"ECONOMIC ACCEPTANCE SAMPLING BY VARIABLES WITH QUADRATIC QUALITY COSTS"

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A b s t r a c t

An economic model is developed, to assist in the selection of minimum cost acceptance sampling plans by variables. The quadratic Taguchi loss function is adopted to model the cost of accepting items, with quality characteristics deviating from the target value. The case of a normally distributed quality characteristic with known variance is examined, and a simple and efficient optimization algorithm is proposed. Comparisons with other methods of deriving sampling plans reveal that the cost penalties for using an inappropriate plan may be very large.

ECONOMIC ACCEPTANCE SAMPLING BY VARIABLES WITH QUADRATIC QUALITY COSTS

Although in recent years more emphasis is placed on process control and off-line quality control methods, acceptance sampling remains a major ingredient of many practical quality control systems. This is especially true in industries that rely heavily on materials, which are supplied from outside vendors, that may be located far away, even in different countries. In such cases, the buyer cannot easily exert direct control on the suppliers' production processes. Therefore, the buyer has to resort to acceptance sampling methods. A typical example is the European wine industry: a certain major European wine producer, with whom we have practical experience, obtains bottles from Germany, Yugoslavia or Bulgaria, and cork stoppers from Portugal.

The majority of the literature on acceptance sampling plans is devoted to the study of acceptance sampling by attributes (see Wetherill and Chiu [1975] for a review). This can be explained by the relative simplicity of those plans, compared with acceptance sampling by variables. Thus, even when the controlled quality characteristic is a continuous variable, the measurements are often compared against predetermined specification limits and the items are classified as conforming or nonconforming accordingly. This approach is justified under the assumption that the quality cost of accepting a nonconforming item is constant, irrespective of the exact value of the quality characteristic, while there is no quality cost in accepting a conforming item, even if the value of the quality characteristic is far from the target and close to (albeit within) a specification limit. Following this assumption, the economic design of acceptance sampling by attributes has been treated extensively. Most of the important developments in the statistical theory and design of sampling inspection by attributes are presented in an excellent monograph by Hald [1981].

The statistical theory of sampling inspection by variables, along with tables and charts for determination of sampling schemes, was first presented systematically by Lieberman and Resnikoff [1955]. The economic design of acceptance sampling plans by variables has been treated successively by Schmidt, Case and Bennett [1974], Ailor, Schmidt and Bennett [1975] and Schmidt, Bennett and Case [1980]. These papers adopted the aforementioned assumption of stepwise quality cost function for accepted items.

With the advent of Taguchi ideas and methods, it is well accepted nowadays that, in general, there is an optimal target value for every measurable quality characteristic. Any deviation from this target value incurs an economic loss, even if the value of the quality characteristic lies within the specification limits. The quadratic quality cost function $q(x) = kx^2$ (where k is a positive constant and x is the deviation from the target) has been suggested by Taguchi [1984] and Kackar [1985] to represent the economic loss due to deviations from the target, and has been widely adopted thereafter. This new approach has resulted in renewed interest in acceptance sampling by variables, which can no more be justifiably substituted by attributes, and in reconsideration of the economic design problem. Among the several models, that use a quadratic quality cost function in the context of quality control by variables, are those by Tang and Schneider [1987] and Tang [1988] for single quality characteristic, and those by Tang and Tang [1989] and Hui [1990] for multiple quality characteristics. These models, however, deal only with screening inspection, i.e. complete (100%) inspection plans, and not with sampling inspection.

Although there certainly exist situations, in which complete inspection is feasible, there are also many cases, where sampling is inevitable, either because inspection is destructive, or because lot sizes are large and inspection is expensive and time consuming. The purpose of this paper is to present and analyze an economic model for the selection of acceptance sampling plans by variables, under the assumption of quadratic quality costs. The model formulation is followed by an optimization procedure, which is illustrated through a numerical example. The sampling procedures, which are suggested by the present quadratic cost model, are compared with respective sampling plans suggested by previous cost models and by international standards, which are used extensively in practice. The paper also examines how the quadratic cost model relates to Bayesian schemes for tests of the mean. Such schemes, as those presented by Wetherill and Campling [1966] for one-sided tests and by Dayananda and Evans [1973] for two-sided tests, follow a decision theory approach to address a similar problem. The conclusions are summarized in the last section of the paper.

