PERFORMANCE MEASURES FOR DYNAMIC PORTFOLIO MANAGEMENT

by

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Performance Measures for Dynamic Portfolio Management

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Abstract

This paper proposes instantaneous versions of the Sharpe ratio and Jensen's alpha as performance measures for managed portfolios. Both are derived from optimal portfolio selection theory in a dynamic model. The instantaneous Sharpe ratio equals the discrete Sharpe ratio plus half of the volatility of the fund. Hence, it does not penalize fund managers as much for taking risks as the discrete ratio does. This is justified by dynamic portfolio theory. Unlike their discrete versions, the instantaneous performance measures take leverage correctly into account in a dynamic setting, and they take into account the fact that investors rebalance their portfolios over time. We calculate the discrete and instantaneous Sharpe ratios for a sample of six mutual funds and the S & P 500 index. They illustrate that the two versions of the Sharpe ratio may rank funds differently, and that funds with high volatility look better when evaluated by the instantaneous than by the conventional Sharpe ratio.
1 Introduction

This paper proposes and analyzes instantaneous versions of the Sharpe ratio and Jensen's alpha as performance measures for managed portfolios. Both are derived from optimal portfolio selection theory in a dynamic model.

The Sharpe ratio is the ratio between expected or average excess return and risk, where risk is measured as standard deviation of return. According to static mean-variance portfolio theory, an investor optimally combines individual risky securities into a portfolio of risky assets so as to maximize the Sharpe ratio. He then combines this portfolio with holdings of the riskless asset, or alternatively uses leverage, so as to achieve his desired level of risk.

The Sharpe ratio was proposed by Sharpe (1966) as a performance measure for managed portfolios. If an investor faces an exclusive choice among a number of managed funds, then the funds can unambiguously be ranked on the basis of their Sharpe ratios, in the sense that a fund with higher Sharpe ratio will enable the investor to achieve a higher expected utility.

Merton (1971) has extended the simple mean-variance portfolio composition rule to a dynamic, continuous-time setting, under the assumption that the interest rate and the first and second moments of returns to all securities are constant. In that case, investors optimally combine individual risky securities into a portfolio of risky assets so as to maximize the instantaneous Sharpe ratio. The instantaneous Sharpe ratio is effectively the same as the discrete Sharpe ratio, but with the rates of returns over finite time intervals replaced by instantaneous rates of return. Nielsen and Vassalou (1997) have shown that Merton's result holds even with time-varying and stochastic means, variances, and covariances, provided that the interest rate and the maximum instantaneous Sharpe ratio are constant or deterministic.

In this paper, we show how the instantaneous Sharpe ratio can be used as a performance measure for managed portfolios in a dynamic context. If investors face an exclusive choice among a number of managed funds, each of which has constant instantaneous Sharpe ratio, and if they are able to dynamically reallocate wealth between their chosen fund and a money market account, then the funds can unambiguously be ranked on the basis of their instantaneous Sharpe ratios. A fund with a higher Sharpe ratio will enable
the investor to achieve a higher expected utility through a dynamic strategy.

In the case where the value of a fund follows a geometric Brownian motion, we can compare the instantaneous Sharpe ratio with a discrete Sharpe ratio calculated from continuously compounded rates of return. It turns out that the instantaneous Sharpe ratio equals the discrete Sharpe ratio plus half of the volatility of the fund.

We find that the volatility adjustment is sizeable relative to the Sharpe ratio itself, and that the relative size of the adjustment does not depend on whether returns are expressed per day, per month, or per year.

The fact that the instantaneous Sharpe ratio differs from the discrete Sharpe ratio by half of the volatility of the fund has important implications for performance evaluation, particularly for high-volatility, active funds.

First of all, the discrete and instantaneous Sharpe ratios may well produce different rankings of funds. The instantaneous Sharpe ratio does not penalize fund managers as much for taking risks as the discrete ratio does. In particular, if two funds have the same discrete Sharpe ratio but different volatilities, then the fund with higher volatility will be the better performer. We illustrate these differences by a comparison of the two ratios calculated for a small number of funds.

Unlike the discrete ratio (calculated from continuously compounded rates of return), the instantaneous Sharpe ratio takes into account the effects of compounding on risk and return and the fact that investors rebalance their portfolios over time. It also takes leverage into account in a dynamic setting. We show that although the discrete Sharpe ratio is unaffected by leverage in a static setting, it is not unaffected in a dynamic setting, where, in fact, it decreases with leverage. The instantaneous Sharpe ratio, by contrast, is unaffected by leverage in a dynamic model.

Although the Sharpe ratio is known and used by practitioners, most academic studies of the performance of managed portfolios do not use it. Rather, they evaluate performance relative to an index and use variations of the so-called Jensen's alpha as performance criterion. Examples include Grinblatt and Titman (1989a), Cumby and Glen (1990), Ferson and Schadt (1996), and Cai, Chan, and Yamada (1997).
We provide an interpretation of Jensen’s alpha in terms of optimal portfolio choice. The literature seems to have been silent on this point. The question is this. If an investor identifies a fund which has a positive alpha, then what exactly does that tell him about what he should do to maximize his expected utility? The answer is as follows. Suppose the investor initially holds a combination of the riskless asset and an index portfolio. He considers whether to tilt his portfolio holdings towards an actively managed fund by investing a small proportion of his wealth in it. He should do so only if it raises the Sharpe ratio of his overall portfolio. We show that Jensen’s alpha is proportional to the first derivative of the overall Sharpe ratio with respect to the proportion invested in the active fund. Hence, a positive alpha means that the investor can increase his expected utility by investing at least a small amount in the fund.

