,1 + \frac{1}{i}\left(\frac{1}{a} - \frac{1}{n}\right) k\right] = g - \frac{k}{n}\,,$$

and therefore,

$$\left.\begin{aligned} i &= \frac{g - k/n}{1 + \theta . k} \\ \text{where} \quad \theta &= \frac{1}{i}\left(\frac{1}{a} - \frac{1}{n}\right) \end{aligned}\right\} \qquad (14)$$

Equation (14) can be employed as a starting point for the investigation of alternative formulas for the approximate bond yield to maturity.

It can be seen that Todhunter's formula (9) is given by equation (14) for the case when $\theta = (n+1)/2n$, a result that follows directly from the expansion of θ which neglects all terms involving powers of i above the first. Also, Worger's formula is given for θ equal to the term between brackets in the denominator of equation (11). Henderson's empirical approximation (12) is a special case of equation (14) for which $\theta = .6$. For the textbook approximation (13), $\theta = .5$.

It is clear that the behavior of the function $\theta = \theta(n,i)$ should be investigated in order to determine the relative accuracy of formulas such as (9), (11), (12), and (13). Table 1 gives the value of the function θ for various values of the term to maturity n and the exact yield i. In Figure 1, the function $\theta(n,i)$ is drawn for i = 1%, 4%, and 25%. For positive rates i the function $\theta(n,i)$ is always positive (since $a < n$) and equals one for $n = 1$, and $n = \infty$ regardless of the value given to the positive rate i. Also, for positive rates i the value of $\theta(n)$ is always above one-half. The attractive property of the function θ is that it varies very slowly with i. This property justifies the use of formulas such as (9) which are derived using Method I. Also, the function θ varies slowly with n and hence formulas such as (12) in which θ is given a constant value provide adequate first approximations.

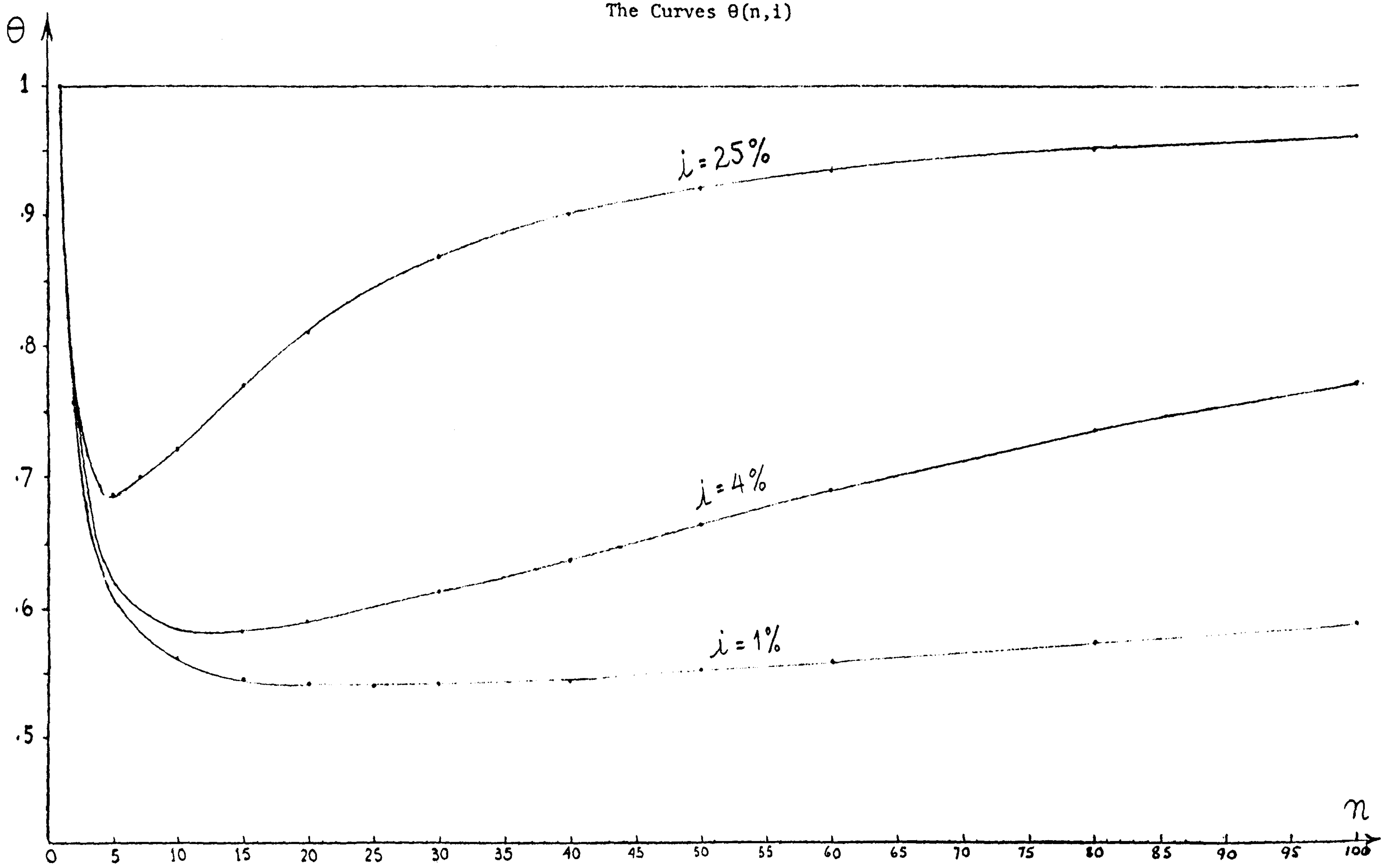
An examination of the last column of Table 1, which gives the value of the ratio $(n+1)/2n$, indicates that for positive yields the ratio $(n+1)/2n$ underestimates the exact value of $\theta(n,i)$. This implies that Todhunter's formula (9) overestimates (underestimates) the exact yield to maturity for bonds selling at a premium (discount). This conclusion