It must be mentioned here that Moskowitz and Tang [1992], in concurrent work that was independent of this piece of research, also consider optimal variables acceptance sampling plans with quadratic quality cost function and obtain similar theoretical results. In addition to the basic theoretical results, however, the present paper examines the relationship with Bayesian tests of the mean and concentrates on comparisons with other sampling plans (notably those suggested by the popular international standards) and on better understanding the practical implications of the mathematical model through analysis of concepts such as MPSD (maximum process standard deviation) in the variables acceptance sampling context. On the other hand, Moskowitz and Tang [1992] discuss extensively other issues, like the robustness of the optimal sampling plan with respect to the form of the prior distribution of the mean and to mis-specification of its parameters. Thus, the two papers can be viewed as complementary.

Notation and Assumptions

μ_o	target value of the variable quality characteristic to be controlled
X	deviation of the quality characteristic of an individual unit from the target value
μ	deviation of the mean of the quality characteristic in a given inspection lot from the target μ_o
σ^2	variance of X in a given inspection lot
N	lot size
n	sample size
$\bar{X}$	sample mean
L	lower acceptance limit for $\bar{X}$
U	upper acceptance limit for $\bar{X}$
$f(x\|\mu)$	probability density function of X, given that the deviation of the mean of the inspection lot from μ_o is μ
$g(\bar{x}\|\mu)$	probability density function of $\bar{X}$, given that the deviation of the mean of the inspection lot from μ_o is μ
$h(\mu)$	probability density function of μ
σ^2_μ	variance of μ
c_s	constant cost of preparing the inspection
c_i	variable sampling and inspection cost per unit
c_r	rejection cost per unit
k	constant of the quality cost $q(x) = kx^2$.

The following assumptions are made:

1. The variance of X, σ^2, is known and constant.

2. The variance of μ, σ^2_μ, is known and constant. To simplify notation, we express the variance of μ in terms of σ^2, namely $\sigma^2_\mu = \sigma^2/D$, where D is a positive constant, expected to be larger than 1.

3. The measurements are free of error.

4. The distribution of X, $f(x|\mu)$, is normal.

5. The distribution of μ, $h(\mu)$, is normal, with mean $E(\mu) = 0$, i.e., on the average the mean of the quality characteristic in a lot equals the target value μ_0.

Assumptions 1, 2, 3, are identical to those made by Schmidt, Case and Bennett [1974] in the development of their cost model, except that we do not necessarily assume that inspection is not destructive. Assumptions 4 and 5, which are made in the examples of the same paper, are adopted here right away, because they occur very often in practice, they facilitate the exposition and simplify the notation (from the combination of assumption 5 with the symmetry in the quality cost kx^2, it follows easily that $L = -U$ at optimality). As a matter of fact, the normality assumption 4 was supported by data on quality characteristics of materials used for bottling wine (diameter of cork stoppers, for example); the data was collected by the Quality Control department of the European wine producer mentioned in the introduction, in the process of determining acceptance sampling plans for incoming materials. However, the general approach and the cost formulation remain valid, even if assumptions 4 and 5 are relaxed. Moskowitz and Tang [1992] explicitly allow for $E(\mu)\neq0$ in their formulation. Moreover, sensitivity analyses reported in the same paper suggest that the optimal sampling plan is robust with respect to the form of the prior distribution $h(\mu)$, as well as to mis-specification of its mean and variance, provided that the tail specification reasonably approximates that of a normal distribution.

The Cost Model

When a lot of size N is received for inspection, a random sample of size n is taken, the values of the quality characteristic X of every unit in the sample are measured and the sample mean $\bar{X}$ is computed. If $\bar{X}$ lies in the interval (L,U), then the lot is accepted as is, otherwise the lot is rejected. The decision variables are the sample size n and the acceptance limits -U (L) and U.