This relation between Jensen’s alpha and the Sharpe ratio holds in a dynamic model as well as in a static model. In a dynamic model, the relevant version of alpha is the instantaneous alpha. It is effectively the same as the discrete alpha, but with the rates of returns over finite time intervals replaced by instantaneous rates of return. We show that the instantaneous alpha is equal to the discrete alpha plus half the variance of the portfolio minus half the covariance of the portfolio with the benchmark.

The rest of the study is organized as follows. Section 2 shows how the instantaneous Sharpe ratio can be used as a performance criterion. Section 3 derives the relation between the instantaneous and the discrete Sharpe ratio. Section 4 discusses the effects of leverage on both ratios. Section 5 provides some simple calculations that illustrate the difference in ranking resulting from the two ratios. Section 6 derives the explicit relation between Jensen’s alpha and the Sharpe ratio. Section 7 discusses the instantaneous Jensen’s alpha and its relation to the discrete Jensen’s alpha. We conclude in Section 9.

2 The Instantaneous Sharpe Ratio

The discrete Sharpe ratio of a portfolio or fund is the ratio between the expected or average excess rate of return on the fund and the volatility or
standard deviation of the rate of return:

\[ S = \frac{E[r_p] - r_f}{\sqrt{\text{var}(r_p)}} \]

where \( r_p \) is the rate of return on the portfolio, and \( r_f \) is the riskless rate.

In this section, we present the instantaneous Sharpe ratio and show how it can be used for performance evaluation of managed portfolios in a dynamic context and how it is related to the discrete Sharpe ratio. We begin by describing the continuous-time mathematics of the model. For a general introduction to continuous time finance models, see Nielsen (1997).

The time horizon is \([0, T]\) for a fixed \( T > 0 \). The investors' information structure is represented by a filtration \( F = (\mathcal{F}_t)_{t \in [0,T]} \) on an underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The interpretation is that \( \mathcal{F}_t \) is the information set available to the investors at time \( t \). Random fluctuations in securities prices are driven by a \( K \)-dimensional process \( W \), which is a \( K \)-dimensional Wiener process (a vector of \( K \) independent one-dimensional Wiener processes) with respect to the filtration \( F = (\mathcal{F}_t) \).

There is a short-term riskless asset such as a money market account whose value \( M(t) \) at time \( t \) has the form,

\[ M(t) = M(0) \exp \left\{ \int_0^t r \, ds \right\} \]

where \( r \) is the instantaneously riskless interest rate. This rate may change over time, but at each instant it is known, and the value of the money market account grows at that rate:

\[ \frac{dM}{M} = r \, dt \]

The value process \( F \) of a portfolio or fund, with dividends reinvested, is assumed to be a positive Itô process with differential

\[ \frac{dF}{F} = \mu \, dt + \sigma dW \]

where \( \mu \) is the instantaneous expected rate of return, and \( \sigma \) is the \( K \)-dimensional row vector of instantaneous relative dispersion coefficients. The
volatility of the fund will be $\sqrt{\sigma \sigma^*}$. The *instantaneous Sharpe ratio* of the fund, denoted by $S_{\text{inst}}$, is defined as

$$S_{\text{inst}} = \frac{\mu - r}{\sqrt{\sigma \sigma^*}}$$

We shall assume that the investor chooses one fund and then splits his wealth between this fund and the money market account. The way in which he splits it may change over time in response to new information. In other words, he implements a *portfolio strategy*, which in this context is a one-dimensional adapted process $q$. The interpretation is that the investor puts the fraction $q$ of his wealth in the fund and $1 - q$ in the money market account. The resulting wealth process $V$ has dynamics

$$\frac{dV}{V} = (q(\mu - r) + r)\, dt + q\sigma\, dW$$

The investor chooses $q$ so as to maximize his expected utility of final wealth $V(T)$.

The following proposition is the theoretical foundation for using the instantaneous Sharpe ratio for performance measurement in a dynamic framework. It says that an investor who splits his wealth between a money market account and a fund can get a higher expected utility the higher is the instantaneous Sharpe ratio of the fund that he chooses, provided that the interest rate varies in a deterministic manner and that the instantaneous Sharpe ratio of the funds under consideration are constant. It follows that if the investor is choosing among a number of funds, each of which has a constant instantaneous Sharpe ratio, then he will prefer the one which has the highest instantaneous Sharpe ratio. Hence, the instantaneous Sharpe ratio can be used to rank funds in the dynamic framework exactly like the discrete Sharpe ratio in a static model.

**Proposition 1** Suppose the interest rate $r$ is deterministic. Consider two funds whose price processes $F_1$ and $F_2$ have differentials

$$\frac{dF_i}{F_i} = \mu_i\, dt + \sigma_i\, dW$$

for $i = 1, 2$. Suppose the instantaneous Sharpe ratios

$$S_{\text{inst},i} = \frac{\mu_i - r}{\sqrt{\sigma_i \sigma_i^*}}$$
\( i = 1, 2, \) are positive constants. Given the investor’s utility function, the maximum expected utility he can get from a portfolio strategy which involves only the fund \( F_1 \) and is adapted to \( \mathcal{F} \) is strictly larger than the maximum expected utility he can get from a portfolio strategy which involves only the fund \( F_2 \) and is adapted to \( \mathcal{F} \), if and only if \( \text{S}_{\text{inst},1} > \text{S}_{\text{inst},2} \).

The proof of Proposition 1 is not entirely simple, for several reasons: (1) the Wiener process \( W \) is potentially high-dimensional, which allows the two funds to be less than perfectly instantaneously correlated, (2) the investor’s trading strategy may in principle be contingent on much more information than just observing the value of the fund he is trading, and (3) the relative dispersion vector \( \sigma_i \) of the fund may be stochastically time-varying.

