A TWO-FACTOR GAUSSIAN
MODEL OF DEFAULT RISK

by

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Abstract

This paper presents a reduced-form model of default risk. The market value of the firm's assets, discounted by a comparable observable measure, is the state variable that affects both the arrival and the magnitude rates of default. This state variable is assumed to exhibit a mean-reverting behavior. If the loss in case of default leaves the investor with a fraction of her claim, corporate bonds can be priced in a two-factor Vasicek model. Under a simple capital structure, the subsequent analysis of equity shows that its rate of return may also show mean-reversion, whose intensity is linearly related to financial leverage. Alternatively, considering that the bondholder recovers a fraction of par or of a risk-free bond when the firm defaults leads to slightly more complex pricing formulas, but in the latter case one can factorize out the price of a riskless bond. The three regimes are numerically compared in terms of yields and of sensitivity to credit risk characteristics. Extensions to a general model for magnitude risk and a two-factor model for the term structure are also proposed.
1 Introduction

There are now various continuous-time models that integrate credit risk in the valuation of interest rate sensitive securities and in the term structure of interest rates. They can be classified in two categories, depending on the way default is assumed to be triggered.

The first current, called the “structural” approach, considers that the event of default happens whenever the firm value becomes equal to a lower bound. The continuous-time dynamics for those values take different forms (see Longstaff and Schwartz [45], Nielsen, Saá-Requejo and Santa-Clara [53] and Merton [51]). Basically, the continuity of the sample paths of $V$ and $K$ means that default time is predictable\(^1\), i.e. there is an increasing sequence of stopping times that converges to the actual default time. This is consistent with the fact that liquidation (by means or bankruptcy or any other method) is often triggered when the firm value falls below some level, which can be predetermined or not.

The other current, called the “reduced-form” approach, assumes that default can be exogenously triggered by some random event, on which firm value may have no impact. This approach avoids the need of the barrier, which is in practice hard to set, and that in some cases bankruptcy might not be predictable. The assumption that the stopping time is “inaccessible” yields models for which the arrival rate of default (hazard rate) becomes crucial for the pricing of interest rate sensitive securities subject to default risk. Examples of this current are the papers by Duffie and Singleton [22], Duffie, Schroder and Skiadas [21] and Lando [39], [40].

Each approach has drawbacks. The ones of the structural approach are twofold: first, the firm value process is difficult to assess empirically, and requires a thorough analysis; second, such an approach completely ignores other causes of default, i.e. arrival of new information about value-independent determinants of default (the most obvious would be cash-related factors), for which continuous processes are hard to justify. On the other hand, the main weakness of inaccessible stopping time models is the lack of fundamental

\(^1\)A noticeable exception is the model proposed by Mason and Bhattacharya [49], who propose a simple structure of jumps in firm value.
meaning of the assumptions: the default rate is a one-dimensional random variable that synthesizes all determinants of default. Thus, if one wants to build some analytically tractable model of this kind, it is either to the cost of simplifications that might make it economically hard to defend, either by considering a very general framework for which empirical analysis is almost impossible to carry out. Moreover, merely giving up firm value throws away some crucial information, since its impact is probably the most important in the analysis of default.

We believe that both types of models are useful. The first ones allow to explicitly take into account the event that a decline in firm value makes the failure more likely, and they provide an economic justification of the event of default; the second ones capture the uncertainty about the exact timing of default due to exogenous, possibly observable sources of risk, and provide a simple analytic framework. So far, there has been little effort towards reconciliation between these approaches. In the first section of his paper about credit risk derivatives, Das [14] manages to introduce firm value as a state variable impacting credit spreads, but he can only recover it implicitly from options prices, without identifying the effect of firm value on the spread. In this case, the attempt is highly unsatisfactory with respect to the weaknesses of each approach, since the state variable used is not observable and is produced as an output, which is hardly sustainable.

Our goal, in this paper, is to propose a model of credit risk that conciliates the analytical simplicity of the reduced-form approach and the recognition of firm value as the main determinant of default. This yields a simple, flexible model, where default risk is translated into a credit spread, which keeps the valuable advantages of tractability and testability of the inaccessible stopping time approach. Thanks to this choice, the criticisms of the structural form approach can be met, since the state variable affecting the likelihood and severity of default may be observable, and the model leaves an open way to other possible determinants of default. This is the reason why our framework has two distinguishable features: it involves the observability of the variable used for valuation of credit risky securities, and it has a very flexible structure so as to secure easy extensions to several state variables. Yet, a special care is put on the ability to constantly justify the assumptions made about firm value. This motivates the particular choice of the state variable as a ratio
involving the market value of the firm and the mean-reverting stochastic process for its behavior. This will lead to a two-factor Ornstein-Uhlenbeck process, one factor for the riskless interest rate process, and one for the default spread, which can easily be extended to other term structure models.

In the development, we adopt in the basic model a framework similar to the one put forward by Duffle and Singleton [22], which is in their paper very general. Our identification of the state variable is much more than a mere specialization of their model, for various reasons.

First, the match between the state variable chosen and its empirical behavior allows to consistently derive the functional form of the credit spread, which is a major improvement on general credit risk models. The identification of a state variable, related to firm value, but whose stochastic process is independent of debt value is indeed of prime importance, because it allows a safe use of an equivalent martingale measure and of the pricing equations for credit risky securities. The change in probability measure for the risk-adjusted pricing is taken into account, in our setup, by an equivalent adaptation of the stochastic process for the state variable. This makes the pricing of corporate securities possible without requiring further restrictions.

Second, because of the parametrization performed here, the sensitivity of the price of defaultable securities with respect to the parameter of the stochastic process for firm value or on its correlation with interest rate risk can be analyzed. This remark is also valid for alternative specifications of the loss experienced in case of default.

Third, it opens the way to the pricing of other corporate securities which are not directly affected by default risk, such as equity: the use of the market value of the firm so as to price risky debt allows to endogenously price the firm's equity by difference. The ability to find out its stochastic process induces the precious possibility to interpret the impact of default risk on

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2Among one-factor models, there is considerable evidence that the Vasicek and the extended Vasicek models do often at least as well as the CIR or the Ho and Lee [29] models. However, as shown by Canabarro [8] and Chen et al. [10], the use of time-series data reveals that one-factor models perform generally bad, and the Vasicek is indeed mainly reliable with cross-sectional data (see de Munnik and Schotman [52]). This issue reveals to be critical for the pricing of interest rate derivatives, for which a multi-factor model seems to be necessary.
the rate of return and volatility of stock.

Finally, this gives a basis for the analysis of defaultable swaps between counterparties with asymmetric default risk, since the major determinant of their asymmetry should indeed be the state variable standing for firm value.

The implications of the model that we propose concerning the pricing of securities issued by the firm are numerous. We provide pricing formulas for discount and coupon bonds, and for derivatives written on these securities, but also for equity under a simple capital structure. In this case, we manage to emphasize the critical role of default risk for the mean-reverting behavior of equity returns and for the correlation between equity returns and the spot interest rate.

But the reduced-form approach adopted here has another valuable advantage over structural form approaches, in that it can take into account another component of credit risk: the magnitude risk of default, which is complementary to the arrival risk of default. The paper by Madan and Unal [47] represents a recognition of this kind of risk. However, Duffie and Singleton [22] point out that models of credit risk are very sensitive to the specification of the loss rate in case of default. In the literature, there are generally two ways to describe this function. A first approach, advocated by Brennan and Schwartz [7], Fons [26], Longstaff and Schwartz [45] and Madan and Unal [47], considers that the investor receives a fraction of the par value of her claim at maturity. In the same spirit, in the papers by Jarrow and Turnbull [35], Nielsen, Saá-Requejo and Santa-Clara [53], Lando [40] and Jarrow, Lando and Turnbull [36], the investor recovers a fraction of the default-free version of the same claim. Duffie and Singleton introduce a third way to model this magnitude risk: when the firm defaults, the investor is left with a fraction of the value of her claim just prior to default. This is typically the case in off balance sheet contracts, such as swaps. But this approach, as shown in our developments of the basic model, can be sustained for the valuation of corporate bonds, because such a specification of magnitude risk is economically consistent with cases where the firm is unable to meet some debt service, but is still running and thus keeps the same risk characteristics posterior to default.

We motivate here that all three specifications can be justified, depending
on the application: the first two roughly correspond to firm liquidation, whereas the last one is more likely to lead to a restructuration. Clearly, they coexist. Due to the flexibility of the modelling choice, we manage to show how to build a Gaussian model with “irreversible default” of either kind, with a possibly very close arrival risk, under the alternative characterizations of the loss rate, defined either as a fraction of the par value, either as a fraction of the default-free version of the same claim, corresponding roughly to the liquidation of the defaulted company. The pricing equations developed by Duffie, Schroder and Skiadas [21] and Lando [40], among others, allow to find analytical expressions for risky bonds under well-behaving arrival and loss rates. Subsequently, numerical simulations of the implied yield-to-maturity in each regime and its sensitivity to credit risk characteristics, for equal parameter values, shows that all three approaches has great flexibility but very different behaviors. This leads to a necessary multidimensional determination of “default risk”

Of course, our ultimate concern is to derive the necessary conditions for reaching a global model, mixing the alternative regimes, which is most likely to occur in reality.

This paper will be organized as follows. Section 2 proposes the basic model. The third Section presents implications concerning the pricing of corporate claims, such as equity and bond options. In Section 4, the role of magnitude risk is discussed, and two alternative models of default risk are presented, compared with the basic model and mixed in a general model. Section 5 analyzes some extensions and limitations of this framework. Section 6 concludes the article.

2 The Basic Model

2.1 The State Variable

In this section, we are primarily interested in the value of a defaultable bond issued by a given firm. This is a zero-coupon bond, to be valued at time \( t \), and maturing at some time \( T \). We shall see later how to price coupon bonds using the same analysis.
Primarily for the purposes of tractability, it is assumed that the interest rate process follows a Gaussian process as described by the Vasicek [61] model:

$$dr(t) = a_r[\bar{b}_r - r(t)]dt + \sigma_r dZ_r(t)$$  \hspace{1cm} (1)

Of course, this very simple setup for the interest rate does not rule out negative values, but they have sufficiently low probability so that the model can be considered to hold.

Our main assumption is that both the arrival and the magnitude risks of default depend on the value of a state variable $X(t)$ — which can also be made dependent on $r(t)$, the level of the spot interest rate — where $X(t) = V(t)/K(t)$, $V(t)$ representing the market value of the firm's assets and $K(t)$ represents an economically comparable counterpart.

The process for the market value of the firm is assumed to follow a Brownian motion depicted by the following stochastic differential equation:

$$\frac{dV(t)}{V(t)} = (r(t) + \mu - s \ln X(t))dt + \sigma_v dZ_v(t)$$  \hspace{1cm} (2)

where $Z_v$ is a standard Brownian motion and has a correlation coefficient $\rho$ with $Z_r$.

The process for $K(t)$ is defined by a stochastic differential equation whose only source of uncertainty comes from the introduction of the spot interest rate in the drift term:

$$\frac{dK(t)}{K(t)} = (r(t) + k)dt$$  \hspace{1cm} (3)

Using (3) and (2), we directly obtain the stochastic differential equation for $X(t) = V(t)/K(t)$:

$$\frac{dX(t)}{X(t)} = (\mu - k - s \ln X(t))dt + \sigma_v dZ_v(t)$$  \hspace{1cm} (4)

The process $K(t)$ is often encountered in structural form models. Under their assumptions, it stands for a barrier whose hitting by the firm value process
automatically triggers default.\(^3\) As such, and referring to the approach of equity as a call option on firm value (see Black and Scholes [6]), it has clearly there the economic meaning of debt. It is often considered as constant, as in Longstaff and Schwartz [45] or in Kim, Ramaswamy and Sundaresan [38], primarily in order to simplify exposition and computation. In Nielsen et al. [53], this process is considered as stochastic, depending on both interest rate and asset value uncertainty, and with positive drift. The latter feature is justified by the increase of market value of debt as time to maturity decreases. The stochastic behavior of \(K(t)\) derives from its (natural) dependence over the term structure of interest rates, and also from the fact that the intensity of production uncertainty impacts on this process for the critical level of firm value. Since default might occur either because of flow-based insolvency (in-capacity to face cash-payments) or because of stock-based insolvency (due to a too low value of the assets), those causes coincide only in a frictionless world (see Nielsen et al. [53]): the stochastic behavior of the barrier alleviates the difficulty to model the possible shift in a real world, with frictions. However, even with these efforts to match reality, this is obviously far from being satisfactory, because the process \(K(t)\) still does not correspond to anything observable in structural form models.

The use of a similar process in the reduced-form approach yields a much more appealing interpretation, for two reasons. The first one results from the flexibility of this approach: the goal of the modelling is to mimic reality as well as possible, instead of primarily seeking economic intuition as in the structural form framework. Therefore, the difficulty to interpret and justify the choice of the barrier is not central anymore. Second, giving up the strict requirement that \(K(t)\) is a lower bound for firm value that triggers default opens the way to another interpretation of the process. Here, the meaning of \(K\) is completely tied to the meaning of \(X\), and so it must be taken as an economic counterpart of \(V(t)\), which reflects a market value. If we interpret the value process as the market price of all the firm’s investments (on the

\(^3\)In the structural approach, \(K(t)\) is taken as the frontier for which default is triggered whenever \(V(t) = K(t)\). We do not want to give up this idea completely, and will assign a special meaning to this event. An alternative way to consider the problem is to assign special values to the event \(X(t) = V(0)/K(0)\), where \(V(0)\) and \(K(0)\) are the values of \(V\) and \(K\) when debt was issued: here, the meaning of those values refers to the starting default characteristics of risky debt. Results do not change through this shift; simply we can replace \(X(t)\) by \((V(t)/K(t))(K(0)/V(0))\) in the further developments.
asset side), $K(t)$ can be seen as another measure, less sensitive to market fluctuations, of the same reality.

We propose three possible, empirically testable interpretations of the process for $K(t)$, which all lead to a mean-reverting behavior of the firm value: an opportunity cost which leads to the overreaction effect (close to a size effect), a measure of earnings opening up the way to the P/E hypothesis and a measure of book value leading to the P/B effect.

In the most economic-oriented view, $K(t)$ relates to the rate of return that should globally reward all the securities issued by the firm in order to finance its activity. If we refer to Modigliani-Miller theory, we know that the opportunity rate of return on those securities depends on the class of risk to which the company belongs. Similarly, the required rate of return resulting from an equilibrium model such as the CAPM depends on the company’s beta. It is immediately clear that such an interpretation for $K(t)$ fits perfectly to the underlying implication in the model for credit risk: the comparison between $V(t)$ and $K(t)$ is an indicator of the firm’s ability to raise funds on the market, and thus contributes to explain flow-based insolvency; but the fact that $K(t)$ has the role of a discount factor, gives it an explanatory role for stock-based insolvency too. Moreover, the fact that it has a constant excess return over the riskless interest rate makes it independent of the value of defaultable debt, which reveals to be a crucial technical condition for the model to work.

But $K(t)$ can also be another measure of the firm value. When viewed as a measure of earnings (or, as put forward by Hawawini and Keim [28], a measure of cash-flows, which is overlapping, but slightly better), then it has clearly the meaning of the discounted value of the firm’s futures earnings, which reflects a firm value in terms of future gains. In the same spirit, one can also take this process as the observable book value of the firm’s assets. The ratio $V/K$ would then be interpreted as the price-to-book ratio. This has the interesting characteristic of allowing the intervention of the accounting dimension, which is often quoted in the determination of credit risk.

The fact that the components of $X$ are two comparable measures has a fundamental implication for the validity of the model in the case of recurrent

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4In this case, it is reasonable to consider that it has a diffusion term, as in Section 5.
default risk, i.e. when default leads to a market value of debt lowered by a fraction \( l \): in this case, \( V(t) \) and \( K(t) \) experience a reduction by the same factor, so that \( X(t) \) remains unchanged and the credit risk applied to the residual claim is not modified. This result suggests that this model is consistent with the assumptions made by Duffie and Singleton [22].

This definition of the stochastic process for \( V(t) \) deserves some discussion, because of the intervention of the term \( -s \ln X(t) \) in the drift component. First, we have to notice that the presence of a size effect or related empirically involves a negative relationship between a measure of size and the rate of return; therefore, it is legitimate to assume that \( s \) is positive.

Second, we have not taken \( V \) as the variable that penalizes the rate of return, but \( X = V/K \). The reason for this comes from our interpretation of the mean-reverting effect: if \( K(t) \) represents an economic measure of cost of capital, then it must correspond to the opportunity cost of investing in the firm: the amount by which \( K(t) \) exceeds \( V(t) \) represents thus the loss in market value of investments with respect to the required payoff to suppliers of resources, which triggers a market reaction to make \( V \) converge to \( K \). Raw firm value alone does not provide information about the excess market return generated by the firm over its economic measure, and thus does not give sufficient legitimacy to this effect. Each time \( V(t) \) exceeds \( K(t) \), it is assumed that the market recognizes that it has overvalued the firm, and tries to correct the anomaly created. This is similar to considering that the hypothesized effect is indeed not due to market capitalization, but to its relative importance with respect to a reference set by the market. So, a firm whose market value is higher than expected tend to experience lower returns in subsequent periods. This phenomenon, which is often reported as overreaction, first documented by de Bondt and Thaler [16] and confirmed by Chopra, Lakonishok and Ritter [12] using the empirical SML as a reference, has indeed been identified as the prime source of size effect by Chan and Chen [9], because a low market value is shown to be mainly due to poor performance. Interestingly, they identify that financial leverage differences have some explanatory power: this paper provides a simple justification of their empirical finding.

Instead, it is possible to consider \( V(t) \) as a measure of price, and \( K(t) \) as a measure either of earnings, consistent with the findings of Basu [4], [5] and
Reinganum [56], either of book value, in the line of the evidence reported by Stattman [59], De Bondt and Thaler [15], Keim [37] and Fama and French [24], [25]. In both cases, a high value of \( X \) leads to lower returns if the effect exists.

The empirical evidence of a mean-reverting effect for equity indeed implies, as shown in the analysis of the firm's stock behavior, a mean-reverting effect for the firm value, but the opposite is not true. Thus, the assumption of mean reversion in firm value returns is weaker than the testable hypothesis of a similar effect for stock returns. This is a sound empirical justification for the postulated stochastic process.

On the theoretical side, this behavior is well justified too. As shown by Leland [41], Toft [60] and Leland and Toft [42], the assumption that the value of the firm's assets follows a geometric Brownian motion implies, under the presence of a default risk, that the market value of the firm is equal to the asset value plus the tax shield of debt minus bankruptcy costs. These costs are negatively related to the ratio \( V/K \), so that it is natural to consider that firms with a large value of this ratio exhibit lower financial risk than firms with a low ratio, and thus should earn a lower expected rate of return: the risk argument explains the mean-reverting behavior. This also implies, as in the model, that \( V \) should be interpreted as a market value, and is in essence observable.

Third, we do not use a linear function of \( V/K \), but we take a monotonically increasing function, namely the natural logarithm. The behavior of this function accounts extremely well for the postulated pattern of rates of return: relative size effect is particularly strong for firms experiencing a reduction in market value, which is illustrated by the concavity of the \( \ln \); in our view, the most neutral value of \( X \) with respect to postulated effect is 1, because then firm value should behave like a geometric brownian motion with drift \( r(t) + \mu \) corresponding to the expected rate of return of investements. If \( X(t) > 1 \), the effect plays negatively, and reversely if \( X(t) < 1 \). Finally, we have to stress that this equation can only be a valid approximation for reasonable values of \( V(t) \): if it goes to zero, the rate of return would go to infinity, and if it is very high, the drift could be less than \( r(t) \), the riskless interest rate. Thanks to the intrinsic mean-reverting behavior of firm value, this is indeed an issue analogous to the one of extreme values for Gaussian
processes, which can be considered as negligible in practice for a reasonable
initial parametrization.