Rejected lots are returned to the supplier or scrapped at a cost of c_r per unit. Therefore, the model is analogous to the scrapping model of Schmidt, Case and Bennett [1974]. Note that the screening option of rejected lots is not considered here for two reasons. First, it has already been stated that sampling is particularly important in cases of large lots and time consuming or destructive inspection, where screening may not be a practically feasible alternative, even after rejection. Second, separating conforming from nonconforming units is in some sense contrary to the notion of quadratic quality costs, whereby there are no specification limits differentiating acceptable from unacceptable units, unless the individual items can be reworked so that x=0. The latter situation of reworkable characteristics, however, is rather uncommon in practice; it is impossible, for example, to correct the dimensions of individual bottles or cork stoppers, used in a bottling process.

There are three cost components that need be considered in this setting: cost of inspection, cost of acceptance (quality cost), and cost of rejection.

The expected cost of inspection per lot is simply

$$C_I = c_s + nc_i. \tag{1}$$

The expected cost of acceptance of an inspection lot, with deviation μ of its mean from the target value μ_o, is

$$C_A(\mu) = N \int_{-U}^{U} \left[\int_{-\infty}^{\infty} q(x) f(x|\mu)\, dx \right] g(\bar{x}|\mu)\, d\bar{x} \quad ,$$

assuming nondestructive inspection. Since $q(x) = kx^2$,

$$\int_{-\infty}^{\infty} q(x) f(x|\mu)\, dx = k\ E(X^2|\mu) = k(\mu^2+\sigma^2)$$

and $C_A(\mu)$ can be written as

$$C_A(\mu) = Nk(\mu^2+\sigma^2)P_a(\mu) \quad ,$$

where

$$P_a(\mu) = \int_{-U}^{U} g(\bar{x}|\mu)\, d\bar{x} \tag{2}$$

is the probability of acceptance of a lot with given μ. Taking the expectation of $C_A(\mu)$ with respect to μ leads to the following expression for the expected cost of acceptance per lot:

$$C_A = Nk \int_{-\infty}^{\infty} (\mu^2+\sigma^2) P_a(\mu) h(\mu) d\mu$$

$$= Nk \int_{-\infty}^{\infty} \mu^2 P_a(\mu) h(\mu) d\mu + Nk\sigma^2 \int_{-\infty}^{\infty} P_a(\mu) h(\mu) d\mu \ , \qquad (3)$$

where $P_a(\mu)$ is given by (2). If inspection is destructive, N must be replaced by N-n in (3).

Since $f(x|\mu)$ is assumed normal with mean μ and variance σ^2, $g(\bar{x}|\mu)$ is also normal with mean μ and variance σ^2/n. In that case, the probability of acceptance $P_a(\mu)$ is simplified to

$$P_a(\mu) = \Phi([U-\mu]\sqrt{n}/\sigma) - \Phi([-U-\mu]\sqrt{n}/\sigma) \ , \qquad (4)$$

$\Phi(\cdot)$ being the cumulative standard normal function.

The expected cost of rejection of an inspection lot, with deviation μ of its mean from the target value μ_o, is

$$C_R(\mu) = Nc_r \left[1 - \int_{-U}^{U} g(\bar{x}|\mu) d\bar{x}\right] = Nc_r [1 - P_a(\mu)] \ .$$

Taking the expectation of $C_R(\mu)$ with respect to μ results in the following expected cost of rejection per lot:

$$C_R = Nc_r \int_{-\infty}^{\infty} [1 - P_a(\mu)] h(\mu) d\mu$$

$$= Nc_r - Nc_r \int_{-\infty}^{\infty} P_a(\mu) h(\mu) d\mu$$

$$= Nc_r - Nc_r P_a \ , \qquad (5)$$

where P_a is the probability of acceptance, computed from

$$P_a = \int_{-\infty}^{\infty} P_a(\mu)h(\mu)d\mu \quad . \tag{6}$$

If inspection is destructive, N must be replaced by N-n in (5). If $f(x|\mu)$ is normal, $P_a(\mu)$ is obtained from (4).

The expected total cost per inspected lot, for given n and U, ETCI(n,U) , is

$$ETCI(n,U) = C_I + C_A + C_R$$

$$= c_s + nc_i + Nk\int_{-\infty}^{\infty} \mu^2 P_a(\mu)h(\mu)d\mu + Nk\sigma^2 P_a + Nc_r - Nc_r P_a. \tag{7}$$

If inspection is destructive, N is replaced by N-n in (7) above.