The proof of Proposition 1 relies on the following lemma:

**Lemma 1** Suppose the interest rate \( r \) is deterministic. Let \( B \) be a one-dimensional standard Brownian motion, and let \( \mathcal{F}^B \) be the augmented filtration generated by \( B \). Consider two funds whose price processes \( F \) and \( \tilde{F} \) have differentials

\[
\frac{dF}{F} = \mu dt + \sigma dW
\]

and

\[
\frac{d\tilde{F}}{F} = (s^2 + r) dt + dB
\]

where

\[
s = \frac{\mu - r}{\sqrt{\sigma^2} i}
\]

is assumed to be constant. Given the investor’s utility function, the maximum expected utility he can get from a portfolio strategy which involves only the fund \( F \) and is adapted to \( \mathcal{F} \) is the same as the maximum expected utility he can get from a portfolio strategy which involves only the fund \( \tilde{F} \) and is adapted to \( \mathcal{F}^B \).

**Proof:**

Set

\[
\phi = \frac{s}{\sqrt{\sigma^2} i} = \frac{\mu - r}{\sigma^2 i}
\]
and

\[ \lambda = \phi \sigma \]

Then

\[ \lambda \lambda^\top = s^2 \]

Define a one-dimensional standard Brownian motion \( C \) by

\[ C(t) = \int_0^t \frac{1}{s} \lambda \, dW \]

and let \( \mathcal{F}^C \) be the augmented filtration generated by \( C \). It follows from the results in Nielsen and Vassalou (1997) that the optimal portfolio strategy when trading the fund \( F \) has the form

\[ q = a \phi \]

where \( a \) is a process which is adapted to \( \mathcal{F}^C \). A strategy of this form gives the following dynamics of wealth:

\[
\begin{align*}
\frac{dV}{V} &= (q(\mu - r) + r) \, dt + q \sigma \, dW \\
&= (a \phi s \sqrt{\sigma^2 + r} + r) \, dt + a \phi \sigma \, dW \\
&= (a s^2 + r) \, dt + a \lambda \, dW \\
&= (a s^2 + r) \, dt + a \, dC
\end{align*}
\]

Consider the fund \( F^\phi \) which arises from trading the fund \( F \) using the portfolio strategy \( \phi \). It has dynamics

\[
\frac{dF^\phi}{F^\phi} = (s^2 + r) \, dt + dC
\]

The wealth dynamics arising from trading the fund \( F \) using the portfolio strategy \( a \phi \) is

\[
\frac{dV}{V} = (a s^2 + r) \, dt + a \, dC
\]

which is the same as the wealth dynamics arising from trading the fund \( F^\phi \) using the portfolio strategy \( a \). Hence, the maximum expected utility that can be achieved from trading the fund \( F \) using portfolio strategies that are adapted to \( \mathcal{F} \) is identical to the maximum expected utility that can be
achieved by trading the fund $F^\phi$ using portfolio strategies that are adapted to $\mathcal{F}^C$. The latter is obviously identical to the maximum expected utility that can be achieved by trading the fund $\hat{F}$ using portfolio strategies that are adapted to $\mathcal{F}^B$.

\[ \square \]

**Proof of Proposition 1:**

Let $B$ be a one dimensional standard Brownian motion, and let $\mathcal{F}^B$ be the augmented filtration generated by $B$. For each $s$, consider a fund whose price processes $\hat{F}[s]$ has differential

$$\frac{d\hat{F}[s]}{\hat{F}[s]} = (s^2 + r) \, dt + dB$$

According to Lemma 1, given the investor’s utility function, the maximum expected utility he can get from a portfolio strategy which involves only the fund $F_i$ and is adapted to $\mathcal{F}$ is the same as the maximum expected utility he can get from a portfolio strategy which involves only the fund $\hat{F}[S_{\text{inst},i}]$ and is adapted to $\mathcal{F}^B$. Therefore, what we need to show is that if $S_{\text{inst},1} > S_{\text{inst},2}$, then the maximum expected utility the investor can get from trading in the fund $\hat{F}[S_{\text{inst},1}]$ with a portfolio strategy which is adapted to $\mathcal{F}^B$ is strictly larger than the maximum expected utility he can get from trading in the fund $\hat{F}[S_{\text{inst},2}]$ with a portfolio strategy which is adapted to $\mathcal{F}^B$.

Let $a$ be the optimal portfolio strategy when he trades the fund $\hat{F}[S_{\text{inst},2}]$. Since this fund has constant instantaneous mean and dispersion, it is known from Merton (1971) that $a$ has the form

$$a = \gamma \frac{\mu_2 - r}{\sigma_2 \sigma_1} = \gamma \frac{S_{\text{inst},2}}{\sqrt{\sigma_1 \sigma}}$$

where $\gamma > 0$ is the relative risk tolerance of the investor’s utility function. Hence, $a > 0$. If the investor uses the portfolio strategy $a$ to trade the fund $\hat{F}[S_{\text{inst},i}]$, then his wealth dynamics is

$$\frac{dV_i}{V_i} = (a S_{\text{inst},i}^2 + r) \, dt + a \, dB$$

The logarithm of his final wealth will be

$$\ln V_i(T) = \ln V(0) + \int_0^t \left( a S_{\text{inst},i}^2 + r - \frac{1}{2} a^2 \right) \, dt + \int_0^t a \, dB$$
Hence,

\[ \ln V_1(T) - \ln V_2(T) = \int_0^t a \left( S_{\text{inst}, i}^1 - S_{\text{inst}, i}^2 \right) \, dt > 0 \]

Therefore, the investor gets a strictly higher expected utility by using \( a \) to trade the fund \( F[S_{\text{inst}, 1}] \) than by using it to trade the fund \( F[S_{\text{inst}, 2}] \).

\[ \square \]

3 Discrete and Instantaneous Sharpe Ratios

In what follows, we derive a relation between the instantaneous Sharpe ratio and the discrete Sharpe ratio calculated from continuously compounded rates of return. We assume that the volatility \( \sigma \) and the expected instantaneous excess rate of return \( \mu - r \) of the fund are constant, and that the interest rate \( r \) is deterministic. Note that the instantaneous Sharpe ratio of the fund will be constant. Although \( \mu - r \) is constant, \( \mu \) itself may not be constant.