2.2 Arrival and Magnitude Rates of Default

The event of default is assumed to follow, under a risk-neutral hazard rate,
an exponential distribution, whose cdf is

\[ F(t) = P_Q[0 < \tau \leq t] = \int_0^t h e^{-h\tau} d\tau \]

where \(\tau\) is the timing of default, \(Q\) is a risk-neutral probability measure and \(h\) is the associated arrival rate of default, or hazard rate.\(^5\)

It is natural, in the spirit of the model, to consider that the process for \(X(t)\) affects the likelihood of default. The hazard rate is thus expressed as \(h(X(t))\), a decreasing function of the state variable. Since \(\ln X(t)\) is mean-reverting, this means that \(K(t)\) must have some kind of economic meaning for the market (otherwise the effect would quickly disappear): it is highly likely that the assessment of credit risk starts from the same economic reference, because this is the way the market should rationally behave. Moreover, the use of an exponential distribution with a stochastic arrival rate guarantees a high flexibility, and is perfectly in line with existing reduced-form models.

The magnitude risk of default is represented by the loss rate \(l\): when the firm defaults, the current market value of the bond is instantaneously reduced by a fraction \(l\), which remains subject to similar credit risk characteristics: this is why the model is called with "recurrent credit risk". We also assume that it is a function \(l(X(t))\) decreasing in \(X(t)\). This formulation does not prevent other state variables from affecting the values of \(h\) and \(l\), but simply that we consider only \(X(t)\) explicitly.

This characterization of the loss rate corresponds to the framework developed by Duffie and Singleton [22]. The intuition for the risk-adjusted expectation that has to be computed can be understood with a discrete binomial model:

\(^5\)Artzner and Delbaen [2] determine under what conditions the hazard rate exists under one probability measure iff it exists under an alternative probability measure. We assume this is the case here.
in a short time interval $\Delta t$, the firm can either default with risk-adjusted probability $h(X(t))\Delta t$, yielding a payoff of $1 - l(X(t))$ times the market value of the security, or not default with probability $1 - h(X(t))\Delta t$, leaving unchanged payoff. The proper risk-adjusted discount rate $R(r(t), X(t))$ respects then the condition:

$$\frac{1}{1 + R\Delta t} = \frac{1}{1 + r\Delta t} [h(X(t))\Delta t(1 + l(X(t))) + (1 - h(X(t))\Delta t)]$$

whose solution, taking the limit as $\Delta t$ goes to zero, is simply $R(r(t), X(t)) = r(t) + h(X(t))l(X(t))$.\(^6\)

Under these circumstances, the value $P_c(r(t), X(t), t, T)$ of a pure discount bond with maturity $T$ is given by the following, provided that default has not yet occurred at time $t$:

$$P_c(r(t), X(t), t, T) = EQ \left[ \exp \left( - \int_t^T R(r(u), X(u))du \right) \right] \mathbb{F}_t$$

where $\mathbb{F}_t$ denotes the expectation under some risk-adjusted probability measure, and $R(r(t), X(t))$ is given by the following equation:

$$R(r(t), X(t)) = r(t) + h(X(t))l(X(t))$$

In this model, it turns out that the vector $\Phi(t) = (r(t), X(t))$ respects a condition equivalent to the one imposed by Hull and White [32] for the use of the martingale approach, but here simply expressed as $E[G(T)\Phi(t)] = E[G(T)|\mathbb{F}_t]$ where $G$ represents the payoff of a defaultable security and $\mathbb{F}_t$ is the sub-sigma-algebra available to the market at $t$ of the filtration $\mathbb{F}$. It clearly implies that $\Phi$ is Markovian.

Our goal here is to achieve the modelling of $R(r(t), X(t))$ so that it follows, under reasonable assumptions, a Gaussian distribution under the assumed risk-neutral distribution. Then, the price of the corporate discount bond, and subsequently of other securities issued by the firm, can be discovered.

\(^6\)See Duffle and Singleton [22], Appendix A, for a more rigorous proof in a general continuous-time setup. This equation does not hold if $X(t)$ is a function of the claim value itself, which is not the case since the processes for $V(t)$ and $K(t)$ are assumed to be exogenous from debt valuation.
The formulation of \( R(r(t), X(t)) \) using equation (6) shows that the proper discount rate for the risk-adjusted expectation is indeed the sum of the riskless interest rate and of a product of two functions of \( X(t) \). We already noticed that \( r(t) \) followed an Ornstein-Uhlenbeck process, and thus the first term of the sum will be normally distributed under the alternative probability measure. The only way to ensure that \( R(t) \) follows a normal distribution is to assume that \( h(X(t))l(X(t)) \) is itself normally distributed. This is done by assuming that the following equality is respected under the equivalent probability measure:

\[
h(X(t))l(X(t)) = C_0 + C_1 \ln X(t)
\]

where, by Itô's lemma, \( \ln X(t) \) respects the following differential equation under the original probability measure:

\[
d\ln X(t) = (\mu - k - \frac{1}{2}\sigma_v^2 - s \ln X(t))dt + \sigma_v dZ_v(t)
\]

and, in order to start at economically interpretable values, \( \ln X(0) = 0 \). Under this assumptions, \( \ln X(t) \) follows a Gaussian mean-reverting process. This behavior is important, because it makes the probability of very high values of \( \ln X(t) \) really marginal. This will show up to be quite critical later on.

The issue of the separate specification of the loss rate and the hazard rate is not central in this modelling. There is actually one degree of freedom for their choice, since the functional form of either induces the one of the other. Thus, the usual choice of an exponential function for the hazard rate, as the formulation used in McDonald and Van de Gucht [50], is compatible with this approach: if \( h(X(t)) = a \exp\{bX(t)\} \), it follows then that \( l(X(t)) = a^{-1} \exp\{-bX(t)\}(C_0 + C_1 \ln X(t)) \).

The evidence of a positive aging effect, as reported by McDonald and Van de Gucht [50], can be explicitly taken into account by the same one-factor model, by simply introduce a time-dependence on the arrival rate: \( h(t, X(t)) \). This can become crucial when the model is used when a firm has only one maturity of debt\(^7\), because then default should become more likely near maturity when

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\(^7\)This model works preferably for debt which is rolled-over through time, which is consistent with the time homogeneity of the coefficients of the credit spread.
\(X(t)\) is low, leading to a hazard rate which is a negative function of time-to-maturity. Taking a linear loss rate, for instance, does not prevent from considering a time dependence on \(h\).

The meanings of the arrival rate, \(h(X(t))\), and of the magnitude risk, \(l(X(t))\), may help to determine which are the critical values that they should take.

- If \(X(t) \to 0\), then \(h(X(t))l(X(t)) \to \infty\), because the arrival rate is an increasing function of \(X(t)\), and the probability of default when the value of the firm goes to zero must go to one, which corresponds to an arrival rate of \(h(X(t)) \to \infty\). Concerning the loss rate, we have more flexibility, but it should reasonably be equal to 1.

- If \(X(t) = 1\), we set that the arrival rate should be equal to a constant \(\lambda\), which was the arrival rate prevailing at debt issuance, whereas the loss rate should be equal to \(1 - \alpha\), \(\alpha\) being the recovery rate of reference, as in models for which the value of the firm does not directly affect magnitude of default. The value \(X(t) = 1\) can thus be considered as the point where the fraction of market value left to debtholders in case of default is constant, and determined from the beginning. Consequently, the product \(h(1)/l(1) = \lambda(1 - \alpha)\).

- If \(X(t) = \pi\) for some constant \(\pi > 1\) such that \(\Pr(Q(X(t) > \pi))\) can be considered as negligible, it is reasonable to assume that, even if there is still a positive probability that default occurs, the loss rate should go to zero. This directly follows from the previous discussion: considering that the recovery rate is increasing and even linear in \(X(t)\), debtholder get to lose \(1 - \alpha\) of the value of their claim if \(X(t) = 1\). If default occurs at a higher value of \(X\), they are likely to hold a more valuable claim.

But what if the value of the firm at time of default is so high, that after all expenses due to this event — under our assumptions about the loss rate, the scenario is the one of a restructuration —, there is still enough to pay back the debtholders? Although we do not require that the arrival rate of default should be equal to zero, there must be a critical value of \(X(t)\) where debtholders are not worse off because of default.\(^6\) But of course, since in this case the firm is virtually riskless,

\(^6\)This possibility is explicitly accounted for in some multiperiod models of debt analysis, such as in Diamond [17] and Rajan [55].
we should have the third condition $h(\pi)I(\pi) = 0$. The constant $\pi$ is thus interpreted as the threshold level of debt risklessness. In the model, this also means that the likelihood that a firm which issued defaultable debt becomes riskless, while its value process remains stochastic, is extremely low. This is consistent with the fact that the credit spread of AAA firms is low, but never null.

These conditions together lead to the following expression:

$$h(X(t))l(X(t)) = \lambda(1 - \alpha) + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{\pi}} \ln X(t) \quad (8)$$

The crucial assumption behind those functions is that the probability that $X(t) > \pi$ is negligible, because negative values of the spread imply negative values for either the loss rate or the arrival rate: none of them is economically sustainable. This issue is exactly the same as for the Vasicek model.

2.3 Pricing Corporate Discount Bonds: The Independent Case

Thanks to equations (5),(6) and (8), and knowing the processes for $X(t)$ and $r(t)$, the valuation of risky bonds in a risk-neutral framework is straightforward, provided that the credit spread is identified under the risk-adjusted probability measure. The transformation of the arrival rate from the original to a risk-neutral probability measure is performed here through the introduction of a risk premium for $X$, which is chosen so as to fit the spread to risky bond prices. Notice that this adjustment is indeed sufficient for risk-neutral valuation, as shown by Duffie and Singleton [22] although the occurrence of default implies a discontinuity in the payoffs.

If the correlation between $Z_v(t)$ and $Z_r(t)$ were zero, equation (5) could be rewritten as a product of expectations:

$$E_Q \left[ \exp \left( - \int_t^T r(u) du \right) \right] \mathcal{F}_t$$

$$E_Q \left[ \exp \left( - \int_t^T (\lambda(1 - \alpha) + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{\pi}} \ln X(u)) du \right) \right] \mathcal{F}_t \quad (9)$$
which is a product of expectations of lognormal, mean-reverting random variables, under the following processes for \( r(t) \) and \( \ln X(t) \):

\[
dr(t) = a_r(\bar{r} - r(t))dt + \sigma_r dZ_r^*(t)
\]

\[
d\ln X(t) = (\bar{\gamma} - \frac{1}{2}\sigma_v^2 - s \ln X(t))dt + \sigma_v dZ_v^*(t)
\]

where \( \bar{r} = b_r - \sigma_r \lambda_r/\sigma_r \), \( \bar{\gamma} = \mu - k - \sigma_v \lambda_v \), \( Z_r^*(t) = Z_r(t) + \lambda_r t \) and \( Z_v^*(t) = Z_v(t) + \lambda_v t \).

The economic interpretation of \( \bar{\gamma} \) is extremely interesting. Having assumed, as for the Vasicek model for the interest rate, a constant price of corporate risk, we have adjusted the long run mean of the process \( \ln X(t) \), which is equal to \((\mu - k - \frac{1}{2}\sigma_v^2)/s\) under the true probability measure, to a value \((\bar{\gamma} - \frac{1}{2}\sigma_v^2)/s\). This implies that the market price of industrial risk taken by the firm is \( \lambda_v = \frac{\mu - k - \bar{\gamma}}{\sigma_v} \). Now, if there had been no size effect in the value process, we have already noticed that \( K(t) \) would indeed be the risk-adjusted discount factor, leading to \( \lambda_v = \frac{\mu - k}{\sigma_v} \) and \( \bar{\gamma} = 0 \); then, \( X(t) \) would be a martingale under \( Q \), which is consistent with \( K \) being considered as a discount factor. However, the presence of mean-reversion reduces the variance of \( \ln X(t) \), and so it should have a nonpositive price of risk. Hence, it is plausible to have \( \mu - k > \bar{\gamma} \geq 0 \), and we can decompose \( \lambda_v = \lambda_{\text{industry}} + \lambda_{\text{size}} = \frac{\mu - k}{\sigma_v} - \frac{\lambda_v}{\sigma_v} \).

This yields computations identical to the ones performed by Vasicek [61]:

\[
P_c(r(t),X(t),t,T) = P(r(t),t,T)D(X(t),t,T)D(X(t),t,T)
\]

\[
= P(r(t),t,T)A_2(t,T)A_1(t,T)X(t)^{-B_2(t,T)}
\]

\[
= A_1(t,T)e^{-B_1(t,T)r(t)}A_2(t,T)X(t)^{-B_2(t,T)}
\]

where

\[
B_1(T - t) = \frac{1 - e^{-a_r(T-t)}}{a_r}
\]

\( ^9 \)Imposing that \( X(t) \) be a martingale under \( Q \) would even imply that \( \bar{\gamma}(t) \) is time-varying, and respects the condition \( E_Q[\ln X(s)|\mathcal{F}_t] = -\frac{1}{2}\text{Var}_{\mathcal{F}_t}[\ln X(s)|\mathcal{F}_t] \), which would destroy the mean-reverting character of \( X(t) \). The fact that \( X(t) \) is not a traded asset allows not to make the model that strict, and the issue of choosing a constant price of risk indeed boils down to assuming that the spread is mean-reverting. For the economic justification of the corresponding pricing kernel, an equilibrium model would indeed be necessary.
This function is a simple function, clearly increasing in \( X(t) \), because \( B_2 \) is negative. \( D(X(t), t, T) \) plays a nice role, since formula (10) shows that the price of a defaultable bond bears a decomposition between the price of a riskless bond and a discount factor uniquely affected by the default characteristics of the firm. This separation result will be useful later on. However, it is disturbing to assume that the correlation coefficient between the two sources of uncertainty is zero. Therefore, we have now to derive the pricing formula in the case of correlated brownian motions.

### 2.4 Pricing Corporate Discount Bonds: The Correlated Case

Now we introduce a correlation coefficient \( \rho \neq 0 \) between \( Z_r(t) \) and \( Z_v(t) \). This situation is of course more realistic, and will lead to a pricing formula which does not completely separate the discounting due to the riskless interest rate and the discounting due to the default characteristics of the firm.

Equations (5) and (6) still hold in this framework, but it has become impossible to rewrite the expectation of the exponential function as a product of expectations. However, Proposition 1 states that the formula for the discount bond in this case will look similar to the one in a one-factor Vasicek model.

**Proposition 1** The price at time \( t \) of a corporate discount bond maturing at \( T \), when credit risk is recurrent, where the default spread is given by (8) and
the processes for \( r(t) \) and \( X(t) \) are given by (1) and (4), respectively, has the following form:

\[
P_c(r(t), X(t), t, T) = \exp\left[\frac{1}{2} K_R^2(t, T) - N_R(r(t), X(t), t, T)\right] \quad (14)
\]

where

\[
K_R^2(t, T) = \kappa_1 e^{-a_r(T-t)} + \kappa_2 e^{-s(T-t)} + \kappa_3 e^{-2a_r(T-t)} + \kappa_4 e^{-2s(T-t)} + \kappa_5 e^{-(a_r+s)(T-t)} + \kappa_6 (T-t) + \kappa_7
\]

\[
N_R(r(t), X(t), t, T) = (\bar{r} + C_1 \frac{\tilde{r} - \sigma_r^2/2}{s})(T-t) + \frac{1}{a_r} (1 - e^{-a_r(T-t)}) (r(t) - \bar{r}) + \frac{1}{s} C_1 (1 - e^{-s(T-t)}) (\ln X(t) - \frac{\tilde{r} - \sigma_v^2/2}{s}) + \lambda (1 - a_r) (T-t)
\]

\[
k_1 = \frac{2}{a_r^2} \left( C_2 \rho \sigma_r + \frac{\sigma_r^2 (1 - \rho^2)}{a_r} \right)
\]

\[
k_2 = \frac{2}{s^2} C_1 C_2 \sigma_v
\]

\[
k_3 = -\frac{\sigma_r^2}{2a_r^3}
\]

\[
k_4 = -\frac{C_1 \sigma_v^2}{2s^3}
\]

\[
k_5 = -\frac{2C_1 \rho \sigma_v \sigma_r}{a_r s (a_r + s)}
\]

\[
k_6 = C_2^2 + \frac{\sigma_r^2 (1 - \rho^2)}{a_r^2}
\]

\[
k_7 = \frac{\sigma_r^2}{2a_r^3} + C_1^2 \frac{\sigma_v^2}{2s^3} - \frac{2}{a_r^2} C_2 \rho \sigma_r - \frac{2\sigma_r^2 (1 - \rho^2)}{a_r^3} - \frac{2}{s^2} C_1 C_2 \sigma_v + \frac{2C_1 \rho \sigma_v \sigma_r}{a_r s (a_r + s)}
\]

\[
C_1 = \frac{\lambda (1 - a_r)}{\ln \frac{1}{r}}
\]

\[
C_2 = \left( \frac{\sigma_r}{a_r} + \frac{C_1 \sigma_v}{s} \right)
\]
with $K^2(t,T) = \text{var}_Q(Y(t,T) \mid \mathcal{F}_t)$ and $N(r(t),X(t),t,T) = E_Q(Y(t,T) \mid \mathcal{F}_t)$, $Y(t,T)$ being the yield to maturity of the bond.

Proof: see Appendix 1.

**Corollary 1** The price of a corporate discount bond can be decomposed as follows:

$$P_c(r(t), X(t),t,T) = P(r(t), t, T)D(X(t), t, T) \exp\left\{ \frac{2p C_1 \sigma_\nu \sigma_r}{s a_r} \left[ \frac{1}{a_r} \left( 1 - e^{-a_r(T-t)} \right) - \frac{1}{s} \left( 1 - e^{-s(T-t)} \right) \right] \right\}$$

where $P(r(t), t, T)$ and $D(X(t), t, T)$ are the same as in equation (10).

Proof: Equation (10) can be written as the following exponentials:

$$P_c(r(t), t, T) = e^{\frac{1}{2} k^2_2(t,T)-n_r(r(t),t,T)} e^{\frac{C^2_2}{2} k^2_2(t,T)-C_1 n_X(X(t),t,T)}$$

where each exponential corresponds to the bond pricing formula proposed in Jamshidian [34] for the processes $r(t)$ and $h(X(t))l(X(t))$, respectively. Equation (14) is just the same exponential plus a few additional terms, that form the third factor of expression (18). \( \square \)

**Proposition 1** uses the classical two-factor bond pricing formula in the Gaussian framework. The interest of this approach lies in the integration of the adjusted value of the firm $X(t)$, of the reference parameter for the arrival rate $\lambda$, of the reference parameter for the recovery rate, $\alpha$, and of the threshold of risklessness $\pi$.

As noticed in Corollary 1, this approach has the valuable advantage of separating risky bond prices in three components: the price of the riskless discount bond, a discount factor only related to the default risks, and an adjustment driven by the sign of the correlation coefficient.
To consider the impact of the correlation, it is interesting to study the sign of the argument of the exponential in (18). The sum in brackets is always positive. Since \( C_1 < 0 \), the argument of the exponential in the covariance term is driven by the sign of \( \rho \): if \( \rho \) is positive, the argument is negative, and thus the price of the discount bond in the presence of positive correlation is lower than in the uncorrelated case. The intuition behind this is that positive correlation induces that when the interest rate increases, firm value is likely to increase too, and these two effects act in opposite directions. A negative correlation means that the term and default risks reinforce each other, so that the variance of the bond yield is higher than the sum of the variances of the yield of a riskless security and of the yield of the spread. Since the variance of the risky interest rate has a positive impact on bond prices, bond prices are positively affected by this correlation. This finding of a significant sensitivity of bond prices to the correlation coefficient is consistent with Longstaff and Schwartz [45], and counters the (rather surprising) sensitivity analysis of Nielsen and Ronn [54]. Moreover, our approach allows to express the exact effect of this correlation, confirming Longstaff and Schwartz's interpretation concerning positive values of the correlation coefficient. Yet, our symmetric interpretation for \( \rho < 0 \) differs from their analysis, because they include \( r(t) \) in the drift of \( X(t) \), contrarily to the framework suggested here.

3 Valuation of Corporate Securities

3.1 Coupon Bonds

The framework that we have considered allows to price coupon bonds in a very simple way. As mentioned by Longstaff and Schwartz [45], the fact that default is triggered for all debt issued when it occurs implies that all maturities receive the same treatment. This is confirmed by Duffie and Singleton [22] within this context of magnitude risk. For a bond paying coupons \( c_i \) at times \( \tau_i, i = 1, \ldots, n \), we have that the price at time \( t \) of this security is given

\[ \text{To see this, we simply notice that the second order derivative of this function with respect to } T - t \text{ is positive for } (T - t) \geq 0. \text{ Since the function and the first derivative equal 0 at } T - t = 0, \text{ we can conclude that the function is positive for positive time to maturity.} \]
Thus, similarly to the Longstaff and Schwartz model, a coupon bond is identical to a simple portfolio of zero-coupon bonds. Yet, this result does not easily extends to the analysis of coupon bond options.