The expected total cost per lot in the case of acceptance without sampling, ETCA, is computed from

$$ETCA = N\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} q(x)f(x|\mu)dx \right] h(\mu)d\mu$$

$$= Nk\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x^2 f(x|\mu)dx \right] h(\mu)d\mu$$

$$= Nk\int_{-\infty}^{\infty} (\sigma^2+\mu^2)h(\mu)d\mu$$

$$= Nk(\sigma^2+\sigma^2{}_\mu) \quad ,$$

since $E(\mu) = 0$. Using $\sigma^2{}_\mu = \sigma^2/D$, ETCA is written

$$ETCA = Nk\sigma^2(1 + 1/D). \tag{8}$$

In the case of rejection without sampling, the expected cost per lot, ETCR, is given by

$$ETCR = Nc_r. \tag{9}$$

Optimization

The optimal quality control policy is the one that yields the minimum expected cost per lot:

$$ETC = \min\{ETCI,\ ETCA,\ ETCR\} \quad , \qquad (10)$$

where ETCI = min{ETCI(n,U)}, and ETCI(n,U), ETCA, ETCR are given by (7), (8) and (9) respectively. Since ETCA and ETCR are constant, an optimization procedure is needed only for the determination of n and U, which minimize ETCI(n,U).

To find the optimal n^* and U^*, the optimal value U(n) of U for a given sample size n is first determined, solving the first-order condition for U. Then, a unidimensional search over the integer n is carried out, using U(n) as the appropriate value of U at each step.

The first-order condition for U is

$$\frac{\partial ETCI(n,U)}{\partial U} = 0 \quad , \quad \text{or}$$

$$\frac{\partial}{\partial U}\left[c_s + nc_i + Nk\int_{-\infty}^{\infty}\mu^2 P_a(\mu)h(\mu)d\mu + Nk\sigma^2 P_a + Nc_r - Nc_r P_a\right] = 0.$$

Since

$$\frac{\partial P_a(\mu)}{\partial U} = \frac{\partial}{\partial U}\int_{-U}^{U} g(\bar{x}|\mu)d\bar{x} = g(U|\mu) + g(-U|\mu) \quad ,$$

the first-order condition for U becomes

$$k \int_{-\infty}^{\infty} \mu^2 [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu =$$

$$= (c_r - k\sigma^2) \int_{-\infty}^{\infty} [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu \qquad (11)$$

for both cases of nondestructive and destructive inspection. Note that the integrals in (11) are always positive. Therefore, (11) has a solution, if and only if $c_r - k\sigma^2 > 0$. If $c_r \le k\sigma^2$, then rejection without sampling is more economical than any sampling plan.

Assume now that $c_r - k\sigma^2 > 0$. The integrals in (11) can be written (see Appendix) as

$$\int_{-\infty}^{\infty} \mu^2 [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu = 2\varphi(U) \left[\frac{n^2U^2}{(n+D)^2} + \frac{\sigma^2}{n+D} \right] ,$$

$$\int_{-\infty}^{\infty} [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu = 2\varphi(U) ,$$

where $\varphi(U) > 0$. The first-order condition then becomes

$$k \left[\frac{n^2U^2}{(n+D)^2} + \frac{\sigma^2}{n+D} \right] = c_r - k\sigma^2$$

and the optimal U for given n is easily computed from

$$U(n) = \sqrt{[c_r(n+D) - (n+D+1)k\sigma^2]\ (n+D)/kn^2} \ . \qquad (12)$$

The second derivative of ETCI(n,U) with respect to U is

$$\frac{\partial^2 ETCI(n,U)}{\partial U^2} = \frac{\partial}{\partial U} \left[2\varphi(U) \left\{ k \left[\frac{n^2U^2}{(n+D)^2} + \frac{\sigma^2}{n+D} \right] - (c_r - k\sigma^2) \right\} \right]$$

$$= 2\varphi(U) \frac{2kn^2U}{(n+D)^2} + 2\varphi'(U) \left\{ k \left[\frac{n^2U^2}{(n+D)^2} + \frac{\sigma^2}{n+D} \right] - (c_r - k\sigma^2) \right\}.$$

When U(n) is given by the first-order condition (12), the second summand above vanishes and we have

$$\frac{\partial^2 ETCI(n,U)}{\partial U^2} = 2\varphi(U) \frac{2kn^2U}{(n+D)^2} > 0 ,$$

implying that the second-order condition for U(n) of (12) to minimize ETCI(n,U) is satisfied.