The continuously compounded rate of return \( r_f \) on the money market account over the time interval \([t, t + \tau]\) is

\[ r_f = \ln M(t + \tau) - \ln M(t) = \int_t^{t + \tau} r \, ds \]

It is deterministic.

Since we are now considering only one fund at a time, we can assume that the Wiener process is one-dimensional and that \( \sigma \) is one-dimensional.

The continuously compounded rate of return \( r_p \) on the portfolio over the time interval \([t, t + \tau]\) is

\[ r_p = \ln F(t + \tau) - \ln F(t) \]

\[ = \int_t^{t + \tau} \left( \mu - \frac{1}{2} \sigma^2 \right) \, ds + \int_t^{t + \tau} \sigma \, dW \]

\[ = \int_t^{t + \tau} r \, ds + \int_t^{t + \tau} \left( \mu - r - \frac{1}{2} \sigma^2 \right) \, ds + \int_t^{t + \tau} \sigma \, dW \]

\[ = r_f + m \tau + \sigma(W(t + \tau) - W(t)) \]

where

\[ m = \mu - r - \frac{1}{2} \sigma^2 \]
It follows that $r_p$ is normally distributed with mean $r_f + m\tau$, variance $\sigma^2\tau$, and standard deviation $\sigma\sqrt{\tau}$. Continuously compounded rates of return over successive time intervals are independent of each other, because the increments $W(t + \tau) - W(t)$ of the Brownian motion are independent.

The discrete Sharpe ratio is the ratio between the mean and standard deviation of excess rates of return over a discrete period. Sharpe (1966) used discretely compounded returns, but continuously compounded returns are often used in practice, and we shall use continuously compounded returns here. If the rates of return are expressed per period of length $\tau$, then the discrete Sharpe ratio is

$$\frac{E(r_p - r_f)}{\sigma} = \frac{m\tau}{\sigma\sqrt{\tau}} = S\sqrt{\tau}$$

where

$$S = \frac{m}{\sigma}$$

is the discrete Sharpe ratio based on annualized rates.

By substituting the definition of $m$ into the definition of the instantaneous Sharpe ratio, we find the relation between the discrete Sharpe ratio and the instantaneous Sharpe ratio:

$$S_{\text{inst}} = \frac{\mu - r}{\sigma} = \frac{m + \frac{1}{2}\sigma^2}{\sigma} = S + \frac{1}{2}\sigma$$

So, the instantaneous Sharpe ratio differs from the discrete Sharpe ratio by a bias equal to $\sigma/2$. This bias of course comes from the difference of $\sigma^2/2$ between the instantaneous mean excess return $\mu - r$ and the discrete mean excess return $m$.

It is important to recognize that (1) the bias is sizeable relative to the size of the Sharpe ratio itself, (2) while the discrete and instantaneous Sharpe ratios do depend on whether returns are expressed per day, per month, or per year, the ranking of portfolios that they produce does not, (3) the relative size of the bias does not depend on whether returns are expressed per day, per month, or per year, and (4) when the Sharpe ratios are estimated from data, the importance of the bias is independent of the frequency of the data.

Point (1) is illustrated by the sample that we present in Section 5, where the bias is about 8% of the discrete Sharpe ratio for the S & P 500 index,
and larger for most of the funds we consider, with a fair amount of variation across funds.

To make points (2) and (3), we calculate the instantaneous and discrete Sharpe ratios for returns expressed per period of length $\tau$, and then we express the bias as a fraction of the discrete Sharpe ratio.

Observe that the definition of the instantaneous Sharpe ratio as 
$$S_{\text{inst}} = \frac{\mu - r}{\sigma}$$
is based on instantaneous returns per period of length one, say one year. The instantaneous Sharpe ratio corresponding to rates of return per time period of length $\tau$ is 
$$\frac{\mu \tau - r \tau}{\sigma \sqrt{\tau}} = S_{\text{inst}} \sqrt{\tau} = S \sqrt{\tau} + \frac{1}{2} \sigma \sqrt{\tau}$$

It is clear that the rankings of funds produced by $S_{\text{inst}} \sqrt{\tau}$ and $S \sqrt{\tau}$ are independent of $\tau$, which was point (2). The size of the bias is 
$$S_{\text{inst}} \sqrt{\tau} - S \sqrt{\tau} = \frac{1}{2} \sigma \sqrt{\tau}$$

which of course goes to zero as the length $\tau$ of the time interval goes to zero. However, expressed as a fraction of the discrete Sharpe ratio $S \sqrt{\tau}$, the bias is 
$$\frac{S_{\text{inst}} \sqrt{\tau} - S \sqrt{\tau}}{S \sqrt{\tau}} = \frac{S_{\text{inst}} - S}{S}$$

which is independent of $\tau$. This was point (3). The relative bias can also be written as 
$$\frac{S_{\text{inst}} \sqrt{\tau} - S \sqrt{\tau}}{S \sqrt{\tau}} = \frac{\mu - r - m}{m} = \frac{\mu \tau - r \tau - m \tau}{m \tau}$$

In other words, it equals the difference between the instantaneous and the discrete expected excess return per period of length $\tau$, expressed as a fraction of the discrete expected excess return.