3.2 Equity

The framework of recurrent default, developed in Section 2, is remarkable in that it proposes a process for the value of the firm and reaches a closed-form solution for the value of risky debt. By Itô calculus, the formula for the value of defaultable debt can easily be used in order to find the stochastic process for debt. If one considers a firm that has issued only debt and equity, the process for the stock of this firm is thus fully determined by the processes for \( r(t) \) and \( X(t) \), conditional on no default. Thus, this reduced-form model achieves an analysis that is usually strictly reserved to structural form models (see Toft [60] and Leland [41] for analyses of the firm's capital structure with default risk).

From formula (14), we have determined that the value of corporate debt maturing at \( T \) in the case of no default is function of three variables: \( t, r(t) \) and \( \ln X(t) \). Using Itô's lemma, we find out the stochastic differential equation for the valuation of equity, and derive some useful corollaries:

**Proposition 2** The stochastic process for the equity of the undefaulted firm whose capital structure is only composed of equity and debt with face value \( F \) maturing at \( T \), where the default spread is given by (8) and the processes for \( r(t) \) and \( X(t) \) are given by (1) and (4), respectively, is described by the following differential equation:

\[
    dS(t) = \beta(r(t), X(t), t, T)dt \\
    + \frac{\sigma_r P_c(t)F}{a_r} (1 - e^{-a_r(T-t)})dZ_r(t) \\
    + \sigma_s \left( \frac{C_1 P_c(t)F}{s} (1 - e^{-s(T-t)}) + V(t) \right)dZ_s(t) 
\]  

(20)
where \( \beta(t, T, r(t), X(t)) \) is given in Appendix 2, whenever \( P_c(t)F \leq V(t) \), and is equal to 0 otherwise.

Proof: see Appendix 2.

**Corollary 2** Under the assumptions of Proposition 2, the correlation between equity returns and the spot interest rate:

\[
\text{corr}(S, r) > \text{(resp. =, <)} 0 \text{ if }
\end{equation}

\[
\rho \sigma_v \left[ \frac{C_1 P_c(t)F}{s} (1 - e^{-s(T-t)}) + V(t) \right] > \text{(resp. =, <)} -\frac{\sigma_r P_c(t)F}{a_r} (1 - e^{-a_r(T-t)})
\]

(21)

Proof: Using two orthogonal Brownian motions, (20) can be written:

\[
dS(t) = \beta(r(t), X(t), t, T)dt + \left( \frac{\sigma_r P_c(t)F}{a_r} (1 - e^{-a_r(T-t)}) \right) + \rho \sigma_v \left( \frac{C_1 P_c(t)F}{s} (1 - e^{-s(T-t)}) + V(t) \right) dU_r(t)
\]

(22)

where \( U_r(t) \) is the only brownian motion affecting the process for \( r(t) \). The sign of the diffusion term in \( dU_r(t) \) is driven by the value of \( \rho \) as in Corollary 2.

**Corollary 3** Under the assumptions of Proposition 2, and if \( s = s_v + s_k \) where \( s_v \) is the speed of mean-reversion of \( V(t) \) and \( s_k \) is the speed of mean-reversion of \( K(t) \), and writing \( B(t) = P_c(t)F \), the expected rate of return on equity is:

\[
\mu_{S(t)} = -\beta'(r(t), t, T) \frac{B(t)}{S(t)} + (\rho(t) + \mu) \frac{V(t)}{S(t)} + (-C_1 \frac{B(t)}{S(t)} - s_v \frac{V(t)}{S(t)}) \ln X(t)
\]

(23)
Proof: follows from isolating the terms in $\ln X(t)$ from the equation for $\beta$ in Appendix 2, and replacing $s$ by $s_\nu$ in equation (59). □

An interesting point here is the clear and easy relationship that exists between the firm's equity and the sources of risk that we have assumed in the model. Since $C_1$ is negative, it is clear that $S(t)$ is less sensitive to $Z_\nu$ than the value process of the firm for low values of $s$ or for low leverage.

Corollary 2 presents a highly interesting result, because the sign of the expression between brackets is not determined. The second term of this bracket is just the direct effect of firm value risk, and is of course positive since equity is part of firm value. The first term is negative. The intuition is as follows: imagine that $\rho$ is positive. Then, a high interest rate is likely to go along with a high firm value. This implies lower credit risk, and thus increases debt value. By substitution, this penalizes the value of stock. If the sum between brackets is negative, it leverages $\rho$ negatively, with the strange effect that the higher the correlation between firm value and spot interest rate, the lower the correlation between $S(t)$ and $r(t)$. The sign of the correlation does not simply "transfer" from $V$ to $S$, because of credit risk.

From the same framework, derivatives written on the firm's equity can obviously be priced using the ordinary valuation methods, since the process for equity is a mere diffusion process depending on two state variables. However, the discount rate for risk-neutral valuation must be then $r(t) + h(X(t))$ instead of $r(t)$, since equity is a defaultable security too whose loss rate, for acceptable values of $X(t)$, is equal to 1.

The most important result concerning equity is given in Corollary 3. Actually, thanks to the simplicity of the bond pricing formula, the instantaneous rate of return for equity shows a simple structure in $\ln X(t)$: since $C_1 < 0$, equity has a mean-reverting component due to the firm value process, and a mean-averting component due to the sensitivity to the rate of return on corporate debt. If $s_\nu = 0$, the intensity of mean-aversion of equity is equal to the mean-reverting intensity of debt.\footnote{This comes from the fact that $X(t) = \frac{s(t) + P(t)F}{K(t)}$, increasing in $S(t)$.} So, with the observable stock prices, the structure of mean-reversion can be found. In addition, the strength of this effect is negatively related to financial leverage: although the second factor is the clear repercussion of the mean-reverting character of $V(t)$ on...
$S(t)$, the first one represents its negative impact on the drift of $P_c(t)$ when $X(t)$ increases.

Thanks to this structure, several other teachings can be driven from this corollary. First, the mean-reverting character is absent if $s_v = 0$ and $s_k > 0$, i.e. if firm value does not exhibit mean-reversion, but its associated measure well. This scenario would mean that the explanation of this effect would be purely structural: when $V(t)$ increases, the cost of capital is revised upwards because the growth of $V$ is assumed to represent a structural change of class of risk. For the all-equity financed firm, this effect on equity would be zero. This also implies that equity may not exhibit mean-reversion even if $K$ does. Second, intensity of mean-reversion is negatively related to $\lambda$ and $\pi$ and positively to $\alpha$: the stronger the arrival and loss rates, the higher the threshold of risklessness, the stronger the mean-reverting behavior of debt. We thus expect to find a less pronounced cyclical behavior of equity among firms severely exposed to credit risk (low rated firms). Finally, it is interesting to notice that the degree of mean-reversion of equity is only related to $s_v$, whereas its degree of mean-aversion is only related to the credit risk characteristics of the firm. Since the presence of mean-reversion decreases the variance of returns, this would mean that high leverage, high credit risk and low mean-reversion of firm value increase the volatility of equity. These are very intuitive phenomena.

Of course, all these results have a nice economic interpretation. The key point is that the rate of return on the firm’s market value is negatively related to the level of $X$, and thus of $V$. A high firm value is translated into the market price of debt, whereas the rate of return on debt is affected by the rate of return on firm value, revised downwards, through an increased likelihood of default. The equity process “steals” a premium from the decreasing return on debt, simply because the firm is becoming more risky. The higher the arrival and loss rates, the greater the mean-reverting effect on debt, and the more severe the “substitution” effect to the benefit of equity. However, this effect is completely unaffected by the rate of mean-reversion of firm value, which is a remarkable separating result. On the other hand, the fact that equity is one of the components of firm value naturally involves that equity should exhibit mean-reversion proportionally to its contribution to the value of the firm: the intensity of mean-reversion only depends on $s_v$ and on leverage.
through what we call a “direct” effect.

The importance of this Corollary becomes clear when we give up the assumption of a mean-reverting process for $X$. Consider an alternative model of default risk involving the value of the firm’s assets, which is assumed to follow a simple geometric Brownian motion. If $V$ (or $X$) is the sole state variable affecting default spread, does one observe such direct and substitution effects? Obviously, the direct effect of mean-reversion cannot be present, since firm value does not exhibit this behavior. Concerning the presence of a substitution effect, it may exist only if the rate of return on debt is affected by the level of the state variable. With no mean-reversion in firm value, its rate of return is independent of its level, and thus the effect of the size of the state variable is only included in the price of corporate debt, not in its expected return. Therefore, it is unlikely that any substitution effect exists in favor of equity. All this means that both the mean-reverting and the mean-averting effects on equity only occur when the state variable used in order to explain default characteristics of corporate debt exhibits mean-reversion. We thus draw three important conclusions out of this subsection:

1. In this model where the price of equity is endogenously determined through the values of the firm assets and debt, mean-reversion of the stock price exists only if the firm value itself is mean-reverting.

2. There might still be mean-reversion in the firm value process even if equity is not mean-reverting.

3. Mean-reversion of stock returns is lower, ceteris paribus, for firms having a higher credit risk.

Thus, evidence of overreaction, P/E or P/B effects on the stock market would imply that, under the assumptions of the model concerning the behavior of default risk and the capital structure of the firm, this mean-reversion would be triggered by firm value. However, failure to observe one of these effects on stock returns would not lead to the rejection of the model. This is why the third statement is very helpful for any attempt to test the model, because this is also an output of the model in the presence of mean-reversion of firm value: if the substitution effect were absent, we could reject the joint assumptions of the model.
3.3 Corporate Bond Options

This subsection presents results concerning options on discount bonds and on coupon bonds. The methodology remains valid for other derivatives subject to similar credit risk characteristics.

3.3.1 Options on Discount Bonds

Using the same model, it is also possible to price derivatives written on bonds issued by the firm. The methodology used for this task still refers to the risk-neutral framework.

First, we have to specify the expectation that we take under the risk-adjusted probability measure $Q$. Equation (5) is indeed a special case of the following general pricing equation for any security $U(r(t), X(t), t, T)$ that depends on the states variables $r(t)$ and $X(t)$, yielding a payoff of $\Phi$ at time $T$ if default does not occur, and $\Phi(1 - l(X(t)))$ if default occurs:

$$U(r(t), X(t), t, T) = E_Q \left[ \exp \left( - \int_t^T R(r(u), X(u)) du \right) \Phi \mid \mathcal{F}_t \right]$$

This equation is similar to the ones proposed by Duffie and Singleton [22] and Madan and Unal [47], among others, for the specific issue of defaultable securities. The basic interest of this equation is that the possibility that default occurs between time $t$ and $T$ is already integrated in the default-adjusted discount rate $R$. Therefore, it is not necessary to condition this expectation to the case of no default. Equation (5) simply assumes that the payoff to the corporate bond is $\Phi = 1$ if no default.

The issue that we consider here is the pricing of a European call option maturing at time $T$ on the corporate discount bond maturing at $T_2$, $T_2 > T$. The strike price of the option is $\phi$, and the firm is subject to the default

This assumption is necessary in order to justify the use of $R$ as a discount rate. It means economically that, in case of default, holders of those securities experience a loss of their claim which is proportionally similar to the loss of debtholders. This view is consistent with options that are attached to corporate bonds and have the same seniority. If seniority of those securities were less than the one of corporate debt, this reasoning would not hold.
characteristics depicted in the second section. Denote \( \Gamma(r, X, t, T, T_2, \phi) \) as the price of this option at time \( t \). Its parametric form is given in the following Proposition:

**Proposition 3** The price \( \Gamma(r, X, t, T, T_2, \phi) \) of a European call option maturing at \( T \) with strike price \( \phi \) on the corporate discount bond maturing at \( T_2 \) is given by the following formula:

\[
\Gamma(r, X, t, T, T_2, \phi) = P_c(r(t), X(t), t, T_2)N(d_1) - \phi P_c(r(t), X(t), t, T)N(d_2)
\]

where

\[
d_1 = \frac{\ln \left[ \frac{P_c(r(t), X(t), t, T_2)}{\phi P_c(r(t), X(t), t, T)} \right]}{\Sigma_P} + \frac{\Sigma_P}{2} \quad \quad d_2 = d_1 - \Sigma_P \quad \quad \Sigma_P^2 = \frac{\sigma^2}{2a_r^2} \left( 1 - e^{-a_r(T_2-T)} \right)^2 \left( 1 - e^{-2a_r(T-T)} \right) + \frac{C_1^2 \sigma^2}{2s^3} \left( 1 - e^{-s(T_2-T)} \right)^2 \left( 1 - e^{-2s(T-T)} \right) + \frac{2C_1 \rho \sigma_r \sigma_r}{a_r s(a_r + s)} \left( 1 - e^{-a_r(T_2-T)} \right) \left( 1 - e^{-s(T_2-T)} \right)^2 \left( 1 - e^{-(a_r+s)(T-T)} \right) \]

where \( N(.) \) denotes the cdf of the standard normal distribution.

**Proof.** See Appendix 3.

Moreover, from the above valuation formula, we can also determine the forward rate:

**Corollary 4** The forward rate at time \( T_1 \), denoted \( f_c(r(t), X(t), t, T) \), has the following form:

\[
f_c(r(t), X(t), t, T) = \frac{\partial P_c(r(t), X(t), t, T)}{\partial T_1} = M(r(t), X(t), t, T) - J(t, T)
\]

27
where

\[ M(r(t), X(t), t, T) = E_Q[R(T) | \mathcal{F}_t] \]

\[ = \bar{r}(1 - e^{-\alpha r(T-t)}) + e^{-\alpha r(T-t)} r(t) + \lambda(1 - \alpha) \]

\[ + C_1(1 - e^{-s(T-t)}) \frac{\bar{q} - \sigma_s^2/2}{s} \]

\[ + C_1 e^{-s(T-t)} \ln X(t) \]

\[ J(t, T) = \text{cov}_Q[R(T), Y(t, T) | \mathcal{F}_t] \]

\[ = -\frac{\sigma_s^2}{2a_s^2} (1 - e^{-2\alpha r(T-t)}) - \frac{C_1^2 \sigma_v^2}{2s^2} (1 - e^{-2s(T-t)}) \]

\[ + \left( \frac{\sigma_s^2 + \alpha C_1 \rho \sigma_s \sigma_v}{a_s^2} \right) (1 - e^{-\alpha r(T-t)}) \]

\[ + \left( \frac{C_1^2 \sigma_s^2 + C_1 \rho \sigma_s \sigma_v}{a_s s^2} \right) (1 - e^{-s(T-t)}) \]

\[ - \frac{C_1 \rho \sigma_s \sigma_v}{a_s s} (1 - e^{-(\alpha r + s)(T-t)}) \]

Proof: see Appendix 3.

This Proposition is extremely important and useful, because it does not suppose the knowledge of the value of \( X(t) \) in order to price the option on a corporate bond. The expression for \( \Sigma_P \), indeed, does not directly depend on \( X(t) \) or \( r(t) \), but only on the parameters of these processes. Thus, observation of the corporate bond prices and proper calibration of the parameters are sufficient to yield an option pricing formula, although we have not specified that \( X(t) \) had to be observed. We remember that observation of the firm value, \( V(t) \), and a fortiori of the ratio \( X(t) \), was one of the major drawbacks of the "first hitting time" approach described in the introduction; here, we have managed to take the firm value explicitly into account in the pricing of derivative securities without having to observe it directly.

3.3.2 Options on Coupon Bonds

Contrarily to the simple method used by Jamshidian [34] in order to price options on coupon bonds in a one-factor model, the presence of two additive factors does not allow to price this option as a portfolio of single options on
the coupons. Using the same notation as equation (19) and Proposition 3, the exercise region for the call option maturing at \( T \) is now the set of pairs \((r(T), X(T))\) which respect this inequality:

\[
P_c(r(T), X(T), T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n) \geq \phi
\]

and denote \( L \subset \mathbb{R}^2 \) the set of all these pairs. It corresponds to the exercise region, and the risk-adjusted expectation of the payoff to the options has to be computed for this region. Unfortunately, the frontier \( L^* \subset \mathbb{R}^2 \) made of the pairs respecting the corresponding equality is nonlinear, so that the solution is not straightforward.

Using the same method as in Chen and Scott [11], the probability functions for options on coupon bonds can be expressed through single numerical integrals.

First, the risk-adjusted expectation under consideration has to be identified. the price of the call option is equal to:

\[
\Gamma(r, X, t, T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n, \phi)
\]

\[
= E_0\left[\left(\sum_{i=1}^{n} e^{-\int_{t}^{T} R(u) \, du} c_i - e^{-\int_{t}^{T} R(u) \, du} \phi\right) 1_{\{r(T), X(T) \in L\}}\right] \quad (25)
\]

The solution to this expectation is given by a sum of integrals given in Proposition 4.

**Proposition 4** The price \( \Gamma(r, X, t, T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n, \phi) \) of a european call option maturing at \( T \) with strike price \( \phi \) on the corporate bond paying coupons \( c_1, \ldots, c_n \) at times \( \tau_1, \ldots, \tau_n \) is given by the following formula:

\[
\Gamma(r, X, t, T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n, \phi) = P_c(r, X, t, \tau_1, \ldots, \tau_n) \sum_{i=1}^{n} w_i N^*(d_i, d_i^*) - \phi P_c(r, X, t, T) N^*(d_0, d_0^*)
\]

\[
(26)
\]

where

\[
w_i = \frac{c_i P_c(r, X, t, \tau_i)}{P_c(r, X, t, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n)}
\]
for $i = 0, \ldots, n$ and $N$ and $N'$ denote, respectively, the cdf and the pdf of the univariate standard normal distribution. The identification of the normal variates is given in Appendix 4.

Proof: see Appendix 4.

Thanks to the Gaussian specification of the additive factors of the risky rate, we have been able to achieve a result which was not obtained by Chen and Scott [11] under a two-factor CIR model for the spot interest rate: the correlation between the two sources of risk can now be explicitly taken into account in the default risk model. The high distribution flexibility of the multivariate normal is of course the key of this result, which can again be easily extended to a multi-factor model with a general deterministic correlation structure.

The computation algorithm for the pricing of discount bond options reveals to involve the resolution of an implicit equation and a probability function which is a unidirectional numerical integral of standard normal variates.

4 The Role of Magnitude Risk

4.1 Two Similar Models with Irreversible Credit Risk

We have considered so far that the event of default leaves the debtholder with a fraction of the value of her claim just prior to default, with the same credit risk. However, the most widespread view of the loss rate considers that default might happen once for all, and leaves the investor with a fraction of par or of the value of the undefaultable version of her claim.
In both cases, this event can be thought as a liquidation. The difference arises from the timing of the reimbursement of the claim. In the first hypothesis, which we call “immediate liquidation”, the assets of the firm are supposed to be directly liquidated in case of default, and bonds are paid back before maturity. The second hypothesis involves that the corporate bond is replaced by a claim with the same maturity: the firm is in liquidation, and thus cannot default twice, but maturities are respected. This is the “delayed liquidation” hypothesis, which is likely to apply to shorter term bonds.

It is still possible to work out the same developments as before, but we will unfortunately be left with a simple time integral for the value of the zero-coupon bond. This can be thought as the simplest expression for which a defaultable bond can be expressed in this framework, since the normal distribution assumed here is indeed among the most tractable realistic assumptions. This gives an idea of the difficulties that are likely to occur for more complex stochastic processes, like the CIR for the spot interest rate.

Defining as before the recovery rate \( l(X(t)) \), i.e. the fraction of par value left in case of default, and the arrival rate of default, which is an event assumed to follow an exponential distribution, by \( h(X(t)) \), the following formula for the pricing of corporate bonds in case of immediate liquidation is given, among others, by Duffie, Schroder and Skiadas [21]:

\[
P_c(r(t), X(t), t, T) = \mathbb{E}_Q \left[ \int_t^T h(X(u))(1 - l(X(u))) \exp\{-\int_u^T r(v) + h(X(v))dv\} du \bigg| \mathcal{F}_t \right] 
+ \mathbb{E}_Q \left[ \exp\{-\int_t^T r(u) + h(X(u))du\} \bigg| \mathcal{F}_t \right] 
\]

The requirements about arrival and loss rates are the following in order to stay in a two-factor Gaussian framework:

- The product \( h(X(t))(1 - l(X(t)) \) should be a power of \( X(t) \)
- The arrival rate \( h(X(t)) \) should be linear in \( \ln X(t) \).