Once U(n) can be readily obtained for every n, a Fibonacci search over n can be employed to identify the pair n, U(n), which minimizes ETCI(n,U(n)) of equation (7). The optimal control policy follows immediately from (10).

It is useful to note the boundary condition, following from (12). If

$$c_r < \frac{n+D+1}{n+D} k\sigma^2 , \tag{13}$$

for a given n, then the decision must be to outright reject inspection lots rather than use a sampling plan with sample size n. The limiting case is that of $c_r = k\sigma^2(n+D+1)/(n+D)$, where U=0 and the lot is accepted only if the sample mean exactly equals the target value. As it has already been mentioned, if $c_r \le k\sigma^2$, then (13) holds for every n and the optimal decision is to outright reject inspection lots, because then ETCR will also be lower than ETCA, from (8) and (9).

Expression (12) for U(n) provides also several interesting and intuitive insights. For a given sample size, U(n) is decreasing in σ and k, and increasing in c_r and D (dU/dD > 0).

Thus, the acceptance limits $L=-U$ and U must be tight (small U), when the product is very variable (large σ), the lot quality is very variable (small D, large σ_μ), the cost of quality is high (large k) and/or the rejection cost is low (low c_r).

Examples

Let

$\sigma = 1.0$

$D = 5.0$

$N = 100{,}000$

$c_s = 10.00$

$c_i = 1.00$

$c_r = 2.50$

$k = 2.00$

and inspection be nondestructive. We first not that

$$c_r - k\sigma^2 = 2.5 - 2(1)^2 = 0.5 > 0 \quad ,$$

meaning that sampling inspection is not dominated by rejection without sampling. The expected costs of acceptance and rejection without sampling are

$$ETCA = Nk\sigma^2(1 + 1/D) = 240{,}000$$

$$ETCR = Nc_r = 250{,}000 \quad .$$

A Fibonacci lattice search over n, using (12) for $U(n)$, identifies $n^* = 303$ and $U^* = 0.506$ as the optimal combination of n, U, yielding $P_a = 0.737$ and

$$ETCI = ETCI(303, 0.506) = 224{,}159.06 = ETC.$$

Therefore, the optimal control policy is sampling inspection with $n = 303$ and $U = 0.506$.

If the unit rejection cost is doubled to $c_r = 5.00$, then

$$c_r - k\sigma^2 = 3 > 0 \quad ,$$

$$ETCR = 500,000 \quad ,$$

rejection is very undesirable and the optimal solution is $n^* = 89$ and $U^* = 1.289$, resulting in $P_a = 0.995$ and

$$ETCI = ETCI(89,1.289) = 239,748.87 = ETC.$$

The above control policy approaches acceptance without sampling. It is interesting to note that, if U(n) is selected through (12), ETCI(n,U(n)) is very flat in the neighbourhood of n^*.

Comparisons with other Statistical and Economic Sampling Plans

The examination of the practical implications of adopting the quadratic quality cost function in the acceptance sampling setting is of particular importance. In this section we explore how the optimal sampling plans suggested by the current approach (quadratic cost plans) may differ from the respective sampling plans that are derived from the optimization of the cost model of Schmidt, Case and Bennett [1974] (SCB plans), and from the international standard ISO 3951 [1989] (ISO plans).