Finally, (4) if the Sharpe ratios are estimated from data, then the quality of the estimate will of course be better the more data is used, and in particular, the higher the frequency of the data. However, the true underlying values of the ratios are unaffected, provided that they are expressed in terms of returns.
per period of a fixed length, such as a year, independently of the sampling frequency.

When estimating the instantaneous Sharpe ratio, we have to take into account the fact that while the parameters $\mu$ and $\sigma$ refer to instantaneous returns, we can actually only observe returns over discrete time periods such as days, weeks, months or years. The equation

$$S_{\text{inst}} = \frac{m + \frac{1}{2} \sigma^2}{\sigma}$$

has the virtue of expressing the instantaneous Sharpe ratio in terms of discrete time moments of the rates of return, since $m$ is the expectation of the annualized discrete time rate of return and $\sigma$ is the standard deviation.

In Section 5, we present some illustrative calculations of the discrete and instantaneous Sharpe ratios based on a subset of funds studied by Modigliani and Modigliani (1997).

The fact that the instantaneous Sharpe ratio equals the discrete Sharpe ratio plus half of the volatility of the fund implies that the ranking of funds based on the discrete and the instantaneous Sharpe ratios may well be different. In particular, if two funds have the same discrete Sharpe ratio but one has higher volatility than the other, then they will be ranked as equal by the discrete Sharpe ratio while the one with higher volatility will be ranked higher by the instantaneous Sharpe ratio. In other words, given the mean annualized excess rate of return $m$, the instantaneous ratio penalizes the fund manager less than does the discrete ratio for taking risk in the form of volatility. Therefore, the fund with higher volatility will enable the investor to achieve a higher expected utility.

The intuition behind this result is that the static one-period theory on which the discrete Sharpe ratio is based overestimates the riskiness of high-volatility funds, because it does not take into account the possibility of portfolio rebalancing.

Take as an example an investor who wants to hold 50 percent of his wealth in the fund and 50 percent in the riskless asset, and whose investment horizon $T$ is one year. In the static framework, he initially invests half of his money in the fund and half in the riskless asset, and then he waits for a year to see what happens. However, already after a month, the value of the fund may
have gone up so that he actually holds 60 percent in the fund and only 40 percent in the riskless asset. During the course of the year, this situation may be further exacerbated.

By contrast, in the dynamic framework, the investor will immediately react to the increase in the value of the fund by selling some of it and investing the proceeds in the riskless asset, so that he always holds exactly 50 percent in each. This lowers the overall riskiness of his strategy. The difference is reflected in the modification of the Sharpe ratio.

The fact that the modified Sharpe ratio penalizes the fund manager less for taking risk does not mean that it rewards him for taking risks without regard to the expected rate of return. If he increases the volatility of the fund, then he has to raise the instantaneous excess rate of return \( \mu - r \) at least proportionally in order to keep the same modified Sharpe ratio. However, this requires less than a proportional increase in the mean annualized excess rate of return \( m \).

4 The Effect of Leverage

We now show that in the dynamic context, the instantaneous Sharpe ratio is unaffected by leverage, while the discrete Sharpe ratio, calculated using continuously compounded returns, decreases with increasing leverage.

Suppose an investor uses a portfolio strategy \( q \). This means that he puts a fraction \( q \) of his wealth in the portfolio in question and the remaining fraction \( 1 - q \) in the money market account. Notice that the fraction \( q \) does not have to be constant. It is an adapted process.

Let \( V \) denote the investor’s wealth process:

\[
V(t) = V(0) \exp \left\{ \int_0^t \left( q(\mu - r) + r - \frac{1}{2} q^2 \sigma^2 \right) ds + \int_0^t q \sigma dW \right\}
\]

or

\[
\frac{dV}{V} = (q(\mu - r) + r)dt + q \sigma dW
\]
The instantaneous Sharpe ratio of wealth will be

\[ S_{\text{inst}}^q = \frac{q(\mu - r)}{q\sigma} = \frac{\mu - r}{\sigma} = S_{\text{inst}} \]

which shows that the instantaneous Sharpe ratio is unaffected by leverage.

The discrete Sharpe ratio of wealth, using continuously compounded annualized rates, is

\[ S^q = \frac{q(\mu - r) - \frac{1}{2}q^2\sigma^2}{q\sigma} \]

\[ = \frac{\mu - r}{\sigma} - \frac{1}{2}q\sigma \]

\[ = \frac{\mu - r}{\sigma} - \frac{1}{2}\sigma + \frac{1}{2}(1 - q)\sigma \]

\[ = S + \frac{1}{2}(1 - q)\sigma \]

where \( S \) is the discrete Sharpe ratio of the fund. This shows that the discrete Sharpe ratio does depend on leverage, unlike what the discrete theory presumes. In fact, the discrete Sharpe ratio decreases with increasing leverage (increasing \( q \)).

Actively managed funds are often found to have a higher volatility and a higher average rate of return than the index but a lower discrete Sharpe ratio. They are then said to fail to outperform the index. The argument is that a levered position in the index could lead to the same volatility as the fund but a higher average rate of return, or to the same average rate of return as the fund with a lower volatility. This argument is now seen to be incorrect in the dynamic framework, because leveraging the index will in fact decrease its discrete Sharpe ratio. Evaluation of a fund relative to the index should be based on the instantaneous Sharpe ratios, which are independent of leverage.