Using the same method as before, we wish to have the following properties which will help us to parametrize these rates:
• if $X(t) = 1$, $h(1)(1 - l(1)) = \lambda \alpha$ and $h(1) = \lambda$

• if $X(t) \to 0$, $h(0) \to \infty$

• if $X(t) = \pi$, $l(\pi)$ should be reasonably close to zero, and the arrival rate $h(\pi) = 0$.

As before, we are not interested in values of $X(t)$ above $\pi$, whose probabilities are considered as negligible.

The corresponding functions have the following form:

\[
\begin{align*}
    h(X(t)) &= \lambda + C_3 \ln X(t) \\
    h(X(t))(1 - l(X(t))) &= \lambda \alpha X(t)^{-\beta}
\end{align*}
\]

(28)

where $C_3 = \lambda / \ln \frac{1}{\pi} < 0$.\(^{13}\) This leads to the following loss rate:

\[
l(X(t)) = 1 - \frac{\lambda \alpha X(t)^{-\beta}}{\lambda + C_3 \ln X(t)}
\]

(29)

Under these assumptions, we can express the bond option formula in the following Proposition:

**Proposition 5** The price at time $t$ of a corporate discount bond maturing at $T$, when credit risk is irreversible, where the arrival risk of default is given by (28), the loss rate on par value is given by (29) and the processes for $r(t)$ and $X(t)$ are given by (1) and (4), respectively, has the following form:

\[
P_c(r(t), X(t), t, T) = \int_t^T \exp \left[ \frac{1}{2} K_2^2(t,u) - N(r(t), X(t), t, u) \right] du + \exp \left[ \frac{1}{2} K_2^2(t,T) - N(r(t), X(t), t, T) \right]
\]

(30)

where

\(^{13}\)This product should be equal to 0 when $X(t) = \pi$, but as before, we are not interested in this extreme values, and thus the loss rate is technically adapted so as to respect this set of equations at $\pi$, i.e. it goes to infinity for this value.
• \( K_1(t, T) = K_2(t, T) \) with \( C_1 \) replaced by \( C_3 \) and \( \lambda \) replaced by \( \lambda \alpha \)
• \( N_i(r(t), X(t), t, T) = N_R(r(t), X(t), t, T) \) with \( C_1 \) replaced by \( C_3 \) and \( C_0 \) replaced by \( \lambda \)
• \( K_1^2(t, u) = K_2^2(t, u) \) with \( C_1 \) replaced by \( C_3 - \beta s \) and \( C_2 \) replaced by \( \left( \frac{\eta^0}{\alpha} + \frac{C_0}{\alpha} \right) \)
• \( N_i(r(t), X(t), t, u) = N_R(r(t), X(t), t, u) + \beta \ln X(t) - \ln \lambda - \ln \alpha + \beta s \left( \frac{\eta^2 + 2}{\alpha^2} \right) (u - t) \) with \( C_1 \) replaced by \( C_3 - \beta s \) and \( C_0 \) replaced by \( \lambda \)

and \( C_3 = \lambda / \ln \frac{1}{\epsilon} \).

Proof: see Appendix 5.

There is a deep difference between this formula and the one given in Proposition 1, because of the intervention of a time integral. Indeed, the meaning of this integral is, as outlined by Duffie et al. [21], the present value of a continuous flow of dividends with rate \( h(X(t))(1 - l(X(t))) \), representing the time \( t \) distribution of default payments. This formula separates the bond value in two components: the second term represents the value of the bond if default never occurs, considering that the claim is completely lost in case of default, and the first term corrects the severity of this assumption by adding the residual payment at default times.

Now, still in the same framework, considering that the payoff upon default is a fraction of the risk-free claim (delayed liquidation hypothesis), formula (27) is adapted to result in:

\[
P_c(r(t), X(t), t, T) = \mathbb{E}_Q \left[ \int_t^T h(X(u))(1 - l(X(u))) \exp \left\{ - \int_u^T r(v) dv \right\} \exp \left\{ - \int_t^u (r(v) + l(X(v)) dv \right\} du | \mathcal{F}_t \right] + \mathbb{E}_Q \left[ \exp \left\{ - \int_t^T (r(u) + h(X(u)) du \right\} | \mathcal{F}_t \right]
\]

(31)

Keeping the same \( h \) and \( l \), the bond pricing formula is now the following:
Proposition 6 The price at time \( t \) of a corporate discount bond maturing at \( T \), when credit risk is irreversible, where the arrival risk of default is given by (28), the loss rate on a default-free bond is given by (29) and the processes for \( r(t) \) and \( X(t) \) are given by (1) and (4), respectively, has the following form:

\[
P_c(r(t), X(t), t, T) = P(r(t), t, T) \int_t^T \exp \left[ \frac{1}{2} K_{1''}^2(t, u, T) - N''(X(t), t, u) \right] du + \exp \left[ \frac{1}{2} K_1^2(t, T) - N_1(r(t), X(t), t, T) \right]
\]

(32)

where \( P(r(t), t, T) \) is the price of an undefaultable zero-coupon bond, \( K_1''(t, T) \) and \( N_1(r(t), X(t), t, T) \) are the same as in equation (30), and

\[
K_{1''}(t, u, T) = \kappa_1' e^{-\alpha(t-T)} + \kappa_2' e^{-\alpha(T-u)} + \kappa_3' e^{-s(u-t)} + \kappa_4' e^{-2s(u-t)} + \kappa_5' e^{-(\alpha(T-t)-s(u-t))} + \kappa_6' (u-t) + \kappa_7' \n\]

\[
N''(X(t), t, u) = \beta \ln X(t) - \ln \lambda - \ln \alpha + C \frac{\gamma - \sigma_v^2/2}{s} (u-t)
\]

+ \left( \frac{C_3}{s} - \beta \right)(1 - e^{-s(u-t)})(\ln X(t) - \frac{\gamma - \sigma_v^2/2}{s})

where

\[
\kappa_1' = \frac{2C_3 \rho s \sigma_v}{a_r^2 s}
\]

\[
\kappa_2' = \left( \frac{-2C_3 \rho s \sigma_v + 2(C_3 - \beta s) \rho s \sigma_v}{a_r^2 s} \right)
\]

\[
\kappa_3' = \frac{2(C_3 - \beta s) C_4 \sigma_v}{s^2}
\]

\[
\kappa_4' = \frac{-2(C_3 - \beta s)^2 \sigma_v}{2s^3}
\]

\[
\kappa_5' = \frac{-2(C_3 - \beta s) \rho s \sigma_v}{a_r \sigma_v (a_r + s)}
\]

\[
\kappa_6' = \frac{C_3 \sigma_v^2}{s^2} + \frac{2C_3 \rho s \sigma_v}{a_r s}
\]

\[
\kappa_7' = \frac{(C_3 - \beta s)^2 \sigma_v^2}{2s^3} - \frac{2(C_3 - \beta s) C_4 \sigma_v}{s^2}
\]

\[
C_3 = \lambda / \ln \frac{1}{\pi}
\]
\[ C_4 = \left( \frac{\rho \sigma_r}{a_r} + \frac{C_3 \sigma_v}{s} \right) \]

\textit{Proof}: see Appendix 6.

What is striking in this result, is that we have managed to factorize out of the time integral the price of a risk-free bond, in spite of the correlation between the spot interest rate and \( X(t) \). The formula can thus really be interpreted as a sum of the risk-free bond, weighted by the flow of the expected discounted fraction of this bond at each point in time, and the undefaulted risky bond. Here again, the only computational difficulty comes form the one-dimensional time integral, which is in practice easy to solve numerically.

\section*{4.2 Comparison of Alternative Approaches for Magnitude Risk}

It is difficult to compare the three regimes directly through the observation of the pricing formulas, because the irreversible credit risk hypothesis does not produce fully analytical results. Our goal here is to analyze their characteristics through numerical simulations, with the requirement that the parameters used for this purpose be comparable across the regimes. Two types of comparisons, based on yields-to-maturity and sensitivity with respect to critical parameters, are performed, with a stress on the impact of the correlation coefficient for each of them: the Gaussian framework has the valuable advantage of allowing the sources of risk to be correlated without adding disturbing complexity.

\subsection*{4.2.1 Yield-to-maturity}

The first goal of these simulations is to detect whether, for a reasonable set of initial parameters, the two-factor framework allows various shapes of the yield curve for different initial conditions, i.e. values of the state variables. In Figures 1 to 8, simulation results are shown for the recurrent regime, with values of \( \rho \) ranging from -1 to 1. Time-to-maturity goes from 0.1 to 10 years. The choice of \( \pi \) corresponds to the concern of having asymptotic probabilities
of negative instantaneous spreads equal to 1% at each point in time. We notice that the initial instantaneous credit spread is given by $C_0 + C_1 \ln X(0)$.

The curves can obviously take various shapes simply by playing on the initial conditions, so that it can be thought that, in spite of the presence of only two factors, the model can fit many observed situations.

But this simulation is also interesting when the two other approaches for magnitude risk are put in the picture. In order to get the most comparable results, $\beta$ has been set to 0 in both regimes, so that the meanings of the arrival and loss rates are really similar. However, due to the characteristics of the liquidation approach, and contrarily to the recurrent regime, the initial conditions does not allow to compute the spot default spread simply by plugging in the values for $R(0)$, because of the intervention of the first term in the bond pricing formulas.

Figures 9 to 16, showing the same curves for the immediate liquidation regime, exhibit a strong tendency of the yields to decline with respect to the recurrent regime, and to have a more concave curvature. Yet, the yield can still be upwards-sloping in cases of low interest rates. In contrast, when one looks at Figures 17 to 24 for delayed liquidation, the behavior of the yield curves looks very similar to the one of the recurrent regime. The main difference with the recurrent risk regime seems to lie in the fact that, in the latter case, the yield is consistently higher.

From these figures, two conclusions can thus be drawn: the framework shows very flexible results, which can result in a good fit to reality, and the delayed liquidation regime is close to the recurrent one, but the latter induces higher yields, and thus higher spreads, than irreversible credit risk models. This can be easily understood by the facts that the payoff in case of default occurs at the same time in the recurrent and the delayed liquidation regimes, but in the latter case default makes this payoff riskless, and thus decreases the implied yield from the time of default on.

---

$^{14}$The sensitivity analysis shows that yields are not very sensitive to changes in $\pi$, so that the issue is not central in this framework.

$^{15}$The only parameter difference between the Figures is in $\pi$, which is greater in the liquidation regimes. If $\pi$ was raised for the recurrent regime, the difference would have been greater.
Yet, the next simulation results show that these conclusions are indeed myopic, in that the differences between the regimes should be appreciated with respect to all the components that create default risk. A sensitivity analysis can bring light on more subtle, but also important, differences between the approaches, and leads to a redefinition of what is meant by “credit risk”.

### 4.2.2 Sensitivity

With the same methodology of simulating numerical values from chosen sets of initial parameters, our goal is to present the values taken by the first order derivative of the yield-to-maturity with respect to some critical parameters. Since this is solely for comparison purposes, the scale of each derivative is not very important, and the alternative approach of looking at bond price elasticities does not bring any useful additional information in this context.

The analysis is performed in two stages: first, sensitivity of the 10 year yield with respect to interesting parameters is presented for $\rho = 0$ in order to determine the contribution of each possible source of default risk, in three cases: neutral initial conditions (interest rate and spread equal to their unconditional risk-adjusted mean), high interest rate and low spread, and low interest rate and high spread. Then, for each regime, the effect of different values of $\rho$ on these derivatives is studied for the most neutral set of initial conditions.

From Tables 1 to 6, four sources of default risk are put forward: the stochastic behavior of $r$, the one of $\ln X$, the correlation, and the intrinsic risk characteristics $\lambda$, $1 - \alpha$ and $\pi$, the latter being positively related to the spread through $C_1$.

The first table already clarifies the critical differences between the three models. The impact of the interest rate is not very different. However, the parameters $s$ and $\sigma_\omega$ of the process for the state variable, as well of the correlation coefficient, seem to have a greater impact in the liquidation hypothesis, as well as the loss rate of reference $1 - \alpha$ and the threshold of risklessness $\pi$. The main source of yield variability in the recurrent regime comes from the loss rate, compared with irreversible credit risk models. The differences for this parameter are huge, and it turns out that the weight of the loss rate vs.
arrival rate is highest for the immediate liquidation.

Contrarily to the previous analysis of yields, it seems that the two models of irreversible credit risk have similar characteristics with respect to comparative statics. The only marked difference concerns the relative importance of the loss rate, with a ratio of .35 to .13 in Table 1; it increases noticeably when initial credit risk is low (Table 2), because then the loss rate has only a very marginal impact in case of immediate liquidation. The reason for this is the following: favorable initial conditions induce that credit losses, if any, are likely to occur after a long period. But, for the immediate liquidation regime, the relative weight of dividends near maturity is lower than for the delayed liquidation regime, because the payoff upon default is constant through time in the first case and increasing in the second one. Thus, the arrival rate has much more importance for the delayed liquidation hypothesis.

The sign of the derivatives is also very instructive for some parameters. for highly risky initial conditions as in Table 3, increases in \( s \) and \( \pi \) have a negative impact on the yield, i.e., raises debt value, for all regimes. This is a counterintuitive result, since \( s \) and \( \pi \) are negatively related to the variance of the spread, which itself is positively related to the bond price. The reason for this stems from the fact that the expectation of the spread is lower, thus increasing bond prices, with the shift in these parameters, because \( \ln X(0) \) is sufficiently low. This expectation effect is stronger than the variance reduction effect, and pulls bond prices upwards.

Tables 4 to 6 show, for neutral initial conditions, the impact of changing \( \rho \) in each regime. As expected from the previous results on the impact of \( \rho \), the differences are stronger for the liquidation regimes.

In all frameworks, the role of \( r \) is more pronounced when \( \rho = -1 \), because then the interest rate and the spread have a maximal correlation: the correlation terms in the variance are thus affected by changes in \( \alpha \) and \( \sigma_r \) in the same direction as the terms of interest rate variance. The story is the same with \( s \) and \( \sigma_y \), because in the latter case the total impact, which is positive, is decomposed into a strong positive effect of the risk-adjusted expectation term in \( d \ln X(t) \) (absent in our risk-neutral specification of \( r(t) \)) and a weaker negative variance effect, which is greater as \( \rho \) decreases. The differences in \( \lambda \) and \( 1 - \alpha \) are negligible, and thus are weakly affected by the
correlation pattern. Yet, the importance of $\pi$ is decreasing with $\rho$, because of a variance effect similar to the one for the interest rate.

There are interesting sign reversions occurring when $\rho$ is allowed to be very high: in Table 4, the sensitivity to $s$ and $\pi$ becomes negative; in Tables 5 and 6, the derivatives with respect to $a$, become negative, and with respect to $\sigma$, is positive. The explanation is to be looked for in the components of the variance when $\rho = 1$: this is the value for which the covariance terms in the bond pricing formulas is the lowest. Due to the modelling characteristics of the liquidation approach, the weight of interest rate-related parameters in the covariance is higher than in the recurrent approach. This covariance effect is so strong that the volatility of the interest rate contributes positively to the yield. For the recurrent approach, the weight of the state variable-related parameters is greater, and is materialized through counterintuitive, although marginal, negative contributions of $\pi$ and $s$.

The similarities between Tables 5 and 6 are striking, outlining the close behavior of the yield under the two regimes (although levels and slopes with respect to maturities are very different, as shown in the previous subsection). In general, the yield is slightly more sensitive to credit risk parameters in the delayed liquidation framework, with the noticeable exception of the loss rate.

However, the distribution of effects of arrival and magnitude risks of reference is differentiated. They change by comparable values for both regimes (a scale of .041 to .047 for $\lambda$ and of .0003 to .0005 for $1 - \alpha$), but start at different levels: .30 and .029 for delayed liquidation versus .09 and .04 for immediate liquidation. This impacts on the relative weight of changing parameters for each regime, depending on the correlation. Arrival risk’s impact changes more with a higher $\rho$ in the immediate liquidation regime.

This set of simulations is instructive concerning the discrimination between the approaches. Analysis of yields-to-maturity as functions of time tends to make believe that the recurrent and delayed liquidation regimes are extremely similar. However, a closer look at the various sensitivities of the yields alters this judgment, and would tend to show that both irreversible credit risk models behave in harmony, except for the intrinsic arrival and magnitude risks.

This drives the following considerations concerning the components of credit
risk: recurrent and delayed liquidation regimes have similar term risk for equal parameter values, in that the impact of the passage of time is virtually the same. Thus, the term structure of risky rates look and behave basically the same way. However, the components of riskiness for each yield are essentially different. Recurrent credit risky bonds are much more affected by the arrival risk of default, simply because it intervenes during all the bond’s lifetime. In contrast, the process for the state variable, the correlation between \( V \) and \( r \), and the threshold of risklessness have a stronger impact on the irreversible credit risk models. The main difference among them comes from the relative weight between arrival and magnitude risk: for the delayed liquidation regime, the former risk is greater and the latter is lower, and indeed very similar to the magnitude risk of the recurrent regime.

4.3 A Global Model of Mixed Magnitude Risk of Default

We have been able to price explicitly corporate discount bonds under the alternative loss rates. If we compare the three possible characterizations for the magnitude risk, the following structure comes out:

- If the recovery rate is a fraction of the value of the claim (restructuring hypothesis):
  \[
P_{c,R}(t) = \exp\{\alpha_R(t,T) + \beta_R(t,T)r(t) + \delta_R(t,T)\ln X(t)\}
\]

- If the recovery rate is a fraction of par (immediate liquidation hypothesis):
  \[
  P_{c,IL}(t) = \int_t^T \exp\{\alpha_{1IL}(t,u) + \beta_{1IL}(t,u)r(t) + \delta_{1IL}(t,u)\ln X(t)\} du \\
  + \exp\{\alpha_{2IL}(t,T) + \beta_{2IL}(t,T)r(t) + \delta_{2IL}(t,T)\ln X(t)\}
  
  P_{DL}(t) = P(r(t),t,T) \int_t^T \exp\{\alpha_{1DL}(t,u,T) + \delta_{1DL}(t,u)\ln X(t)\} du \\
  + \exp\{\alpha_{2DL}(t,T) + \beta_{2DL}(t,T)r(t) + \delta_{2DL}(t,T)\ln X(t)\}
\]
Of course, from these three distinct models of magnitude risk, it is interesting to come up with a global model of mixed default risk. This means that the firm under consideration is subject to any of the three possibilities of default, each of them being (naturally) mutually exclusive. For each point in time, we would then have the probabilities $p_R(X(t), t, T)$, $p_{IL}(X(t), t, T)$, $p_{DL}(X(t), t, T)$, with $p_{DL} = 1 - p_R - p_I$, corresponding to each regime prevailing at time $t$, i.e. the probability that the firm belongs to one of type of recovery in case of default from $t$ to $T$.

The most general setup requires the use of a variation of Green's function. In its simplest form, this function is:

$$G(r, x, t, T) = \int_{-\infty}^{\infty} e^{-y} p(r, x, t, T; y, r', x', y') dy$$

where $y = \int_t^T R(s) ds$. This is a shorthand for the probability, conditional on current values of $r$ and $X$, that the interest rate at $T$ takes the value $r'$ and that the state variable $X$ takes the value $x'$, discounted back to $t$ at the risky rate.