Unfortunately, direct comparisons with either of the above approaches are not possible, because of the basic difference in the assumed cost structure. Specifically, both Schmidt, Case and Bennett [1974] and ISO 3951 [1989] define specification limits S_L (lower) and S_U (upper) for the quality characteristic X. The former approach assumes that the quality cost of accepted defective items, c_a, is incurred only if X is outside the specification limits and it is constant, while the latter does not consider quality costs explicitly. In spite of that fact, it is deemed that even some indirect comparisons, based on reasonable assumptions, are useful. The numerical Example 2 of Schmidt, Case and Bennett [1974] will serve to illustrate these comparisons. Using the same notation as before, the parameters of the example are the following:

$S_L = -2.0$

$S_U = +2.0$

$\sigma = 0.75$

$D = 7.0$

$N = 50{,}000$

$c_s = 1.0$

$c_i = 0.12$

$c_r = 0.20$

$c_a = 11.00$

In order to be able to compare a quadratic cost plan with a SCB plan, it is necessary to relate k of $q(x) = kx^2$ with c_a. It is assumed here that

$$c_a = q(\tilde{x}) = k\tilde{x}^2 \quad , \qquad (14)$$

where

$$\tilde{x} = E[x \mid x \geq S_U \, , \, \mu=0] \quad ,$$

i.e., the quality cost of an item with value of the quality characteristic $\tilde{x}$ equals the cost of a nonconforming unit, with $\tilde{x}$ being the average dimension of a nonconforming unit, when $\mu=0$. (A simpler approach might be to set $c_a = q(S_U)$, leading to larger values of k.) It is known (Ryan [1989]) that the mean $\tilde{x}$ of the truncated normal distribution is

$$\tilde{x} = [1 - \Phi(S_U/\sigma)]^{-1} \quad (\sigma/\sqrt{2\pi}) \; \exp[-S_U{}^2/2\sigma^2].$$

Thus, in this case, $\tilde{x} = 2.25$ and $k = c_a/\tilde{x}^2 = 2.173$.

Since $c_r - k\sigma^2 = 0.20 - 2.173(0.75)^2 < 0$, the optimal quadratic cost plan in this case is rejection without sampling, with an expected cost ETCR = 50,000(0.20) = 10,000 per lot.

The optimal solution to the scrapping model of Schmidt, Case and Bennett [1974] was $n^* = 254$, $U^* = 0.425$. Under the assumption of stepwise quality cost, the expected cost of this plan is 6078.04, much lower than the cost of rejection without sampling, which is again 10,000, unaffected by the choice of quality cost function. If the appropriate quality cost function is quadratic with k=2.173, the expected cost per lot for the above SCB plan is computed from (7) and it is found that ETCI(254,0.425) = 58,187.35, a 482% cost increase over the optimal quadratic cost plan. If k were obtained from $c_a = q(S_U)$, ETCI(254,0.425) would be even larger.

Comparisons between quadratic cost plans and ISO plans are possible only after the determination of an "equivalent" acceptable quality level (AQL) for the ISO plans. Here, the AQL that needs to be specified is taken to be equal to the breakeven quality level p^*, which, following Hald [1981], is defined as $p^*=c_r/c_a$. In the example $p^*=0.0182$; for specified AQL=1.82%, the standard AQL Conversion Table suggests the use of AQL=2.5%. For the common general inspection level II and for normal inspection, the ISO plan is n=61, U=0.73. From (7), the expected total cost of the ISO plan is ETCI(61,0.73)=68,188.67, a 582% cost increase over the quadratic cost plan. It is interesting to note that the standard ISO 3951 [1989] proposes to reject without sampling if σ > MPSD (maximum process standard deviation), where MPSD = $0.223(S_U-S_L)$ for AQL=2.5% and normal inspection. Thus, since here $S_U-S_L = 4.0$, MPSD=0.892 > 0.75 = σ, and the standard suggests a sampling plan. However, condition $c_r-k\sigma^2 > 0$ of the quadratic cost model implies that the maximum process standard deviation is actually $\sqrt{c_r/k}$. In the example under study, $\sqrt{c_r/k} = 0.303 < 0.75$. Consequently, the process standard deviation is so large, that the lot must be rejected without samples being taken.

Table 1 depicts the quadratic cost, SCB and ISO plans, their total expected costs, ETC, per lot and the breakdown of ETC into costs of inspection (C_I), acceptance (C_A) and rejection (C_R). Differences in costs may be larger or smaller than those of Table 1, depending on the specific problem parameters. It is obvious, though, that when the quadratic quality cost model is appropriate, deriving sampling plans, which optimize other cost models, or using the sampling plans, which the broadly used standards suggest, may lead to erroneous decisions incurring substantial cost penalties.