5 Sample Calculations: Sharpe Ratios

In this section, we illustrate the potential difference in ranking of funds provided by the discrete and the instantaneous Sharpe ratios. We have calculated the discrete and instantaneous Sharpe ratios for the S & P 500 index and
Table 1 lists the results. Although the data are monthly, the table is based on annualized continuously compounded rates of return. Observe that in all cases, the instantaneous Sharpe ratios are somewhat higher than the discrete ones. The difference is more pronounced for funds with high volatility.

The column "% bias" shows the difference between the discrete and the instantaneous Sharpe ratio, expressed as a percentage of the discrete Sharpe ratio. It is the same as the difference between the discrete and the instantaneous mean, expressed as a percentage of the discrete mean. Notice that the bias is sizeable.

The ranking of the funds is the same for the two measures, except for one of the funds, AIM Constellation A. AIM beats the index according to the instantaneous Sharpe ratio but performs less well than the index according to the discrete Sharpe ratio. This fund has higher average return than the

for six mutual funds. The funds are the same as those studied by Modigliani and Modigliani (1997), except that we have omitted Fidelity Puritan, for which we did not have data. We have used monthly data from November 1987 to September 1996. The mutual fund data are from Morningstar while the return on the S & P 500 and the US T-bill with 30-days to maturity are from Datastream.

Table 1: Discrete and instantaneous Sharpe ratios

<table>
<thead>
<tr>
<th>fund</th>
<th>Mean</th>
<th>Std.Dev</th>
<th>S</th>
<th>S_{inst}</th>
<th>% bias</th>
<th>S rank</th>
<th>S_{inst} rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>S &amp; P500</td>
<td>0.143</td>
<td>0.119</td>
<td>0.760</td>
<td>0.819</td>
<td>7.8</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>AIM Constellation A</td>
<td>0.201</td>
<td>0.196</td>
<td>0.755</td>
<td>0.853</td>
<td>13.0</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>20th Century Vista Investors</td>
<td>0.171</td>
<td>0.233</td>
<td>0.507</td>
<td>0.623</td>
<td>22.9</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>T. Rowe Price New Horizons</td>
<td>0.183</td>
<td>0.180</td>
<td>0.725</td>
<td>0.815</td>
<td>12.4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Fidelity Magellan</td>
<td>0.161</td>
<td>0.133</td>
<td>0.815</td>
<td>0.881</td>
<td>8.2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Vanguard Windsor</td>
<td>0.131</td>
<td>0.138</td>
<td>0.564</td>
<td>0.633</td>
<td>12.2</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Income Fund of America</td>
<td>0.121</td>
<td>0.067</td>
<td>1.010</td>
<td>1.044</td>
<td>3.3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>30-day T. Bill</td>
<td>0.053</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Annualized continuously compounded rates.
The results are expressed as pure numbers, not percent.
index but also higher volatility. Both the discrete and the instantaneous Sharpe ratio penalize it for its high volatility, but the instantaneous Sharpe ratio penalizes it less.

6 Jensen versus Sharpe

There are various versions of Jensen’s alpha, corresponding to different asset pricing models. Here we will only deal with the original Jensen’s alpha, which corresponds to the mean-variance CAPM, and its dynamic modification.

The usual interpretation of alpha is that it is a risk-adjusted performance measure which adjusts expected or average returns for beta risk. However, this interpretation does not explicitly relate alpha to optimal portfolio choice or say precisely what an investor should do if he identifies one or more funds with positive alpha.

This section gives a precise interpretation of Jensen’s alpha in terms of portfolio optimization and explains the relation between Jensen’s alpha and the Sharpe ratio. The rates of return in the formulas to follow can be interpreted either as rates of return over discrete time periods, as will be appropriate in a static model, or as instantaneous rates of return, for use in a dynamic model. The expectations, variances, and covariances should be interpreted accordingly.

Jensen’s alpha of a portfolio, relative to an index \( x \), is defined as

\[
\alpha = E(r_p) - r_f - \beta(E(r_x) - r_f)
\]

where \( r_f \) is the riskless rate, \( r_p \) and \( r_x \) are the rates of return on the portfolio \( p \) and on the index \( x \), and

\[
\beta = \frac{\text{cov}(r_p, r_x)}{\text{var}(r_x)}
\]

is the beta of the portfolio with respect to the index.

If indeed the index \( x \) is efficient, then the true alpha of every security and every portfolio will be zero, although of course an estimated alpha may be
different from zero because of estimation error. However, alpha can be calculated and be given a precise interpretation in terms of portfolio optimization even if the index is not efficient.

Suppose the investor initially holds a combination of the riskless asset and an index portfolio tracking the index \( x \), in proportions \( 1 - \nu \) and \( \nu \). He now considers whether to tilt his portfolio a little bit in the direction of the fund \( p \). In other words, he considers taking a small fraction \( \epsilon \) of his wealth and investing it in the portfolio \( p \), while reducing the fractions held in the riskless asset and the index to \((1 - \epsilon)(1 - \nu)\) and \((1 - \epsilon)\nu\) respectively.