Our goal is to express equations (5), (27) and (31) with the help of this kind of function. Therefore, we must define a variant of the following kind:

$$G_i(r, x, t, u, T) = \int_{-\infty}^{\infty} e^{-y_i} p(r, x, t, u, T; i, y_i, y', x') dy_i$$

where $i = R, IL, DL$, $y_i = \int_t^u R_i(s) ds$, $R_R(t) = r(t) + h_R(X(t)) l_R(X(t))$, $R_{IL}(t) = r(t) + h_{IL}(X(t))$, $R_{DL}(t) = r(t) + h_{DL}(X(t))$ and $y' = \int_t^T r(s) ds$. This function denotes the probability, conditional on current values of $r$ and $X$, that the state variable $X(u)$ takes the value $x'$, that the price of the riskless bond maturing at $T$ at time $u$ is $\exp{-y'}$, and that the regime of default by time $u$ is $i$.$^{16}$ With this generic function, the bond pricing formula is the sum of three terms, each corresponding to the prevailing regime:

$$P_c(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_R(r, x, t, T) dx' dy'$$

$^{16}$It is not necessary to introduce $r'$ in this function, since $y'$ and $x'$ are sufficient statistics for this variable.
This formula applies to the developments of the preceding sections, but also to the more general case of arbitrary functions for arrival and magnitude rates, provided that they only depend on \( X \). Moreover, the adaptation to the rates \( h_i \) and \( l_i \), dependent on any other state variable is also straightforward.

The issue of assessing \( p(r, x, t, u, T; i, y_i, y', x') \) is not trivial. The probability at time \( t \), conditional only on \( r(t) \) and \( X(t) \), that the company belongs to regime \( i \) at time \( u \) ("default-unconditional"), which depends on the future realization of the variable \( X \), is not similar to the probability that the company belongs to regime \( i \) conditional on \( r(t) \), \( X(t) \) and that default has not occurred by time \( u \) ("default-conditional"), which is much more intuitive, and can reasonably be taken as a prior. Even if the latter probability were not dependent on \( X \), the former would not be independent of \( X \), since it must take into account the fact that when default occurs, the probability that the firm belongs to regime \( i \) afterwards is either 0 or 1, depending on the actual regime at the time of default (because the event of default resolves uncertainty). Hence, it depends on the arrival rate of default, which is a function of \( X \).

Unfortunately, in the simple hypothesis of constant probabilities for each regime at time \( u \) conditional on no default from \( t \) to \( u \) throughout the period, the formula does not reduce to an appealing pricing formula. It looks much

\[
+ \int_t^T \int_{-\infty}^\infty \int_{-\infty}^\infty G_{IL}(r, x, t, u, u)h_{IL}(x')(1 - l_{IL}(x'))dx'dy'du \\
+ \int_{-\infty}^\infty \int_{-\infty}^\infty G_{IL}(r, x, t, T)dx'dy' \\
+ \int_t^T \int_{-\infty}^\infty \int_{-\infty}^\infty G_{DL}(r, x, t, u, T)h_{DL}(x')(1 - l_{DL}(x'))\exp\{-y'\}dx'dy' \\
+ \int_{-\infty}^\infty \int_{-\infty}^\infty G_{DL}(r, x, t, T)dx'dy' 
\]  

(33)

\[17\] For instance, consider the simple case where the default-conditional probabilities of being in regimes "R", "DL" and "IL" at time \( u \) if the firm has not defaulted yet are \( \frac{3}{4} \), 0 and \( \frac{1}{4} \), respectively. Imagine also that \( h_R \) is very low, and \( h_{IL} \) is very high. Then, the default-unconditional probability of being "R" at time \( u \) is much lower than the one conditional on no default, because default is likely to occur early, at a time where there is a high probability of falling into the regime of immediate liquidation.
simpler if the "default-unconditional" probabilities, i.e. the ones taking into account the fact that default might occur before time $u$, are constant.

According to Bayes's rule, this would be the case with constant conditional probabilities and equal arrival rates $h_i(X(u))$ for each regime $i$ under the risk-adjusted probability measure. The arrival rate must be a linear function of $\ln X(t)$ in the liquidation framework, and this may also be the case in the recurrent default risk framework provided that $l_R$ is constant or a deterministic function of $t$. In this case, formula (33) reduces to:

$$P_c(t) = p_{R}P_{c,R}(t) + p_{IL}P_{c,IL}(t) + p_{DL}P_{c,DL}(t)$$

which is simply the weighted average of the bond prices computed using each method.

Two conclusions can thus be drawn from this subsection: it is possible to mix the three possible regimes of recovery in case of default, but the formula is likely to become very complex, even if the probabilities of each regime conditional on no default are constant throughout the life of debt. A generalized model for magnitude risk is therefore not a straightforward extension of the individual models, but can be obtained with constant conditional probabilities and equal risk neutral arrival rates.

5 Extensions and Limitations

The extensions proposed in this section are developed under the basic model for magnitude risk, and are of course easily implementable under the alternative approaches of irreversible credit risk.

5.1 Extending to a Mean-Reverting Stochastic Process $K(t)$

It is worth considering that the process $K(t)$, which stands for an alternative measure of firm value, is not deterministic anymore. This, of course, can be justified only if $K(t)$ is taken as an opportunity cost of capital, in the line
of the overreaction hypothesis. Since both the value process and the interest rate process are stochastic, it is reasonable that this process fluctuates in accordance to the corresponding Wiener processes. The dependence on interest rate volatility seems natural given that $K(t)$ represents a discount factor. The volatility of the firm’s assets can also impact on the behavior of this discount factor.

In addition, it is also reasonable to consider that $K(t)$ experiences a mean-reverting behavior of the similar kind as $V(t)$, because the effect stemming from the difference between $V(t)$ and $K(t)$ might be explained by the overreaction hypothesis (mean-reversion of $V$) or by a more “structural” hypothesis: the mean-reversion of $K$ translates the revised market expectations due to a shift in the ratio $X(t)$. Instead of being temporary, the shocks in $V(t)$ are then considered as permanent, because $K(t)$ is forced to move in the same direction.

Thus, we rewrite (3) as:

$$
\frac{dK(t)}{K(t)} = (r(t) + k + s_k \ln X(t))dt + \sigma_{kv}dZ_v(t) + \sigma_{kr}dZ_r(t)
$$

This equation is close to the one proposed by Nielsen et al. [53], who do not postulate mean-reversion. By Itô calculus, we can directly find out the corresponding differential equation for $dX(t)/X(t)$:

$$
\frac{dX(t)}{X(t)} = (\mu - k - (s + s_k) \ln X(t) + (\sigma_{kv}^2 + \sigma_{kr}^2 + 2\rho\sigma_{kv}\sigma_{kr}))dt
+ (\sigma_v - \sigma_{kv})dZ_v(t) - \sigma_{kr}dZ_r(t)
$$

and so for $d\ln X(t)$:

$$
\begin{align*}
    d\ln X(t) &= (\mu - k - \frac{1}{2}(\sigma_v^2 - \sigma_{kv}^2 - \sigma_{kr}^2 - 2\rho\sigma_{kv}\sigma_{kr}) - (s + s_k) \ln X(t))dt \\
    &\quad + (\sigma_v - \sigma_{kv})dZ_v(t) - \sigma_{kr}dZ_r(t)
\end{align*}
$$

Of course, in this setup, we keep on considering a constant price of risk for the process $d\ln X(t)$, since we are exactly in the same framework as before. However, the meaning of $\lambda_x$ is slightly different, because equation (35) implies
the relationship \( \mu - k - (\sigma_v - \sigma_{kv})\lambda_x - \sigma_{kr}\lambda_r = \bar{\gamma} \), with \( \lambda_r = a_r(b_r - \bar{r})/\sigma_r \).

This adjusted price of risk will then be rewritten:

\[
\lambda_x = \frac{\mu - k - \bar{\gamma} - \frac{\sigma_{kr}}{\sigma_r} a_r(b_r - \bar{r})}{\sigma_v - \sigma_{kv}}
\]

This shows that the stochastic characterization of \( K(t) \) reduces the price of industry risk if \( \sigma_{kr} \) is sufficiently high with respect to \( \sigma_{kv} \), and increases it otherwise. The covariance of \( K(t) \) with the interest rate allows to explain part of the risk premium by interest rate risk, but the reduced loading of \( X(t) \) on \( Z_v(t) \), for a given level of \( \bar{\gamma} \), makes this source of risk more rewarded.

Risk neutral valuation can take place, but now we have that the differential equation for \( R(t) \) under the risk-adjusted probability measure looks like the following:

\[
dR(t) = dr(t) + C_1 d\ln X(t) + a_r(\bar{r} - r(t))dt + C_1(\bar{\gamma} - s \ln X(t))dt + C_1(\sigma_r - \sigma_{kr})dZ_{\sigma_r}(t) + (\sigma_r - C_1\sigma_{kr})dZ_{\sigma_{kr}}(t)
\]

(36)

where \( \bar{\gamma} = \frac{\gamma}{2} - \frac{1}{2}(\sigma_v^2 - \sigma_{kv}^2 - \sigma_{kr}^2 - 2\rho\sigma_{kv}\sigma_{kr}) \)

Which leads naturally to the following result:

**Proposition 7** The price at time \( t \) of a corporate discount bond maturing at \( T \) where the default spread is given by (8) and the processes for \( r(t) \) and \( X(t) \) are given by (1) and (34), respectively, is given by equations (14) to (17), where \( s \) is replaced by \( s + s_k \), \( \sigma_v \) is replaced by \( \sigma_v - \sigma_{kv} \), \( \sigma_r \) is replaced by \( \sigma_r - C_1\sigma_{kr} \) and \( \bar{\gamma} \) is replaced by \( \bar{\gamma} + \frac{1}{2}\sigma_v^2 \).

**Proof:** This follows simply from equation (36), to which the same valuation methodology as in Appendix 1 can be applied. \( \Box \)

A separation result is here less interesting to obtain. The reason is that the process \( X(t) \) already depends on the process of \( r(t) \), by means of the diffusion term \( \sigma_{kr} \). Therefore, even when one considers that the two sources of uncertainty are uncorrelated, a bond price written as a product of two independent risk-adjusted expectations has no true economic meaning.
5.2 Extending to a Two-Factor Model of Riskless Interest Rates

As we have extensively discussed before, the choice of the original Vasicek model is not out of order concerning one-factor models of interest rates. However, although the model is analytically very tractable, which allows to outline the contributions of the paper concerning default risk, there is some accuracy to gain with the use of a two-factor model. One way to do it would be to add an additive spread to the spot interest rate, following itself a Ornstein-Uhlenbeck process, and price securities in the same way as we have done before but with three additive components to the risky interest rate instead of two. This possibility is extremely easy to implement, and can be used in order to model the LIBOR rate, for example, which differs considerably from the riskless spot rate, empirically proxied to the rate of 3 months Treasury bills. To stay in line with the current framework, it is natural to use a mean-reverting Gaussian spread, but there are many available possibilities in the literature.

The alternative approach is to consider that the two factors are two forward rates, considered at the same moment, but for instantaneously lending (or borrowing) at different points in time. Then, the forward rate for any other point in time would be a linear combination of these two forward rates. This framework is more appealing, because it does not require the presence of an additive spread (which is usually hard to justify, except for default risk), but allows the term structure of interest rates to shift (as in the Vasicek model) and tilt, which yields a closer match to reality.

Following the pioneering work of Duffie and Kan [20], Ho, Stapleton and Subrahmanyan [30] have proposed a model for which the necessary and sufficient condition for the whole term structure to be driven by two distinct factors is that the short rate follows a two-dimensional ARMA process with mean-reversion and lagged memory parameter. We will work through a similar model, derived in continuous-time, that exhibits the same properties except that the short rate is normally instead of lognormally distributed.18

18Thus, we do not rule out negative interest rates, but the gain of tractability is, as we have said before, substantial.
The Ho, Stapleton and Subrahmanyam model (from now on denoted HSS) rests on the following discrete-time formulation for the risk-adjusted19 dynamics of the term structure:

\[ \ln r(t) = \mu(0,t) + [\ln r(t-1) - \mu(0,t-1)](1 - a_r) + \sum_{\tau=1}^{t-1} \nu_{t-\tau} \eta^{t-1} + \epsilon_t \]  (37)

\[ \mu(t-1,t) = \mu(0,t) + [\ln r(t-1) - \mu(0,t-1)](1 - a_r) + \nu_{t-1} \]  (38)

This set of equations means that the log of the spot rate follows a mean-reverting process, with conditional expectations themselves following an autoregressive moving average (ARMA) process. Equation (38) only results from the assumption that the log of the spot rate follows an ARMA process, with memory parameter \( \eta \), depicted in (37). The random variables \( \epsilon \) and \( \nu \) are i.i.d., normally distributed with variances \( \sigma_\epsilon \) and \( \sigma_\nu \), respectively.

The same methodology as in Section 2 is now applicable in order to price riskless as well as risky bonds. The formula for riskless zero-coupon bonds is only given here, but a formula for risky bonds is easily worked out along the lines of the previous sections. In order to limit its length, we present the formula for the price at time 0 of a bond maturing at \( T \). Starting from the same set of equations, but considering that the spot rate follows a normal distribution instead of lognormal boils down to rewriting \( r(t) \) instead of \( \ln r(t) \). Moreover, it is also natural to assess that the expectation of the interest rate at time 0 is the same for all maturities: \( \mu(0,t) = \mu(0,s) = \mu_0 \) \( \forall t,s \). Consider, to avoid any degeneracy, that \( \eta \neq e^{-\sigma} \). These assumptions lead to Proposition 8, which uses the following lemma:

**Lemma 1** The risk adjusted process for the Gaussian short rate starting at \( r(0) \) and following a two-dimensional ARMA process with constant unconditional expectation is described by the following stochastic differential equation:

\[ dr(t) = [a_r(\mu_0 - r(t)) + \sigma_\nu \int_0^t \eta^{t-u} d\tilde{Z}_\nu(u)] dt + \sigma_d \tilde{Z}_d(t) \]  (39)

---

19 The authors assume the existence of an equivalent martingale measure for the original process for the short rate, which is a premise for all no-arbitrage models of the term structure.
which corresponds to the unique Itô process:

\[
    r(t) = r(0)e^{-\alpha t} + \mu_0(1 - e^{-\alpha t}) + \int_0^t \sigma_v \frac{\eta^{-u} - e^{-\alpha (t-u)}}{a_r + \ln \eta} d\tilde{Z}_v(u) + \int_0^t \sigma e^{-\alpha (t-u)} d\tilde{Z}_v(u) \quad (40)
\]

where \(\tilde{Z}_v(t)\) and \(\tilde{Z}_v(t)\) are two uncorrelated standard Brownian motions.

**Proof.** The stochastic differential equation (39) is just the continuous-time counterpart of equation (37) with \(r\) replacing \(\ln r\) and \(\mu_0\) replacing \(\mu(0,t)\).

In order to discover the unique process underlying this differential equation, start from the process \(\zeta(t) = e^{\alpha t} r(t)\). The stochastic differential equation for this function is:

\[
    d\zeta(t) = e^{\alpha t} [a_r \mu_0 - \sigma_v \int_0^t \eta^{-u} d\tilde{Z}_v(u)] dt + e^{\alpha t} \sigma_v d\tilde{Z}_v(t)
\]

Integrating this process from 0 to \(t\), noticing that, following Fubini's Theorem:

\[
    \int_0^t e^{(\alpha t + \ln \eta) u} \int_0^u \eta^{-v} d\tilde{Z}_v(u) dv = \int_0^t \int_0^u e^{(\alpha t + \ln \eta) u} \eta^{-v} dv d\tilde{Z}_v(u) = \int_0^t \frac{\eta^{-v}}{a_r + \ln \eta} (\eta^t e^{\alpha t} - \eta^0 e^{\alpha u}) d\tilde{Z}_v(u)
\]

and rearranging, yields equation (40). \(\square\)

**Proposition 8** The price at time 0 of a riskless bond maturing at \(T\) where the process for the spot interest rate follows a two-dimensional ARMA process as depicted in Lemma 1 has the following form:

\[
    P_{HSS}(r(0), 0, T) = \exp \left\{ \frac{1}{2} K_{HSS}^2(0, T) - N_{HSS}(r(0), \mu_0, 0, T) \right\}
\]

where \(K_{HSS}^2\) and \(N_{HSS}\) are given by equations (83) and (84) in Appendix 6.

**Proof:** see Appendix 7.
The function is fully analytical, and permits the valuation of interest rate derivatives as well as the introduction of a default spread. For instance, considering the reversible form of default risk with no correlation between the source of risk affecting firm value ($Z_v$) and the one affecting the conditional mean of the interest rate ($Z_i$) leaves a formula identical to the one of Corollary 1. A more general analysis, with possible correlation between each Brownian motion, does not cause any technical difficulty, but involves substantially more computations. Since the principles are exactly the same, getting into more details is beyond the scope of this article.

Within the same framework, it is also possible to fall partly into the CIR specification for one factor, and thus get at least one desirable feature: the dependence of the spot interest rate variance over its own level. In order to do this, equation (39) is decomposed into:

$$
\begin{align*}
\Delta r(t) &= \Delta r_1(t) + \Delta r_2(t) \\
\Delta r_1(t) &= a_1[\mu_1 - r_1(t)]dt + \sigma_1 \sqrt{r_1(t)}dZ_i(t) \\
\Delta r_2(t) &= a_2[\mu_2 - r_2(t)]dt + \sigma_2 \int_0^t \eta(t-u)dZ_v(u)dt
\end{align*}
$$

which does not prevent from obtaining negative interest rates but still yields a tractable model if there is correlation between $X(t)$ and $Z_v(t)$. Using the results driven by Cox, Ingersoll and Ross [13], the riskless bond pricing formula can be written as:

$$
P_{HSS,CIR}(r(0),0,T) = A_{CIR}(0,T) \exp\{-B_{CIR}(0,T)r_1(0)\} P_{HSS}(r_2(0),0,T)
$$

where

$$
A_{CIR}(0,T) = \left[ \frac{2\psi(\psi+a_1)^{1/2}}{(\psi+a_1)(e^{\psi T}-1)+2\psi} \right]^{2a_1/\sigma_i^2}
$$

$$
B_{CIR}(0,T) = \frac{2(e^{\psi T} - 1)}{(\psi + a_1)(e^{\psi T} - 1) + 2\psi}
$$

$$
P_{HSS}(r_2(0),0,T) = P_{HSS} \text{ with } \sigma_i = 0
$$

$$
\psi = (a_1 + 2\sigma_i^2)^{1/2}
$$

The interpretation of such a two-factor model is very appealing in the context of a default risk model: although the first factor, consistently with the
CIR assumptions, should account for the real equilibrium interest rate, independent of any economy's conjunctural conditions, the second factor has a decaying memory and behaves as an ARMA process. This could represent a way to integrate the time-varying macro-economic conditions, that affect the credit spread of the whole economy but also each firm's default characteristics through the correlation between \( V \) and \( r_2 \). Therefore, \( r_2 \) can be assigned the role of a second default state variable, summarizing the determinants of defaults which are exogenous to firm value (business conditions). This "economy-wide" spread can be, in turn, supposed to be affected by any other state variable.

5.3 The Basic Limitation: Negative Values of Default Spreads

Clearly, no company's security can be less risky than the riskless one, whose instantaneous rate of return is \( r(t) \). Therefore, negative values for the default spread are not welcome at all. In the framework considered before, we had to trust the effect of the mean-reverting component of \( X(t) \) in order to assign negligible probabilities to negative values of the default spread.

If one still wants to keep the joint assumption of the Vasicek model for the spot interest rate and a normally distributed default spread depending on a process \( X(t) \), then it is necessary to posit that the default spread follows a truncated normal distribution. The equation for \( R(t) \) looks like the following:

\[
R(t) = r(t) + \left( \lambda(1 - \alpha) + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{\tau}} \ln X(t) \right) 1_{(X(t) \leq \tau)} \tag{42}
\]

The expectation of the discounted payoff under the risk-neutral measure performed in equation (5) is then the one of an exponential of a truncated normal variable. We believe that the gain of tractability because of not truncating the variable, conditional on well behaving parameters, makes the previous version more attractive. The reason is the following: the expected difference

\[20\]This is even a more serious issue than for the process \( r(t) \), because the modelling of real interest rates may, in extreme circumstances, show that inflation triggers empirically negative values. Empirically, the existence of negative real or nominal interest rates is not even heretic [23].
$\delta(t)$ between equations (6) and (42) can be modeled by the expression

$$
\delta(t) = \mathbb{E}_Q[R^*(t) - R(t)]
$$

$$
= \int_{-\infty}^{0} \int_{r}^{\infty} (r + C_1 \ln x)Q(X(t) = x)Q(r(t) = r)dxdr
$$

which leads to saying that, overall, the expected difference between the true yield and the one given by the model is equal to:

$$
\Delta(t, T) = \int_{t}^{T} \delta(s)ds
$$

Starting with reasonable values of $X(t)$ and considering a reversion rate high enough makes the issues for $r$ and $X$ exactly similar if those processes are independent. Now, if we introduce some correlation between the Wiener processes, the story may change. Particularly, a negative value for $\rho$ would mean that $r(t)$ and $X(t)$ are negatively correlated. High values for $X(t)$ would then be likely to go along with low values for $r(t)$. Therefore, the overall effect on the difference would be increased, because the effect negative spreads on the discount factor is high compared to the value of $r(t)$. In the opposite, a positive correlation would alter the weight of negative spread values. As Longstaff and Schwartz [45] find, it turns out empirically that default spreads are negatively related to the level of interest rates, which somewhat reduces the drawback of using Gaussian variables here.