Table 1 about here

Relation with Bayesian Tests of the Mean

The use of quadratic cost penalties for deviations from a target value is not as new in statistics as it is in the field of quality control. Dayananda and Evans [1973] studied the design of two-sided Bayesian statistical tests of the mean of a normal distribution with known variance, using several utility functions. Their model with quadratic utility functions resembles the cost model that has been presented in this paper, although the terminology is not that of acceptance sampling in quality control. It is instructive to examine how the two models are related, so as to bridge the gap between the decision theoretic and the quality control approaches.

The general problem in Dayananda and Evans [1973] is to make a decision on whether or not the parameter θ of a known distribution lies in the interval (θ^-, θ^+), where θ^- and θ^+ are explicitly or implicitly specified. The possible decisions are

$$d_1 \; : \; \theta \in (\theta^-, \theta^+)$$

$$d_2 \; : \; \theta \notin (\theta^-, \theta^+)$$

and the utility resulting from taking the decision d_i for given θ is denoted $u(d_i,\theta)$, $i=1,2$. In acceptance sampling terminology, d_1 means acceptance of the lot and d_2 means rejection.

In the case of testing the deviation, μ, of the mean of a normal distribution with known variance from the target value, the quadratic utility functions take the form

$$u(d_i,\mu) = K_i + k_i\mu^2 \quad , \qquad i=1,2,$$

where $k_1<0$, $k_2>0$, $K_1>K_2$, and

$$(\mu^-)^2 = (\mu^+)^2 = (K_1 - K_2) \;/\; (k_2 - k_1) = K. \tag{15}$$

Note that the utility is a quadratic function of μ (not of X), not only if the decision is to accept the lot, but if the decision is to reject as well.

Making the same assumption about the normality of the prior distribution of μ, Dayananda and Evans [1973] arrive at their conditions (4.10) for the acceptance limits U(n). Using simple algebra, it can be shown that these conditions are equivalent to expressions (12) and (13) of this paper, if and only if

$$K = c_r/k - \sigma^2 \quad . \tag{16}$$

Because of (15), the limits of the hypothesized interval (μ^-, μ^+) can be written as

$$\mu^- = - \sqrt{c_r/k - \sigma^2}$$

$$\mu^+ = + \sqrt{c_r/k - \sigma^2} \quad . \tag{17}$$

Thus, setting the acceptance limits at $\pm U(n)$ according to (12) is equivalent to testing whether μ lies in the interval $(- \sqrt{c_r/k - \sigma^2} , + \sqrt{c_r/k - \sigma^2})$. Naturally, a search over n is still needed, in order to determine the optimal pair n^*, U^*. For example, using the parameter values of our first numerical example, $\pm \sqrt{c_r/k - \sigma^2} = \pm 0.500$. The optimal test of the composite hypothesis $\mu \in (-0.500, +0.500)$ calls for $n^* = 303$ and critical region defined by $U^* = 0.506$, as our previous solution indicates.

Another difference between the two models is that in the decision theoretic model of Dayananda and Evans [1973] μ^- and μ^+ depend solely on cost parameters, through (15), whereas in the present quality control formulation the equivalent values also depend on the variance σ^2 of the quality characteristic, through

(17). This should come as no surprise, since the cost (utility) parameters of the decision theoretic model refer to values of the parameter (μ), while the cost parameters of the quality control model refer to values of the quality characteristic (X) itself. The cost ratio K of (15) indirectly incorporates the variance σ^2 of the quality characteristic.

Conclusions

An economic model for the selection of cost minimizing acceptance sampling plans has been presented, incorporating the Taguchi approach of quadratic quality costs. Optimization of the model is simple and efficient, with negligible computational requirements. It has been shown that the acceptance sampling plan, which is derived from the optimization of this model may differ substantially from the plans that other economic approaches or the relevant international standard suggest. Thus, when the quadratic cost model is truly applicable, following a different cost model, or the extensively used charts and tables of the international standard ISO 3951 [1989] may result in very sizeable cost penalties, coming mainly from the cost of accepting items with values of the quality characteristic away from the target. The relationship between the quadratic cost model and a decision theory model with quadratic utilities has also been investigated. The similarities and differences have been highlighted, facilitating the understanding and consideration of those models from a unified, practical quality control perspective.