Let \( y \) denote the initial portfolio. Its rate of return is

\[ r_y = (1 - \nu)r_f + \nu r_x \]

The expected excess rate of return and variance of the new portfolio will be

\[ E r_y - r_f + \epsilon (E r_p - E r_y) \]

and

\[ (1 - \epsilon)^2 \text{var}(r_y) + \epsilon^2 \text{var}(r_p) + 2\epsilon(1 - \epsilon)\text{cov}(r_y, r_p) \]

respectively, and the Sharpe ratio (or instantaneous Sharpe ratio) will be

\[ S(\epsilon) = \frac{E r_y - r_f + \epsilon (E r_p - E r_y)}{\sqrt{(1 - \epsilon)^2 \text{var}(r_y) + \epsilon^2 \text{var}(r_p) + 2\epsilon(1 - \epsilon)\text{cov}(r_y, r_p)}} \]

**Proposition 2** The derivative of \( S(\epsilon) \) with respect to \( \epsilon \), evaluated at \( \epsilon = 0 \), is

\[ S'(0) = \frac{\alpha}{\nu \sqrt{\text{var}(r_x)}} \]

**Proof:**

Set

\[ E(\epsilon) = E r_y - r_f + \epsilon (E r_p - E r_y) \]

\[ v(\epsilon) = (1 - \epsilon)^2 \text{var}(r_y) + \epsilon^2 \text{var}(r_p) + 2\epsilon(1 - \epsilon)\text{cov}(r_y, r_p) \]

and

\[ \sigma(\epsilon) = \sqrt{v(\epsilon)} \]
Then

\[
S(\epsilon) = \frac{E_{r_p} - r_f + \epsilon(E_{r_p} - E_{r_y})}{\sqrt{(1 - \epsilon)^2 \text{var}(r_y) + \epsilon^2 \text{var}(r_p) + 2\epsilon(1 - \epsilon)\text{cov}(r_y, r_p)}} = \frac{E(\epsilon)}{\sigma(\epsilon)}
\]

To calculate $S'(0)$, first observe the following:

\[
E(0) = E_{r_p} - E_{r_y}
\]

\[
u(0) = \text{var}(r_y)
\]

\[
\sigma(0) = \sqrt{\text{var}(r_y)}
\]

\[
E'(\epsilon) = E_{r_p} - E_{r_y}
\]

\[
v'(\epsilon) = -2(1 - \epsilon)\text{var}(r_y) + 2\epsilon\text{var}(r_p) + 2(1 - 2\epsilon)\text{cov}(r_y, r_p)
\]

\[
v'(0) = -2\text{var}(r_y) + 2\text{cov}(r_y, r_p)
\]

\[
\sigma'(\epsilon) = \frac{1}{2} \frac{v'(\epsilon)}{\sigma(\epsilon)}
\]

and

\[
\sigma'(0) = \frac{1}{2} \frac{v'(0)}{\sigma(0)} = \frac{1}{2} \frac{-2\text{var}(r_y) + 2\text{cov}(r_y, r_p)}{\sqrt{\text{var}(r_y)}} = \frac{-\text{var}(r_y) + \text{cov}(r_y, r_p)}{\sqrt{\text{var}(r_y)}}
\]

Moreover,

\[
E_{r_p} - r_f + \frac{\text{cov}(r_p, r_y)}{\text{var}(r_y)}(E_{r_y} - r_f)
\]

\[
= E_{r_p} - r_f
\]

\[
+ \frac{\nu \text{cov}(r_p, r_x)}{\nu^2 \text{var}(r_x)}(\nu E_{r_x} + (1 - \nu)r_f - r_f)
\]

\[
= E_{r_p} - r_f + \frac{1}{\nu} \beta \nu (E_{r_x} - r_f)
\]

\[
= E_{r_p} - r_f + \beta (E_{r_x} - r_f)
\]

\[
= \alpha
\]
Now,

\[ S'(0) = \frac{E'(0)\sigma(0) - E(0)\sigma'(0)}{\nu(0)} \]

\[ = \frac{1}{\text{var}(r_y)} \left[ (E_{r_p} - E_{r_y})\sqrt{\text{var}(r_y)} - (E_{r_y} - r_f)\frac{-\text{var}(r_y) + \text{cov}(r_p, r_y)}{\sqrt{\text{var}(r_y)}} \right] \]

\[ = \frac{1}{\sqrt{\text{var}(r_y)}} \left[ E_{r_p} - E_{r_y} + (E_{r_y} - r_f) - (E_{r_y} - r_f)\frac{\text{cov}(r_p, r_y)}{\text{var}(r_y)} \right] \]

\[ = \frac{1}{\nu\sqrt{\text{var}(r_x)}} [E_{r_p} - r_f + \beta(E_{r_y} - r_f)] \]

\[ = \frac{\alpha}{\nu\sqrt{\text{var}(r_x)}} \]

\[ \square \]

Proposition 2 leads to the following interpretation of alpha.

If \( \alpha > 0 \), then an investor who basically invests in the index or in a combination of the index and the riskless asset can increase his Sharpe ratio and hence his expected utility by investing a small positive amount in the fund \( p \). Of course, if \( \alpha < 0 \), then he can achieve the same effect by short-selling the fund, if this is possible.

When \( \epsilon \) varies, the standard deviation and mean of the investor’s entire portfolio traces out a hyperbolic curve, which is in fact the risky portfolio frontier generated by two assets, the initial portfolio and the fund \( p \). This frontier should not be confused with the usual frontier constructed from all available securities. We illustrate this in Figure 1, where \( \alpha > 0 \). When \( \epsilon = 0 \), we are at the point \( y \). As \( \epsilon \) increases, we move up along the upper branch of the small hyperbola. The Sharpe ratio \( S(\epsilon) \) is initially increasing and then decreasing. It has a maximum point \( \bar{\epsilon} > 0 \), which represents the optimal fraction of wealth to take out of the initial portfolio and put into the fund \( p \). It corresponds to the point \( \bar{\epsilon} p + (1 - \bar{\epsilon})y \) in the figure.