6 Conclusion

The two-factor model of default risk that we have proposed here has a great advantage over the competing approaches of credit risk literature.

The "structural form" approach, that rests on economical grounds, uses the value of the firm as the sole determinant of default: in addition to being mainly tractable numerically and hard to test empirically, the only intervention of the process of firm value is somewhat disturbing. Our approach tries to overcome those difficulties, by a great analytical simplicity and the possibility to observe the state variable. Moreover, the integration of firm
value into our model of arrival risk only implies an impact of firm value on the likelihood of default, and leaves the field open to other sources of credit risk.\textsuperscript{21}

On the other side, the framework that we used, where the event of default follows an exponential distribution, is well adapted to the financial reality; the main drawback that is often met is the lack of underlying meaning of the models. Here, we tried not to give up on applicability of the model, but introducing explicitly what must be the most important determinant of its occurrence. Thus, this model brings a little of something that usually misses in this approach: a possible source of default.

In order to get nice-looking results, we have been relying on three important, and even maybe questionable, assumptions: the Gaussian distribution of the spot riskless rate, the gaussian distribution of the default spread, and the presence of a size-related effect. In spite of the questions that they can raise, they bring some very interesting outcomes concerning the separation of a loss rate and of an arrival rate of default: if one considers that the parameter choice leads to negligible probabilities of outliers — which are especially unpleasant in Gaussian models of interest rates —, we reach a very nice separation result, that allows to use firm value in order to assess the two kinds of risks.

The most striking harvest of analytical tractability concerns other corporate securities, especially equity and bond options. We have managed let alone to characterize the stochastic differential equation for the stock price process, but also to isolate a very simple structure of mean-reversion. The pricing of options on discount bonds is straightforward, while the formula for coupon bond options only involves a single numerical integration. In a related paper, we also introduce a new approach for the pricing of swaps with bilateral asymmetric default risk, within a similar framework.

Another usefulness of this approach clearly rests in the ability to develop, compare and mix the three competing, and so far mutually exclusive in the

\textsuperscript{21}In the first hitting time approach, the only coherent way to achieve this goal is to consider a discontinuous process for the barrier \( K \). However, as illustrated by the paper by Mason and Bhattacharya [49], interesting results have only been found for a very simple, discrete type of jump process. Any realistic assumption would be hindered by great technical difficulties.
literature, approaches of magnitude risk in a unified framework. Numerical simulations showed that this identification is not trivial, and leads to considering "default risk" as a set of distinct corporate characteristics.

The main question that comes out of this model is whether it is possible to design something comparable without having to assess specific processes for firm value and default spread. Of course, a two-factor term structure consistent CIR model would then be most appreciated, but it is then impossible to combine it with a lognormal firm value.
Appendix

Appendix 1: Proof of Proposition 1

The process for \( r(t) \) is given by the following stochastic differential equation:

\[
dr(t) = a_r[\bar{r} - r(t)]dt + \sigma_r dZ_r(t)
\]

and the process for \( \ln X(t) \) respects the following:

\[
d\ln X(t) = [\mu - k - \frac{1}{2} \sigma_v^2 - s \ln X(t)]dt + \sigma_v dZ_v(t)
\]

with a correlation \( \rho \) between the Brownian motions.

Under the risk-neutral measure \( Q \), using constant prices of risk \( \lambda_r \) and \( \lambda_v \), those processes behave like the following:

\[
dr(t) = a_r[\bar{r} - r(t)]dt + \sigma_r dZ_r(t) + \lambda_r t
\]

\[
d\ln X(t) = \frac{1}{s} [\frac{\gamma - \sigma_v^2/2}{s} - \ln X(t)]dt + \sigma_v dZ_v(t) + \lambda_v t
\]

(43)

(44)

and the correlation coefficient between \( dZ_r(t) \) and \( dZ_v(t) \) is still equal to \( \rho \).

Define now a two-dimensional Brownian motion \( \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)' \) which respects the following identity:

\[
\begin{pmatrix}
\tilde{Z}_1(t) \\
\tilde{Z}_2(t)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
\rho & \sqrt{1 - \rho^2}
\end{pmatrix}
\begin{pmatrix}
\tilde{Z}_1 \\
\tilde{Z}_2
\end{pmatrix}
\]

(45)

This two-dimensional process has been built so that the Brownian motions are orthogonal to each other. From now on, we will work with the following processes, that are identical to (43) and (44):

\[
dr(t) = a_r[\bar{r} - r(t)]dt + \rho \sigma_r d\tilde{Z}_1(t) + \sqrt{1 - \rho^2} \sigma_r d\tilde{Z}_2(t)
\]

\[
d\ln X(t) = \frac{1}{s} [\frac{\gamma - \sigma_v^2/2}{s} - \ln X(t)]dt + \sigma_v d\tilde{Z}_1(t)
\]

54
which behave like Ornstein-Uhlenbeck processes, corresponding to unique processes $r(t)$ and $\ln X(t)$ with initial values $r(0)$ and $\ln X(0)$, whose forms are:

$$
\begin{align*}
r(t) & = (1 - e^{-\alpha t})r(0) + e^{-\alpha t} \int_0^t e^{\alpha u} d\bar{Z}_1(u) + \sqrt{1 - \rho^2}e^{-\alpha t} \int_0^t e^{\alpha u} d\bar{Z}_2(u) \\
\ln X(t) & = (1 - e^{-\sigma^2 t}) \frac{\bar{y} - \sigma^2/2}{s} + e^{-\sigma t} \ln X(0) + \int_0^t e^{\sigma u} d\bar{Z}_1(t)
\end{align*}
$$

From equations (6), (8) and (5), the risk-neutral expectation that we have to take in order to price the corporate discount bond is:

$$
\mathbb{E}_t \left[ \exp \left\{ -\int_t^T r(u) + \lambda(1 - \alpha) + \frac{(1 - \alpha)}{\ln \pi} \ln X(u) du \right\} \bigg| \mathcal{F}_t \right]
$$

where the argument of the exponential is equal to:

$$
Y(t,T) = \int_t^T \left[ -(1 - e^{-\alpha(u-t)})r(t) + e^{-\alpha(u-t)}r(t) + \rho \sigma e^{-\alpha u} \int_t^u e^{\alpha v} d\bar{Z}_1(v) + \sqrt{1 - \rho^2} \sigma e^{-\alpha u} \int_t^u e^{\alpha v} d\bar{Z}_2(v) + \lambda(1 - \alpha) + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{r}} (1 - e^{-\sigma(u-t)}) \frac{\bar{y} - \sigma^2/2}{s} + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{r}} e^{-\sigma(u-t)} \ln X(t) + \frac{\lambda(1 - \alpha)}{\ln \frac{1}{r}} \sigma e^{-\sigma u} \int_t^u e^{\sigma v} d\bar{Z}_1(v) \right] du
$$

where $Y(t,T)$ can be interpreted as the yield to maturity of the corporate bond. The argument of the integral is normally distributed, and so is obviously $Y(t,T)$. To simplify notation, denote $C_1 = \frac{\lambda(1 - \alpha)}{\ln \frac{1}{r}}$. 

55
Starting from:

\[ Y(0, T) = \int_0^T \left[ (\bar{r} + C_1 \frac{\bar{v} - \sigma_t^2/2}{s}) + \lambda(1 - \alpha) + e^{-a_1 u}(r(0) - \bar{r}) \right. \]
\[ + C_1 e^{-s_1 u} (\ln X(0) - \frac{\bar{v} - \sigma_t^2/2}{s}) + \rho \sigma_r \int_0^u e^{-a_1 (u-v)} d\bar{Z}_1(v) \]
\[ + \sqrt{1 - \rho^2} \sigma_r \int_0^u e^{-a_1 (u-v)} d\bar{Z}_2(v) \]
\[ + C_1 \sigma_v \int_0^u e^{-s_1 (u-v)} d\bar{Z}_1(v) \] \] du

where we have that:

\[ \int_0^T \int_0^u e^{-c_1 (u-v)} d\bar{Z}_1(v) du = \int_0^T \left[ \int_v^T e^{-c_1 (u-v)} du \right] d\bar{Z}_1(v) \]
\[ = \int_0^T \frac{1}{c_1} (1 - e^{-c_1 (T-v)}) d\bar{Z}_1(v) \]

This gives, with some simple calculations:

\[ Y(0, T) = (\bar{r} + C_1 \frac{\bar{v} - \sigma_t^2/2}{s}) + \lambda(1 - \alpha))T + \frac{1}{a_r} (1 - e^{-a_1 T})(r(0) - \bar{r}) \]
\[ + C_1 \frac{1}{s} (1 - e^{-s_1 T}) (\ln X(0) - \frac{\bar{v} - \sigma_t^2/2}{s}) \]
\[ + \rho \sigma_r \int_0^T (1 - e^{-a_1 (T-v)}) d\bar{Z}_1(v) \]
\[ + \sqrt{1 - \rho^2} \sigma_r \int_0^T (1 - e^{-a_1 (T-v)}) d\bar{Z}_2(v) \]
\[ + C_1 \sigma_v \int_0^T (1 - e^{-s_1 (T-v)}) d\bar{Z}_1(v) \]
\[ = (\bar{r} + C_1 \frac{\bar{v} - \sigma_t^2/2}{s}) + \lambda(1 - \alpha))T \]
\[ - \frac{1}{a_r} (r(T) - r(0)) - \frac{C_1}{s} (\ln X(T) - \ln X(0)) \]
\[ + \left( \frac{\rho \sigma_r}{a_r} + \frac{C_1 \sigma_v}{s} \right) \bar{Z}_1(T) + \frac{\sqrt{1 - \rho^2} \sigma_r}{a_r} \bar{Z}_2(T) \]
and, since $Y(t, T) = Y(0, T) - Y(0, t)$, rewriting $C_2 = \frac{\rho \sigma_r}{\alpha_r} + \frac{C_1 \sigma_a}{s}$:

$$
Y(t, T) = (\bar{r} + C_1 \frac{\gamma - \sigma^2/2}{s} + \lambda(1 - \alpha))(T - t) \\
- \frac{1}{a_r}(r(T) - r(t)) - \frac{C_1}{s}(\ln X(T) - \ln X(t)) \\
+ C_2((\bar{Z}_1(T) - \bar{Z}_1(t)) + \frac{\sqrt{1 - \rho^2 \sigma_r}}{a_r}(\bar{Z}_2(T) - \bar{Z}_2(t)) (48)
$$

Thanks to equation (48), the computation of the expectation of the yield is straightforward:

$$
N_R(r(t), X(t), t, T) = \mathbb{E}_Q[Y(t, T) \mid \mathcal{F}_t] \\
= (\bar{r} + C_1 \frac{\gamma - \sigma^2/2}{s} + \lambda(1 - \alpha))(T - t) \\
+ \frac{1}{a_r}(1 - e^{-\alpha_r(T-t)})(r(t) - \bar{r}) \\
+ \frac{C_1}{s}(1 - e^{-\alpha_r(T-t)})(\ln X(t) - \frac{\gamma - \sigma^2/2}{s}) (49)
$$

In order to calculate the variance, notice that we have four random variables in this expression: $r(T)$, $\ln X(T)$, $\bar{Z}_1(T)$ and $\bar{Z}_2(T)$; we need the full variance-covariance matrix. Computations are simple:

$$
\text{var}_Q[r(T) \mid \mathcal{F}_t] = \frac{\sigma_t^2}{2a_r}(1 - e^{-2\alpha_r(T-t)}) = v_t^2 \quad (50) \\
\text{var}_Q[\ln X(T) \mid \mathcal{F}_t] = \frac{\sigma_X^2}{2s}(1 - e^{-2\lambda(T-t)}) = v_X^2 \quad (51) \\
\text{var}_Q[\bar{Z}_1(T) \mid \mathcal{F}_t] = \text{var}_Q[\bar{Z}_2(T) \mid \mathcal{F}_t] = T - t \quad (52) \\
\text{cov}_Q[r(T), \ln X(T) \mid \mathcal{F}_t] = \frac{\rho \sigma_r \sigma_a}{\alpha_r + \lambda s}(1 - e^{-(\alpha_r + \lambda s)(T-t)}) = \text{cov}_{rX} \quad (53) \\
\text{cov}_Q[r(T), \bar{Z}_1(T) \mid \mathcal{F}_t] = \frac{\rho \sigma_r \sigma_a}{a_r}(1 - e^{-\alpha_r(T-t)}) = \text{cov}_{rZ_1} \quad (54) \\
\text{cov}_Q[r(T), \bar{Z}_2(T) \mid \mathcal{F}_t] = \frac{\sqrt{1 - \rho^2 \sigma_r}}{a_r}(1 - e^{-\alpha_r(T-t)}) = \text{cov}_{rZ_2} \quad (55) \\
\text{cov}_Q[\ln X(T), \bar{Z}_1(T) \mid \mathcal{F}_t] = \frac{\sigma_X}{s}(1 - e^{-\lambda(T-t)}) = \text{cov}_{XZ_1} \quad (56)
$$

With these intermediary results, the variance of the bond yield can now be expressed:

$$
K_R^2(t, T) = \text{var}_Q[Y(t, T) \mid \mathcal{F}_t]
$$
\[
\begin{align*}
\frac{1}{a^2}v_t^2 + \frac{C_2^2}{s^2}v_X^2 + \left(C_2 + \frac{\sigma^2(1 - \rho^2)}{a^2}\right)(T - t) + \frac{2C_1}{ar}c_{cX}X
- 2\frac{C_2}{a}c_{cZ}Z - \frac{2\sigma^2(1 - \rho^2)}{a^2}c_{cZ}Z
- \frac{2C_1C_2}{s}c_{cXZ}
\end{align*}
\]

Now, since \( Y(t, T) \) is normally distributed, we note that:
\[
\begin{align*}
E_Q[\exp{-Y(t, T)} | \mathcal{F}_t] &= \exp{-E_Q[Y(t, T) | \mathcal{F}_t] + \frac{1}{2} \text{var}_Q[Y(t, T) | \mathcal{F}_t]} \\
&= \exp{-N_R(r(t), X(t), t, T) + \frac{1}{2}K_R^2(t, T)} \quad (58)
\end{align*}
\]

and plugging expressions (49) and (57) into (58) gives the risk-adjusted expectation sought, which is the value of the pure discount bond. This formula is equivalent to the one given in Proposition 1.  

\[ \square \]

**Appendix 2: Proof of Proposition 2**

Denote by \( S(t) \) the price of equity of the undefaulted firm at time \( t \). Consider that debt has face value \( F \) and matures at \( T \): the market value of corporate debt at time \( t \) is equal to \( P_e(r(t), X(t), t, T)F \) or, in short, \( P_e(t)F \). If \( V(t) \leq P_e(t)F \), it is clear from our assumptions that equity is not worth anything, and thus its rate of return is equal to 0. Otherwise, it is given by the following expression:
\[
\frac{dS(t)}{S(t)} = \frac{dV(t)}{V(t)} \left( \frac{V(t)}{V(t) - P_e(t)F} \right) - \frac{dP_e(t)}{P_e(t)} \left( \frac{P_e(t)F}{V(t) - P_e(t)F} \right)
\]

The process for \( \frac{dV(t)}{V(t)} \) is already known, and is equal to:
\[
\frac{dV(t)}{V(t)} = (r(t) + \mu - s \ln X(t))dt + \sigma_c dZ_c(t) \quad (59)
\]

and the only unknown process is thus the one for \( \frac{dP_e(t)}{P_e(t)} \). From Proposition 1, we have that the value of \( P_e(t) \) is given by:
\[
P_e(t) = \exp{-N_R(r(t), X(t), t, T) + \frac{1}{2}K_R^2(t, T)}
\]
and thus, noting that $\frac{\partial^2 P_c(t)}{\partial r^2} = P_c(t)\left(\frac{\partial N_R}{\partial r}\right)^2$ and $\frac{\partial^2 P_c(t)}{\partial \ln X^2} = P_c(t)\left(\frac{\partial N_R}{\partial \ln X}\right)^2$, using Itô's lemma, we have:

$$\frac{dP_c(t)}{P_c(t)} = \left(-\frac{\partial N_R}{\partial t} + \frac{1}{2} \frac{\partial K_R^2}{\partial t} \right)dt - \frac{\partial N_R}{\partial r}dr(t) - \frac{\partial N_R}{\partial \ln X}d\ln X(t)$$

$$+ \frac{1}{2} \left(\frac{\partial N_R}{\partial r}\right)^2 (dr(t))^2 + \frac{1}{2} \left(\frac{\partial N_R}{\partial \ln X}\right)^2 (d\ln X(t))^2$$

(60)

where:

$$\frac{\partial K_R^2}{\partial t} = \frac{2}{\alpha_r} (C_2 \rho \sigma_r + \frac{\sigma_r^2 (1 - \rho^2)}{\alpha_r}) e^{-\alpha_s (T-t)} + \frac{2C_1 C_2 \sigma_v}{s} e^{-s (T-t)}$$

$$- \frac{\sigma_r^2 e^{-2\alpha_s (T-t)}}{\alpha_r^2} - \frac{C_2^2 \sigma_v^2}{s^2} e^{-2s (T-t)}$$

$$- \frac{2C_1 \rho \sigma_r \sigma_v}{\alpha_r s} e^{-(\alpha_r + s) (T-t)} - \frac{C_2^2 - \sigma_r^2 (1 - \rho^2)}{\alpha_r^2}$$

$$\frac{\partial N_R}{\partial t} = \left(\tilde{r} + C_1 \tilde{\gamma} - \frac{\sigma_r^2}{s} / 2 + \lambda (1 - \alpha)\right)$$

$$- e^{-\alpha_r (T-t)} (r(t) - \tilde{r}) - C_1 e^{-s (T-t)} (\ln X(t) - \frac{\tilde{\gamma} - \sigma_r^2 / 2}{s})$$

$$\frac{\partial N_R}{\partial r} = \frac{1}{\alpha_r} (1 - e^{-\alpha_r (T-t)})$$

$$\frac{\partial N_R}{\partial \ln X} = \frac{C_1}{s} (1 - e^{-s (T-t)})$$

Under the original probability measure, the processes for $dr(t)$ and $d\ln X(t)$ are given by (1) and (7), respectively; furthermore, we note that $(dr(t))^2 = \sigma_r^2 dt$ and $(d\ln X(t))^2 = \sigma_v^2 dt$. Hence, we can rewrite equation (60) as:

$$\frac{dP_c(t)}{P_c(t)} = \beta_1 (r(t), X(t), t, T)dt$$

$$- \frac{\sigma_r}{s} (1 - e^{-\alpha_r (T-t)}) dZ_r(t) - \frac{C_1 \sigma_v}{s} (1 - e^{-s (T-t)}) dZ_v(t)$$

where:

$$\beta_1 = \frac{1}{\alpha_r} \left(C_2 \rho \sigma_r + \frac{\sigma_r^2 (1 - \rho^2)}{\alpha_r}\right) e^{-\alpha_r (T-t)} + \frac{C_1 C_2 \sigma_v}{s} e^{-s (T-t)}$$
\[
\begin{align*}
&-\frac{\sigma_r^2}{2a_r^2}e^{-2a_r(T-t)} - \frac{C_1^2\sigma_v^2}{2s^2}e^{-2s(T-t)} \\
&-\frac{C_1\rho\sigma_r\sigma_v}{a_r s}e^{-(a_r+s)(T-t)} - \frac{C_2^2}{2} - \frac{\sigma_v^2(1-\rho^2)}{2a_r^2} \\
&+ (\bar{r} + C_1\frac{\bar{q} - \sigma_v^2/2}{s} + \lambda(1-\alpha)) \\
&+ e^{-a_r(T-t)}(r(t) - \bar{r}) + C_1e^{-s(T-t)}(\ln X(t) - \frac{\bar{q} - \sigma_v^2/2}{s}) \\
&- \frac{1}{a_r}(1 - e^{-a_r(T-t)})(a_r(\tilde{b}_r - r(t))) \\
&- \frac{C_1^2}{s}(1 - e^{-s(T-t)})(\mu - k - \frac{1}{2}\sigma_v^2 - s\ln X(t)) \\
&+ \frac{\sigma_r^2}{2a_r^2}(1 - e^{-a_r(T-t)})^2 + \frac{C_1^2\sigma_v^2}{2s^2}(1 - e^{-s(T-t)})^2
\end{align*}
\]

It turns out from this expression that the rate of return of the corporate bond has a mean-reverting component, since it is negatively related to \(X(t)\) and thus to \(P_c(t)\). It is logical, since \(P_c(t)\) is increasing in \(\ln X(t)\), which is itself mean-reverting.