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Appendix

$$g(U|\mu) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\left[- \frac{(U-\mu)^2}{2\sigma^2/n} \right] ,$$

$$h(\mu) = \frac{\sqrt{D}}{\sigma\sqrt{2\pi}} \exp\left[- \frac{\mu^2}{2\sigma^2/D} \right] .$$

We can write the product $g(U|\mu)h(\mu)$ in the form

$$g(U|\mu)h(\mu) = \frac{\sqrt{nD}}{2\pi\sigma^2} \exp\{-(1/2) \left[\frac{nU^2 - 2nU\mu + n\mu^2 + D\mu^2}{\sigma^2} \right]\}$$

$$= \frac{\sqrt{nD}}{2\pi\sigma^2} \exp\{-(1/2)\left[\frac{\{\mu-[nU/(n+D)]\}^2}{\sigma^2/(n+D)} + \frac{nU^2(n+D)-n^2U^2}{\sigma^2(n+D)} \right]\}$$

$$= \frac{\sqrt{nD}}{2\pi\sigma^2} \exp\{-(1/2)\left[\frac{(\mu-y)^2}{\tau^2} + \frac{nDU^2}{\sigma^2(n+D)} \right]\}$$

where

$$y = nU/(n+D) \quad , \quad \tau = \sigma/\sqrt{n+D} .$$

After some rearrangement,

$$g(U|\mu)h(\mu) = \frac{1}{\tau\sqrt{2\pi}} \exp\left[- \frac{(\mu-y)^2}{2\tau^2} \right] * \frac{\sqrt{nD/(n+D)}}{\sigma\sqrt{2\pi}} \exp\left[- \frac{nDU^2}{2(n+D)\sigma^2} \right]$$

which means that

$$g(U|\mu)h(\mu) = \varphi(U)\omega_+(\mu)$$

where $\omega_+(\cdot)$ is the normal density function with mean y and variance τ^2, while $\varphi(\cdot)$ is a normal density function with mean 0 and variance $\sigma^2(n+D)/nD$, independent of μ.

In a similar manner, it can be shown that

$$g(-U|\mu)h(\mu) = \varphi(U)\omega_-(\mu)$$

where $\omega_-(\cdot)$ is the normal density function with mean -y and variance τ^2.

Therefore, the integral

$$I_1 = \int_{-\infty}^{\infty} \mu^2 [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu$$

can be written as

$$I_1 = \int_{-\infty}^{\infty} \mu^2 g(U|\mu)h(\mu)d\mu + \int_{-\infty}^{\infty} \mu^2 g(-U|\mu)h(\mu)d\mu$$

$$= \int_{-\infty}^{\infty} \mu^2 \varphi(U)\omega_+(\mu)d\mu + \int_{-\infty}^{\infty} \mu^2 \varphi(U)\omega\_(\mu)d\mu$$

$$= \varphi(U)[y^2+\tau^2] + \varphi(U)[(-y)^2+\tau^2]$$

$$= 2\varphi(U)\left[\frac{n^2U^2}{(n+D)^2} + \frac{\sigma^2}{n+D} \right] .$$

Similarly, the integral

$$I_2 = \int_{-\infty}^{\infty} [g(U|\mu)+g(-U|\mu)]h(\mu)d\mu$$

can be written as

$$I_2 = \int_{-\infty}^{\infty} g(U|\mu)h(\mu)d\mu + \int_{-\infty}^{\infty} g(-U|\mu)h(\mu)d\mu$$

$$= \int_{-\infty}^{\infty} \varphi(U)\omega_+(\mu)d\mu + \int_{-\infty}^{\infty} \varphi(U)\omega\_(\mu)d\mu$$

$$= 2\varphi(U) .$$

Table 1. Numerical Example Solutions			
	Quadratic Cost Plan	SCB Plan	ISO 3951 Plan
n^*	0	254	61
U^*	Always reject	0.425	0.730
Inspection Cost C_I	0	31.48	8.32
Acceptance Cost C_A	0	56,764.52	68,033.09
Rejection Cost C_R	10,000.00	1,391.36	147.26
Total Cost ETC	10,000.00	58,187.35	68,188.67