TABLE 1

n \ i	.01	.02	.03	.04	.05	.06	.08	.10	.15	.20	.25	$\frac{n+1}{2n}$
	Values of $\theta = \frac{1}{i}\left[\frac{1}{a_n} - \frac{1}{n}\right]$ for various i and n											
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	.7568	7533	.7540	.7550	.7563	.7574	.7596	.7619	.7675	.7727	.7778	.7500
3	.6729	.6716	.6735	6755	.6776	.6797	.6838	.6878	.6976	.7070	.7158	.6667
4	.6312	.6317	.6344	.6373	.6403	.6433	.6490	.6547	.6684	.6814	.6938	.6250
5	.6065	.6083	.6120	.6157	.6196	.6233	.6307	.6380	.6554	.6719	.6874	.6000
10	.5595	.5665	.5744	.5823	.5901	.5978	.6129	.6275	.6617	.6926	.7203	.5500
15	.5466	.5581	.5701	.5819	.5935	.6050	.6270	.6481	.6957	.7361	.7698	.5334
20	.5422	.5579	.5739	.5896	.6049	.6178	.6482	.6746	.7317	.7768	.8117	.5250
30	.5319	.5659	.5896	.6124	.6344	.6553	.6937	.7275	.7931	.8376	.8679	.5167
40	.5459	.5778	.6088	.6381	.6656	.6910	.7358	.7726	.8371	.8757	.9001	.5125
50	.5515	.5911	.6289	.6638	.6955	.7241	.7718	.8086	.8676	.9001	.9200	.5100
60	.5580	.6051	.6489	.6884	.7232	.7535	.8016	.8366	.8891	.9167	.9333	.5083
80	.5720	.6331	.6871	.7329	.7706	.8012	.8459	.8755	.9167	.9375	.9500	.5063
100	.5867	.6602	.7216	.7702	.8077	.8363	.8755	.9001	.9334	.9500	.9600	.5050
200	.6584	.7694	.8360	.8754	.9001	.9167	.9375	.9500	.9667	.9750	.9800	.5025
∞	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.5000

FIGURE 1

The Curves θ(n,i)



also holds for formula (13) for which $\theta = .5$. The values in Table 1 also indicate that Todhunter's formula (9) will give close results when n and i are not too large, that is, for $n \leqslant 5$ and $i \leqslant .08$. For $n = 5$ formula (9) and formula (12) are identical. For terms between 5 and 10 formula (12) is better than formula (9) if the rate i is about 3 per cent or over. For longer terms, formula (12) is better if the rate i is about 2 per cent or over. In any case, the textbook formula (13) is the least accurate of those surveyed. We recommend that it be abandoned in favor of either Todhunter's formula (9) or Henderson's formula (12).

5. CONCLUDING REMARKS

The purpose of this paper was to trace the historical developments of the efforts to find simple and accurate methods of approximating an annuity's implicit yield and a bond's yield to maturity. We have shown that the history of yield approximations is indeed very rich. Most of the contributions in this area can be attributed to British actuaries who are not well known outside their profession and country. This may partly explain why some of these contributions have not yet reached the finance textbooks. In particular, we have shown that the standard textbook approximation formula for the bond's yield to maturity is the least accurate of a large family of formulas some of which were suggested as early as 1856. In light of these findings we recommend that the standard textbook formula (13) be abandoned in favor of either Todhunter's formula (9) or Henderson's formula (12).

FOOTNOTES

1. See for example the latest contributions of Mahoney (1980) and Hawawini and Vora (1980.a).

2. The tables published by Trenchant were in French and those by Steven were in Dutch (Smith, 1967). It seems that the first tables published in English appeared in Richard Witt's, *Arithmetical Questions, Touching the Buying or Exchange of Annuities in 1613* (Lewin, 1970).

3. See Augustus De Morgan (1859). Michael Dary is the author of *Interest Epitomized Both Compound and Simple*, which was published in 1677. Some of the contributions cited in this paper have been reproduced in Hawawini and Vora (1980b).

4. See for example the textbook by Francis (1980).

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