The process for \(S(t)\) can thus be determined by the following equation:

\[
dS(t) = \beta(r(t), X(t), t, T)dt \\
+ \frac{\sigma_r P_c(t)F}{a_r} (1 - e^{-a_r(T-t)})dZ_r(t) \\
+ \sigma_v \left( \frac{C_1 P_c(t)F}{s} (1 - e^{-s(T-t)}) + V(t) \right) dZ_v(t)
\]

where:

\[
\beta(r(t), X(t), t, T) = -\beta_1(r(t), X(t), t, T)P_c(t)F + (r(t) + \mu - s\ln X(t))V(t)
\]

□

Appendix 3: Proof of Proposition 3

The value \(\Gamma(r(t), X(t), t, T, T_2, \phi)\) of the european call option on the corporate zero-coupon bond is computed in the risk-neutral valuation framework.
Given the assumptions made in case of default before time $T$, we have to compute the following expectation:

$$
E_Q[\exp(- \int_t^T R(r(u), X(u))du) \max[\tilde{P}_c(r(T), X(T), T, T_2) - \phi, 0] | \mathcal{F}_t]
$$

where $\tilde{P}_c(r(T), X(T), T, T_2)$ is a lognormally distributed random variable:

$$
\ln \tilde{P}_c(r(T), X(T), T, T_2) = \frac{1}{2} K^2(T, T_2) - N_R(\hat{r}(T), \ln \tilde{X}(T), T, T_2)
$$

and $N$ has the following form:

$$
N_R(\hat{r}(T), \ln \tilde{X}(T), T, T_2) = (\tilde{r} + C_1 \tilde{\eta} - \sigma_v^2/2s) + \lambda(1 - \alpha)(T_2 - T)
$$

$$
+ C_1 (1 - e^{-s(T_2 - T)})(\ln \tilde{X}(T) - \tilde{\eta} - \sigma_v^2/2s)
$$

$$
+ \frac{1}{\alpha_r} (1 - e^{-a_r(T_2 - T)})(\hat{r}(T) - \tilde{r})
$$

and thus:

$$
\text{var}_Q[\ln \tilde{P}_c(T) | \mathcal{F}_t] = \text{var}_Q[N_R(\hat{r}(T), \ln \tilde{X}(T), T, T_2) | \mathcal{F}_t]
$$

$$
= \frac{2C_1}{a_r s} (1 - e^{-a_r(T_2 - T)})(1 - e^{-s(T_2 - T)}) \text{cov}_{r X}
$$

$$
+ \frac{1}{\alpha_r^2} (1 - e^{-a_r(T_2 - T)})^2 v_r^2 + \frac{C_1^2}{s^2} (1 - e^{-s(T_2 - T)})^2 \text{cov}_{X Y}
$$

$$
\equiv \Sigma_P
$$

where $v_r^2$, $v_X^2$, and $\text{cov}_{r X}$ are the same as in Appendix 1.

With this expression for the variance of the logarithm of the bond price, we can safely apply the Black and Scholes methodology, in a similar way to the calculations performed by Jamshidian [34] to discover the value of the call price, which is given by:

$$
\Gamma(r, X, t, T, T_2, \phi) = P_c(r(t), X(t), t, T_2)N(d_1) - \phi P_c(r(t), X(t), t, T)N(d_2)
$$

where

$$
d_1 = \frac{\ln \left[ \frac{P_c(r(t), X(t), t, T_2)}{P_c(r(t), X(t), t, T)} \right]}{\Sigma_P} + \frac{\Sigma_P}{2}
$$

$$
d_2 = d_1 - \Sigma_P
$$
which proves Proposition 3.

Furthermore, the forward rate at time $T$, denoted $f_c(r(t), X(t), t, T)$, is determined by (see Jamshidian [34]):

$$f_c(r(t), X(t), t, T) = \frac{\partial P_c(r(t), X(t), t, T)}{\partial T}$$

$$= E_Q[R(T) | F_t] - \text{cov}[R(T), Y(t, T)] | F_t]$$

$$= M(r(t), X(t), t, T) - J(t, T)$$

(61)

Since we know the processes for $R(T)$, which is given by $R(T) = r(T) + \lambda(1 - \alpha) + C_1 \ln X(T)$ where $r(T)$ respects equation (46) and $\ln X(T)$ is defined by (47), and for $Y(t, T)$, proposed in equation (48), we find that:

$$M(r(t), X(t), t, T) = E_Q[r(T) | F_t] + \lambda(1 - \alpha) + C_1 E_Q[\ln X(T) | F_t]$$

$$= \tilde{r}(1 - e^{-\alpha r(T-t)}) + e^{-\alpha r(T-t)r(t)} + \lambda(1 - \alpha)$$

$$+ C_1(1 - e^{-\alpha X(T-t)})+ \frac{\tilde{r} - \sigma^2/2}{s} + C_1 e^{-\alpha X(T-t)} \ln X(t)$$

$$J(t, T) = \frac{-1}{a} v^2_r - \frac{C_1^2}{s} v^2_X - C_1(\frac{1}{a} + \frac{1}{s}) \text{cov}_{rX}$$

$$+ C_2 \text{cov}_{rZ_1} + \frac{\sigma_r \sqrt{1 - \rho^2}}{a} \text{cov}_{rZ_2} + C_1 C_2 \text{cov}_{XZ_1}$$

where $v^2_r$, $v^2_X$, $\text{cov}_{rX}$, $\text{cov}_{rZ_1}$, $\text{cov}_{rZ_2}$ and $\text{cov}_{XZ_1}$ are all defined in Appendix 1. This corresponds to the formula given in Corollary 4. □

Appendix 4: Proof of Proposition 4

The price of the call option on the corporate coupon-bearing bond is equal to:

$$\Gamma(r(t), X(t), t, T, c_1, \ldots, c_n, r_1, \ldots, r_n, \phi)$$

$$= E_Q\left[\left(\sum_{i=1}^{n} e^{-\int_{c_i}^{r} R(u) du} - e^{-\int_{T}^{T} R(u) du} \phi\right) 1_{\{(r(T), X(T)) \in L\}} \right] | F_t]$$

(62)

There are $n + 1$ expectations to be computed in this equation. They can all be solved thanks to the following lemma.
Lemma 2 Let \( F(r(T), X(T)) \) denote the payoff function for a contingent claim on \((r(T), X(T))\) maturing at \(T\). Let \( P_c(r, X, t, \tau) \) denote the value, under a two-factor Ornstein-Uhlenbeck process for the instantaneous short rate submitted to recurrent credit risk of a pure discount bond maturing at \(\tau\). Then the value of the claim can be written as

\[
U(r, X, t, T, \tau) = P_c(r, X, t, \tau) E_Q[F(r(T), X(T))|\mathcal{F}_t]
\]

where the expectation is taken with respect to \(r(T)\) and \(X(T)\) which are distributed as:

\[
\begin{pmatrix}
  r(T) \\
  \ln X(T)
\end{pmatrix} \sim N
\begin{pmatrix}
  m_r(r, t, T, \tau) \\
  m_X(r, t, T, \tau)
\end{pmatrix},
\begin{pmatrix}
  v_r^2 & \text{cov}_{rX} \\
  \text{cov}_{rX} & v_X^2
\end{pmatrix}
\]

where \(N\) denotes a bivariate normally distributed random variable with its corresponding vector of expectations and variance-covariance matrix, with

\[
m_r(r, t, T, \tau) = r(t)e^{-ar(T-t)} + (\bar{r} - \frac{\sigma_r^2}{a_r^2} - \frac{C_1\rho\sigma_r\sigma_v}{a_r s})(1 - e^{-ar(T-t)})
\]

\[
+ \frac{\sigma_r^2}{2a_r^2}(e^{-ar(\tau-T)} - e^{-ar(\tau+T-2t)})
\]

\[
+ \frac{C_1\rho\sigma_r\sigma_v}{s(a_r + s)}(e^{-s(\tau-T)} - e^{-s(\tau-t)-ar(T-t)})
\]

\[
m_X(X, t, T, \tau) = \ln X(t)e^{-s(T-t)} + (\tilde{\gamma} - \frac{\sigma_v^2}{s} - \frac{C_1\rho\sigma_r\sigma_v}{a_r s})(1 - e^{-s(T-t)})
\]

\[
+ \frac{\sigma_v^2}{2a_v^2}(e^{-ar(\tau-T)} - e^{-ar(\tau+T-2t)})
\]

\[
+ \frac{C_1\rho\sigma_r\sigma_v}{s(a_r + s)}(e^{-s(\tau-T)} - e^{-s(\tau-t)-ar(T-t)})
\]

and \(v_r^2, v_X^2\) and \(\text{cov}_{rX}\) are given in Appendix 1.

Proof of Lemma 2: In order to price any derivative security \(U(r, X, t, T, \tau)\) whose payoff is \(F(r(T), X(T))\), we notice first that this security must satisfy the usual partial differential equation (PDE):

\[
-U_T - U_r + a_r(\bar{r} - r(t))U_r + (\tilde{\gamma} - \frac{1}{2}\sigma_v^2 - s \ln X(t))U_X
\]

\[
+ \frac{1}{2}(\sigma_r^2U_{rr} + 2\rho\sigma_r\sigma_vU_{rx} + \sigma_v^2U_{xx}) - r(t)U = 0
\]

(66)
where the subscripts $T, \tau, r$ and $X$ represent partial derivatives with respect to time-to-maturity, $r(t)$ and $\ln X(t)$, respectively. Rewriting $U(r, X, T, \tau) = P_c(r, X, t, \tau)G(r, X, T)$ taking into account the fact that $G_T = P_T = 0$, this PDE is decomposed in:

\[
G[-P_r + a_r(\bar{r} - r(t))P_r + (\bar{\gamma} - \frac{1}{2}\sigma_v^2 - s \ln X(t))P_x
+ \frac{1}{2}(\sigma_r^2 P_{rr} + 2\rho \sigma_r \sigma_v P_{rx} + \sigma_v^2 P_{xx}) - r(t)P]
+ P[-G_t + a_r(\bar{r} - r(t)) + \frac{P_r}{P} \sigma_r^2 + \frac{P_x}{P} \rho \sigma_r \sigma_v]G_r
+(\bar{\gamma} - \frac{1}{2}\sigma_v^2 - s \ln X(t)) + \frac{P_x}{P} \sigma_v^2 + \frac{P_r}{P} \rho \sigma_r \sigma_v]G_X
+ \frac{1}{2}[\sigma_r^2 G_{rr} + \rho \sigma_r \sigma_v G_{rx} + \sigma_v^2 G_{xx}] = 0
\]

The term multiplied by $G$ satisfies the PDE for $P$, and is thus equal to 0. Since $P_r/P = -B_1(t - t)$ and $P_x/P = -B_2(t - t)$ as given in equations (12) and (13) respectively, we have the following condition for $G$:

\[
-G_t + a_r(\bar{r} - r(t)) - B_1(t_i - t)\sigma_r^2 - B_2(t_i - t)\rho \sigma_r \sigma_v]G_r
+(\bar{\gamma} - \frac{1}{2}\sigma_v^2 - s \ln X(t)) + \frac{P_x}{P} \sigma_v^2 + \frac{P_r}{P} \rho \sigma_r \sigma_v]G_X
+ \frac{1}{2}[\sigma_r^2 G_{rr} + \rho \sigma_r \sigma_v G_{rx} + \sigma_v^2 G_{xx}] = 0
\]

subject to unchanged maturity conditions. From Friedman's [27] Theorem 5.2, we thus get the standard result:

\[
U(r, X, t, T, \tau) = P_c(r, X, t, \tau)E_{F_t}[F(r(T), X(T))|F_t]
\]

where the expectation is taken with respect to the processes satisfying the following stochastic differential equations:

\[
\begin{align*}
    dr(t) & = a_r[\bar{r} - \frac{\sigma_r^2(1 - e^{-a_r(t-t_i)})}{a_r^2} - \frac{C_1 \rho \sigma_r \sigma_v(1 - e^{-s(t-t_i)})}{a_r s} - r(t)]dt
    + \sqrt{1 - \rho^2 \sigma_r^2} d\tilde{Z}_r(t) + \rho \sigma_r d\tilde{Z}_v(t) \\
    d\ln X(t) & = [\bar{\gamma} - \frac{1}{2}\sigma_v^2 + \frac{C_1 \sigma_v^2(1 - e^{-s(t-t_i)})}{s} + \frac{\rho \sigma_r \sigma_v(1 - e^{-a_r(t-t_i)})}{a_r s} - s \ln X(t)]dt + \sigma_v d\tilde{Z}_v(t)
\end{align*}
\]

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which correspond to a unique value for \( r(T) \) and for \( \ln X(T) \):

\[
\begin{align*}
    r(T) &= r(t)e^{-a_r(T-t)} + \left( \frac{\sigma_r^2}{2} - \frac{C_1 \rho \sigma_r \sigma_u}{a_r s} \right) (1 - e^{-a_r(T-t)}) \\
        &\quad + \frac{\sigma_r^2}{2a_r^2} (e^{-a_r(\tau-T)} - e^{-a_r(\tau+T-t)}) \\
        &\quad + \frac{C_1 \rho \sigma_r \sigma_u}{s(a_r + s)} (e^{-s(\tau-T)} - e^{-s(\tau-t)-a_r(T-t)}) \\
        &\quad + \sqrt{1 - \rho^2 \sigma_r} \int_t^T e^{-a_r(T-u)} d\tilde{Z}_r(u) + \rho \sigma_r \int_t^T e^{-a_r(T-u)} d\tilde{Z}_u(u) \\
    \ln X(T) &= \ln X(t)e^{-s(T-t)} + \left( \frac{\sigma^2 - \gamma^2/2}{s} - \frac{C_1 \sigma_u^2}{s^2} \right) (1 - e^{-s(T-t)}) \\
        &\quad + \frac{\sigma_u^2}{2a_u^2} (e^{-a_u(\tau-T)} - e^{-a_u(\tau+T-t)}) \\
        &\quad + \frac{C_1 \rho \sigma_r \sigma_u}{s(a_r + s)} (e^{-s(\tau-T)} - e^{-s(\tau-t)-a_r(T-t)}) \\
        &\quad + \sigma_u \int_t^T e^{-s(T-u)} d\tilde{Z}_u(u)
\end{align*}
\]

These are normally distributed random variables, whose expectations correspond to expressions (64) and (65), and whose variance-covariance matrix is unchanged with respect to the original probability measure. \( \square \)

Using Lemma 2 with (62), the value of the claim can be written as:

\[
\begin{align*}
    \Gamma = \sum_{i=1}^n P_c(\tau, X, t, \tau_i) c_i E_Q^0 \left[ 1_{\{r(T), X(T) \in L_i \}} | \mathcal{F}_t \right] \\
    - P_c(\tau, X, t, T) \phi E_Q^0 \left[ 1_{\{r(T), X(T) \in L_0 \}} | \mathcal{F}_t \right]
\end{align*}
\]

where the processes for \( r \) and \( \ln X \) for each expectation is given by the SDEs (67) and (68) where \( \tau \) is indexed by \( i \) and \( \tau_0 \equiv T \), and the exercise regions \( L_i \) enclose the pairs for which the option is exercised under the same processes, i.e. \( L_i \) is the set of pairs \( (r(T), \ln X(T)) \) solving:

\[
P_c(r(T), \ln X(T), T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n) \geq \phi
\]

under the SDE corresponding to \( Q^i \). Similarly, \( L_i^* \) defines the set of pairs such that the former expression is satisfied as an equality.
In order to solve each of these expectations, we change variables so express \( R(T) \) as the sum of independent processes:

\[
\begin{align*}
\dot{r}'(T) &= r(T) - \rho \sigma_T \int_t^T e^{-\alpha_r(T-u)} d\tilde{Z}_v(u) \\
\ln X'(T) &= \ln X(T) + \frac{\rho \sigma_T}{C_1} \int_t^T e^{-\alpha_r(T-u)} d\tilde{Z}_v(u)
\end{align*}
\]

which respects \( R(T) = r(T) + C_0 + C_1 \ln X(T) = r'(T) + C_0 + C_1 \ln X'(T) \) and redefines an associated exercise region \( L'_t \) and an associated boundary \( L''_t \). The solution to the implicit equation for \( L''_t \) is a monotonically decreasing function \( \dot{r}'(T) = \Lambda_t'(\ln X'(T)) \), defined over the whole real space \( \mathbb{R}^2 \) because the support of both random variables is the entire real line and bond prices are monotonic functions of each of them. This allows to express each term of (69) as a single integral:

\[
E_Q^i[(r(T), X(T)) \in L_t^i | \mathcal{F}_i] = \int_{-\infty}^{\infty} N(d_i'(\ln x')) N'(d_i'(\ln x')) d\ln x' 
\]

where

\[
\begin{align*}
d_i'(\ln x') &= \frac{C_1(\ln x' - m_X(X, t, T, \tau_i))}{\sqrt{C_1^2 v_X^2 + 2C_1 \text{cov}_{x, X} + \rho^2 v^2}} \\
d_i''(\ln x') &= \frac{\Lambda_t'(\ln x') - m_{r', T, \tau_i}}{\sqrt{(1 - \rho^2)v^2}}
\end{align*}
\]

where \( N \) and \( N' \) denote, respectively, the cdf and the pdf of the univariate standard normal distribution.

The algorithm for computing \( N^*(d_i, d''_i) \) is quite simple: since the distribution of \( \ln X'(T) \) and \( r'(T) \) is known, for a given value of \( \ln X'(T) \), one has to find the value of \( r'(T) \) such that \( P_r(r'(T), \ln X'(T), T, c_1, \ldots, c_n, \tau_1, \ldots, \tau_n) = \phi \) under the corresponding distributions for \( \ln X'(s) \) and \( r'(s) \), \( s \geq T \). The identification of the integrand is completed by taking the above cdf and pdf.

Plugging (70) into (69) gives formula (26) in Proposition 4. □

Appendix 5: Proof of Proposition 6
The price of the corporate bond under irreversible default risk is equal to the sum of two risk-adjusted expectations:

\[
P_c(r(t), X(t), t, T) = \mathbb{E}_Q \left[ \int_t^T h(X(u))(1 - l(X(u))) \exp(\int_t^u (r(v) + h(X(v)))dv)du \mid \mathcal{F}_t \right] + \mathbb{E}_Q \left[ \exp(-\int_t^T (r(u) + h(X(u)))du) \mid \mathcal{F}_t \right]
\]

Using the same method as in Appendix 1, we can easily rewrite, under the alternative measure:

\[
Y_1(t, T) = \int_t^T e^{r(u) + h(X(u))}du
\]

\[
= (\bar{r} + C_3 \frac{\gamma - \sigma_d^2/2}{\sigma_r} + \lambda)(T - t) - \frac{1}{a_r}(r(T) - r(t)) - \frac{C_3}{s}(\ln X(T) - \ln X(t)) + C_4((\bar{Z}_1(T) - \bar{Z}_1(t)) + \frac{\sqrt{1 - \rho^2 \sigma_r}}{a_r} (\bar{Z}_2(T) - \bar{Z}_2(t))
\]

where \( C_4 = (\frac{\sigma_r}{a_r} + \frac{C_3}{\sigma_r}) \). It is immediately clear that \( Y_1(t, T) = Y(t, T) \) as defined in Appendix 1 with \( C_2 \) replaced by \( C_4 \), \( C_0 \) replaced by \( \lambda \) and \( C_1 \) replaced by \( C_3 \).

This gives for the second term of (73):

\[
\mathbb{E}_Q \left[ \exp\left( -\int_t^T (r(u) + h(X(u)))du \right) \mid \mathcal{F}_t \right] = \exp\{-\mathbb{E}_Q[Y_1(t, T) \mid \mathcal{F}_t] + \frac{1}{2} \text{var}_Q[Y_1(t, T) \mid \mathcal{F}_t]\}
\]

where \( N_1 \) and \( K_1^2 \) are computed as in Appendix 1.

To solve the first term of the RHS of (73), notice that it can be written:

\[
\mathbb{E}_Q \left[ \int_t^T h(X(u))(1 - l(X(u)))\exp\{-\int_t^u (r(v) + h(X(v)))dv\}du \mid \mathcal{F}_t \right]
\]
where \( p(r, x, r', x', t, u) \) denotes the (multivariate normal) joint probability density of \( \tilde{r}(u), \tilde{X}(u) \) conditional on \( r(t) = r, X(t) = x \). Because of lognormality of the exponential in (74), we can further characterize this multiple integral by:

\[
\int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(r, x, r', x', t, u) \exp\{\ln \lambda + \ln \alpha - \beta \ln X(u) - Y_f(t, u)\} dr' dx' du
\]

where \( \frac{dr}{dx} \) denotes the conditional expectation of the variable at time \( u \) given information at time \( t \).

Furthermore, \( N_f'(r(t), X(t), t, u) = N_R(r(t), X(t), t, u) - \ln \lambda - \ln \alpha + \beta \ln X(t) + \beta s(\frac{u-t}{2}) \), with \( C_1 \) replaced by \( C_3 - \beta s \) and \( \lambda \) replaced by \( \lambda/\alpha \), and \( K_f'^2(t, u) = K_R^2(t, u) \), with \( C_1 \) replaced by \( C_3 - \beta s \) and \( C_2 \) replaced by \( (\frac{\sigma^2_r}{2} + \frac{\sigma^2_s}{2}) \). This completes the proof. \( \square \)

Appendix 6: Proof of Proposition 7

It is useful to rewrite (31) as the following:

\[
P_c(r(t), X(t), t, T) = E_Q\left[ \int_t^T \exp\{r(u)dv\} \exp\{\ln \lambda + \ln \alpha \} \right]
\]
\[- \int_t^u h(X(v))dv}du | \mathcal{F}_t] \\
+ \mathbb{E}_Q[\exp\{- \int_t^T (r(u) + h(X(u))du) | \mathcal{F}_t] \\
\]

The second term of the RHS is the same as in Appendix 4. Only the first term has to be computed. By the same way as in Appendix 4, we can rewrite it as:

\[
\mathbb{E}_Q[\int_t^T \exp\{- \int_t^T r(v)dv\} \exp\{\ln \lambda + \ln \alpha \\\n- \int_t^u h(X(v))dv}du | \mathcal{F}_t] \\
= \int_t^T \mathbb{E}_Q[u \exp\{\int_t^T r(v)dv + \ln \lambda + \ln \alpha - \beta \ln X(u) \\\n- \int_t^u \lambda + C_3 \ln X(v)dv\}]du \\
= \int_t^T \exp\{\mathbb{E}_Q[\int_t^T r(v)dv + \ln \lambda + \ln \alpha - \beta \ln X(u) \\\n- \int_t^u \lambda + C_3 \ln X(v)dv | \mathcal{F}_t] \\
+ \frac{1}{2} \text{var}_Q[\int_t^T r(v)dv + \ln \lambda + \ln \alpha - \beta \ln X(u) \\\n- \int_t^u (\lambda + C_3 \ln X(v))dv | \mathcal{F}_t]}du \\
\]

(75)

It is useful to remember the following expressions:

\[
\int_t^T r(v)dv = \bar{r}(T - t) - \frac{1}{\alpha_r}(r(T) - r(t)) \\
+ \frac{\rho \sigma_r}{\alpha_r}(\bar{z}_1(T) - \bar{z}_1(t)) + \frac{1 - \rho^2 \sigma_r}{\alpha_r}(\bar{z}_2(T) - \bar{z}_2(t)) \\
\]

\[
\int_t^u (\lambda + C_3 \ln X(v))dv = (\lambda + C_3 \bar{\gamma} - \frac{\sigma_v^2}{s})(u - t) \\
\]
It is straightforward to see that:

\[
E_Q\left[ \int_t^T -r(v)dv + \ln \lambda + \ln \alpha - \beta \ln X(u) \right. \\
- \int_t^u (\lambda + C_3 \ln X(v))dv \mid \mathcal{F}_t \right] \\
= E_Q\left[ \int_t^T -r(v)dv \mid \mathcal{F}_t \right] - N''(X(t), t, u) 
\]  

(76)

where

\[
N''(X(t), t, u) = \beta \ln X(t) - \ln \lambda - \ln \alpha + C_3 \frac{\gamma - \frac{\sigma^2}{2}}{s}(u - t) \\
+ \left( \frac{C_3}{s} - \beta \right)(1 - e^{-s(u-t)})(\ln X(t) - \frac{\gamma - \frac{\sigma^2}{2}}{s}) 
\]  

(77)

In order to compute the variance, it is important to know that:

\[
\text{cov}_Q[r(T), \ln X(u) \mid \mathcal{F}_t] = \frac{\rho \sigma_a \sigma_a}{a_r + s} \left( e^{-a_r(T-u)} - e^{-(a_r(T-t)-s(u-t))} \right) \\
\equiv \text{cov}_{r(T)X(u)} 
\]  

(78)

\[
\text{cov}_Q[r(T), \tilde{Z}_1(u) \mid \mathcal{F}_t] = \frac{\rho \sigma_a}{a_r} \left( e^{-a_r(T-u)} - e^{-a_r(T-t)} \right) \\
\equiv \text{cov}_{r(T)\tilde{Z}_1(u)} 
\]  

(79)

Then, we obtain:

\[
\text{var}_Q\left[ \int_t^T -r(v)dv + \ln \lambda + \ln \alpha - \beta \ln X(u) - \int_t^u (\lambda + C_3 \ln X(v))dv \mid \mathcal{F}_t \right] du \\
= \frac{1}{a_r^2} v_{r(T)}^2 + \frac{\sigma_r^2}{a_r^2} (T - t) - 2 \frac{\rho \sigma_r}{a_r^2} \text{cov}_{r(T)\tilde{Z}_1(T)} - 2 \frac{\sqrt{1 - \rho^2} \sigma_r}{a_r^2} \text{cov}_{r(T)\tilde{Z}_2(T)} \\
+ \left( \frac{C_3}{s} - \beta \right)^2 v_X^2(u) + \frac{C_3^2 \sigma_v^2}{s^2} + \frac{2C_3 \rho \sigma_a \sigma_v}{a_r s}(u - t) 
\]

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where $C_4 = \left( \frac{\sigma_v}{\alpha} + \frac{\sigma_v}{\alpha} \right)$, and the $v$ and $\text{cov}$ factors are as defined in equations (50) to (56) for corresponding maturities, and in (78) and (79) for different maturities. The term $k^2(t, T)$ is the variance of the yield on a risk-free bond, defined in Jamshidian [34].

Using (80) and (76) in (75), we get the formula given in Proposition 5. □

Appendix 7: Proof of Proposition 8

The price of the riskless zero-coupon bond at time 0 can be expressed by the following risk-neutral expectation:

$$P_{HSS}(\tau(0), 0, T) = \mathbb{E}_0[\exp\{-\int_0^T \tau(u)du\}|\mathcal{F}_0]$$  

(81)

where $\tau(u)$ is given by equation (40). Thus, the integral is:

$$Y(0, T) = \int_0^T -[\mu_0 + e^{-\alpha u}(\tau(0) - \mu_0) + \sigma_v \int_0^u e^{-\alpha (u-v)}d\tilde{Z}_v(v) \quad \text{or}\quad \frac{\sigma_v}{\alpha} \int_0^u (\eta^u - e^{-\alpha u})d\tilde{Z}_v(v)]du$$

Rewriting $\eta^u = e^{\ln \eta}$ and applying Fubini’s Theorem allow to rewrite:

$$\int_0^T \int_0^u (\eta^u - e^{-\alpha (u-v)})d\tilde{Z}_v(v)du$$

$$= \int_0^T \left[ \frac{1}{\ln \eta} (e^{\ln \eta(T-v)} - 1) + \frac{1}{-\alpha} (1 - e^{-\alpha (T-v)}) \right]d\tilde{Z}_v(v)$$

$$\int_0^T \int_0^u e^{-\alpha (u-v)}d\tilde{Z}_v(v)du = \int_0^T \frac{1}{\alpha} (1 - e^{-\alpha (T-v)})d\tilde{Z}_v(v)$$

$$\int_0^T$$
which yields:

\[
Y(0, T) = \mu_0 T - \frac{1}{a_r} (e^{-a_r T} - 1) (r(0) - \mu_0) + \frac{\sigma_r}{a_r} \int_0^T (1 - e^{-a_r (T-v)}) d\tilde{Z}_r(v) \\
+ \frac{\sigma_v}{a_r + \ln \eta} \int_0^T \left[ \frac{1}{\ln \eta} (e^{\ln \eta (T-v)} - 1) - \frac{1}{a_r} (1 - e^{-a_r (T-v)}) \right] d\tilde{Z}_v(v)
\]

Now, since \( Y(0, T) \) is normally distributed, equation (81) is similar to:

\[
P_{HSS}(r(0), 0, T) = \exp\{\frac{1}{2} \text{Var}_Q[Y(0, T) | \mathcal{F}_0] - E_Q[Y(0, T) | \mathcal{F}_0] \}
\]  

(82)

The first moment of the distribution of \( Y(0, T) \) is easily found:

\[
E_Q[Y(0, T) | \mathcal{F}_0] = N_{HSS}(r(0), \mu_0, 0, T) \\
= \mu_0 T + \frac{1}{a_r} (1 - e^{-a_r T}) (r(0) - \mu_0)
\]

(83)

For the variance of the bond yield, we start from the following expression:

\[
\text{Var}_Q[Y(0, T) | \mathcal{F}_0] \equiv K_{HSS}^2(0, T)
\]

\[
= \frac{\sigma_v^2}{[\ln \eta (a_r + \ln \eta)]^2} \int_0^T (e^{(T-v)\ln \eta} - 1)^2 dv \\
+ \frac{\sigma_v^2}{[a_r (\ln \eta + a_r)]^2} \int_0^T (1 - e^{-a_r (T-v)})^2 dv \\
- \frac{2 \sigma_v^2}{a_r \ln \eta (\ln \eta + a_r)} \int_0^T (e^{\ln \eta (T-v)} - 1)(1 - e^{-a_r (T-v)}) dv \\
+ \frac{\sigma_r^2}{a_r^2} \int_0^T (1 - e^{-a_r (T-v)})^2 dv
\]

These four time integrals can be simply solved. Skipping the calculations, the expression for the variance is:

\[
K_{HSS}^2(0, T) = \frac{\sigma_v^2}{(a_r + \ln \eta)^2 [\ln \eta]^2} \left( T + \frac{2}{\ln \eta} (1 - e^{\ln \eta T}) - \frac{1}{2 \ln \eta} (1 - e^{2 \ln \eta T}) \right) \\
+ \frac{1}{a_r^2} (T - \frac{2}{a_r} (1 - e^{-a_r T}) + \frac{1}{2a_r} (1 - e^{-2a_r T}))
\]

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\begin{align*}
&+ \frac{2}{a_r \ln \eta} (T + \frac{1}{a_r - \ln \eta} (1 - e^{(\ln \eta - a_r)T}) \\
&\quad + \frac{1}{\ln \eta} (1 - e^{\ln \eta T}) - \frac{1}{a_r} (1 - e^{-a_r T}))] \\
&+ \frac{\sigma^2}{a_r^2} [T - \frac{2}{a_r} (1 - e^{-a_r T}) + \frac{1}{2a_r} (1 - e^{-2a_r T})] \tag{84}
\end{align*}

Plugging (83) and (84) into (82) solves the formula for the bond value. \hfill \Box
References


Figures 1-8
Recurrent regime

Term structure of yields-to-maturity of a corporate discount bond for initial parameter values: 
\( \sigma_x = 5; \gamma = 0.07; \sigma_x = 0.02; \lambda = 0.2; \gamma = 0.03; \alpha_x = 0.2; \lambda = 0.03; \alpha = 0.5; \pi = 1.22 \) and \( \rho = -1, 0, 1 \) for (respectively) the lower, middle and upper curves.
Figure 9-16
Immediate liquidation

Term structure of yields-to-maturity of a corporate discount bond for initial parameter values:
\(a, = .5; F = .07; \sigma_r = .02; \beta = 2; \gamma = .03; \sigma_r = 2; \lambda = .03; \alpha = .5; \pi = 1.4; \beta = 0 \) and \( \rho = -1, 0, 1 \) for (respectively) the lower, middle and upper curves.
Figures 17-24
Delayed liquidation

Term structure of yields-to-maturity of a corporate discount bond for initial parameter values:
\( \sigma = .5; \gamma = .07; \sigma_x = .02; z = .2; \sigma_x = .2; \lambda = .03; \alpha = .5; \pi = 1.4; \beta = 0 \) and \( \rho = -1, 0, 1 \) for (respectively) the lower, middle and upper curves.
Table 1
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond with initial parameter values:

\( a_1 = 0.5; \gamma = 0.07; \sigma_1 = 0.02; s = 0.2; \gamma = 0.02; \sigma_2 = 0.2; \rho = 0; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4; \) and initial conditions: \( r = 0.07; \ln X = 0. \)

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Table 2
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond with initial parameter values:

\( a_1 = 0.5; \gamma = 0.07; \sigma_1 = 0.02; s = 0.2; \gamma = 0.02; \sigma_2 = 0.2; \rho = 0; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4; \) and initial conditions: \( r = 0.08; \ln X = 0.11. \)

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<td>0.4101</td>
<td>0.2295</td>
<td>0.0164</td>
<td></td>
</tr>
<tr>
<td>1-\alpha</td>
<td>0.0246</td>
<td>0.0292</td>
<td>0.0401</td>
<td></td>
</tr>
<tr>
<td>\pi</td>
<td>0.0053</td>
<td>0.0124</td>
<td>0.0119</td>
<td></td>
</tr>
</tbody>
</table>

Table 3
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond with initial parameter values:

\( a_1 = 0.5; \gamma = 0.07; \sigma_1 = 0.02; s = 0.2; \gamma = 0.02; \sigma_2 = 0.2; \rho = 0; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4; \) and initial conditions: \( r = 0.06; \ln X = -0.11. \)

<table>
<thead>
<tr>
<th>Regime</th>
<th>argument</th>
<th>recurrent</th>
<th>del.liq.</th>
<th>im.liq.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0056</td>
<td>0.0056</td>
<td>0.0050</td>
<td></td>
</tr>
<tr>
<td>\sigma_1</td>
<td>-0.0564</td>
<td>-0.0564</td>
<td>-0.0483</td>
<td></td>
</tr>
<tr>
<td>\sigma_2</td>
<td>-0.0051</td>
<td>-0.0064</td>
<td>-0.0061</td>
<td></td>
</tr>
<tr>
<td>\sigma_3</td>
<td>0.0207</td>
<td>0.0313</td>
<td>0.0297</td>
<td></td>
</tr>
<tr>
<td>\rho</td>
<td>0.0009</td>
<td>0.0016</td>
<td>0.0015</td>
<td></td>
</tr>
<tr>
<td>\lambda</td>
<td>0.5443</td>
<td>0.4687</td>
<td>0.2468</td>
<td></td>
</tr>
<tr>
<td>1-\alpha</td>
<td>0.0327</td>
<td>0.0299</td>
<td>0.0401</td>
<td></td>
</tr>
<tr>
<td>\pi</td>
<td>-0.0026</td>
<td>-0.0027</td>
<td>-0.0027</td>
<td></td>
</tr>
</tbody>
</table>
Table 4
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond submitted to the recurrent risk regime with initial parameter values:
\( a_r = 0.5; \tilde{r} = 0.07; \sigma_r = 0.02; s = 0.2; \gamma = 0.02; \sigma_s = 0.2; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4 \)
and initial conditions: \( r = 0.07; \ln X = 0 \), for different values of \( p \).

<table>
<thead>
<tr>
<th>argument</th>
<th>( \rho = -1 )</th>
<th>( \rho = 0 )</th>
<th>( \rho = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_r )</td>
<td>0.0032</td>
<td>0.0018</td>
<td>0.0004</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>-0.0997</td>
<td>-0.0564</td>
<td>-0.0130</td>
</tr>
<tr>
<td>( s )</td>
<td>0.0039</td>
<td>0.0017</td>
<td>-0.0004</td>
</tr>
<tr>
<td>( \sigma_s )</td>
<td>0.0163</td>
<td>0.0207</td>
<td>0.0250</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.0009</td>
<td>0.0009</td>
<td>0.0009</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.4483</td>
<td>0.4772</td>
<td>0.5061</td>
</tr>
<tr>
<td>( 1-\alpha )</td>
<td>0.0269</td>
<td>0.0286</td>
<td>0.0304</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0.0031</td>
<td>0.0014</td>
<td>-0.0004</td>
</tr>
</tbody>
</table>

Table 5
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond submitted to the delayed liquidation regime with initial parameter values:
\( a_r = 0.5; \tilde{r} = 0.07; \sigma_r = 0.02; s = 0.2; \gamma = 0.02; \sigma_s = 0.2; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4 \)
and initial conditions: \( r = 0.07; \ln X = 0 \), for different values of \( p \).

<table>
<thead>
<tr>
<th>argument</th>
<th>( \rho = -1 )</th>
<th>( \rho = 0 )</th>
<th>( \rho = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_r )</td>
<td>0.0044</td>
<td>0.0018</td>
<td>-0.0008</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>-0.1352</td>
<td>-0.0564</td>
<td>0.0223</td>
</tr>
<tr>
<td>( s )</td>
<td>0.0101</td>
<td>0.0063</td>
<td>0.0024</td>
</tr>
<tr>
<td>( \sigma_s )</td>
<td>0.0235</td>
<td>0.0314</td>
<td>0.0392</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.0016</td>
<td>0.0016</td>
<td>0.0016</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.3021</td>
<td>0.3493</td>
<td>0.3963</td>
</tr>
<tr>
<td>( 1-\alpha )</td>
<td>0.0292</td>
<td>0.0295</td>
<td>0.0298</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0.0080</td>
<td>0.0048</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

Table 6
Simulated sensitivity of the yield-to-maturity of a 10-year corporate discount bond submitted to the immediate liquidation regime with initial parameter values:
\( a_r = 0.5; \tilde{r} = 0.07; \sigma_r = 0.02; s = 0.2; \gamma = 0.02; \sigma_s = 0.2; \lambda = 0.03; \beta = 0; \alpha = 0.5; \pi = 1.4 \)
and initial conditions: \( r = 0.07; \ln X = 0 \), for different values of \( p \).

<table>
<thead>
<tr>
<th>argument</th>
<th>( \rho = -1 )</th>
<th>( \rho = 0 )</th>
<th>( \rho = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_r )</td>
<td>0.0039</td>
<td>0.0015</td>
<td>-0.0008</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>-0.1225</td>
<td>-0.0484</td>
<td>0.0254</td>
</tr>
<tr>
<td>( s )</td>
<td>0.0095</td>
<td>0.0059</td>
<td>0.0023</td>
</tr>
<tr>
<td>( \sigma_s )</td>
<td>0.0224</td>
<td>0.0298</td>
<td>0.0371</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.0015</td>
<td>0.0015</td>
<td>0.0015</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.0906</td>
<td>0.1318</td>
<td>0.1728</td>
</tr>
<tr>
<td>( 1-\alpha )</td>
<td>0.0400</td>
<td>0.0404</td>
<td>0.0409</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0.0075</td>
<td>0.0046</td>
<td>0.0017</td>
</tr>
</tbody>
</table>