It is alternatively possible that \( S(\epsilon) \) does not have a maximum but is increasing for all \( \epsilon > 0 \). This occurs if the riskless rate is at or above the expected rate of return on the minimum variance portfolio formed from the
index and the fund \( p \), which corresponds to the top-point of the small hyperbolic curve in the figure. This resembles the situation where the riskless rate is at or above the global minimum variance portfolio formed from the risky securities, as analyzed for example in Huang and Litzenberger (1988).

Observe that alpha alone does not say how much the investor should optimally invest in the fund. In other words, we cannot calculate \( \tilde{\epsilon} \), the optimal value of \( \epsilon \), knowing only the value of alpha. The idiosyncratic variance of the fund also matters. If the investor puts a too large fraction of his wealth into the fund \( p \), then the idiosyncratic risk may result in a lower Sharpe ratio and a lower expected utility.

The analysis above applies not only in a static model but also in a dynamic continuous-time model, when the rates of return over a discrete time interval are replaced by instantaneous rates of return. The Sharpe ratio will be replaced by a instantaneous Sharpe ratio, and alpha will be replaced by a instantaneous alpha, which we shall define in the following section.
The Instantaneous Alpha

In this section, we derive a relation between the instantaneous and the discrete alphas.

Let $F_x$ be the value of the index fund with dividends reinvested, and let $F_p$ be the value of the other fund with dividends reinvested. Assume that they follow the processes

$$F_x(t) = F_x(0) \exp \left\{ \int_0^t \left( \mu_x - \frac{1}{2} \sigma_x^2 \right) ds + \int_0^t \sigma_x dZ_x \right\}$$

and

$$F_p(t) = F_p(0) \exp \left\{ \int_0^t \left( \mu_p - \frac{1}{2} \sigma_p^2 \right) ds + \int_0^t \sigma_p dZ_p \right\}$$

where $\mu_x$ and $\mu_p$ are the instantaneous expected rates of return, $\sigma_x$ and $\sigma_p$ are the instantaneous volatilities or standard deviations of the rates of return, and $Z_1$ and $Z_2$ are two potentially correlated standard Wiener processes with correlation coefficient $\rho$. In differential form,

$$\frac{dF_x}{F_x} = \mu_x dt + \sigma_x dZ_x$$

and

$$\frac{dF_p}{F_p} = \mu_p dt + \sigma_p dZ_p$$

The instantaneous alpha of the fund, denoted by $\alpha_{\text{inst}}$, is defined as

$$\alpha_{\text{inst}} = \mu_p - r - \beta_{\text{inst}} (\mu_x - r)$$

where

$$\beta_{\text{inst}} = \frac{\sigma_p \sigma_x \rho}{\sigma_x^2} = \frac{\sigma_p \rho}{\sigma_x}$$

The instantaneous alpha is effectively the discrete alpha with the rates of return over finite time intervals replaced by instantaneous rates of return.

For the purpose of deriving a relation between the instantaneous and the discrete alpha, assume that the interest rate varies in a deterministic manner, and that the correlation $\rho$, the volatilities $\sigma_p$ and $\sigma_x$, and the instantaneous expected excess rates of return $\mu_p - r$ and $\mu_x - r$ are constant.
The continuously compounded rates of return on \( x \) and \( p \) over the time interval \([t, t + \tau]\) are

\[
rx = r_f + m_x \tau + \sigma_x (Z_x(t + \tau) - Z_x(t))
\]

and

\[
rp = r_f + m_p \tau + \sigma_p (Z_p(t + \tau) - Z_p(t))
\]

where

\[
m_x = \mu_x - r - \frac{1}{2} \sigma_x^2
\]

and

\[
m_p = \mu_p - r - \frac{1}{2} \sigma_p^2
\]

They follow a joint normal distribution with the following means, variances and covariances:

\[
\begin{align*}
E r_p &= r_f + m_p \tau \\
E r_x &= r_f + m_x \tau \\
\sigma_p^2 &= \sigma_p^2 \tau \\
\sigma_x^2 &= \sigma_x^2 \tau
\end{align*}
\]

and

\[
\text{cov}(r_p, r_x) = \sigma_p \sigma_x \rho \tau
\]

The rates of return over successive time intervals are independent of each other, because the increment vectors

\((Z_p(t + \tau) - Z_p(t), Z_x(t + \tau) - Z_x(t))\)

of the Wiener processes are independent.

If the rates of return are expressed per period of length \( \tau \), then the discrete Jensen’s alpha is

\[
m_p \tau - \frac{\text{cov}(r_p, r_x)}{\text{var}(r_x)} m_x \tau = m_p \tau - \frac{\sigma_p \sigma_x \rho \tau}{\sigma_x^2 \tau} m_x \tau = (m_p - \beta m_x) \tau = \alpha \tau
\]

where

\[
\beta = \frac{\text{cov}(r_p, r_x)}{\text{var}(r_x)} = \frac{\sigma_p \sigma_x \rho \tau}{\sigma_x^2 \tau} = \frac{\sigma_p \sigma_x \rho}{\sigma_x^2} = \beta_{\text{inst}}
\]
and
\[ \alpha = m_p - \beta m_x \]
is the discrete Jensen's alpha based on annualized returns.

By substituting the definitions of \( m_p \) and \( m_x \) into the definition of the instantaneous alpha, we find the relation between the discrete alpha and the instantaneous alpha:

\[
\alpha_{\text{inst}} = \mu_p - r - \beta(\mu_x - r) \\
= m_p + \frac{1}{2}\sigma_p^2 - \beta\left(m_x + \frac{1}{2}\sigma_x^2\right) \\
= m_p - \beta m_x + \frac{1}{2}\left(\sigma_p^2 - \beta\frac{1}{2}\sigma_x^2\right) \\
= \alpha + \frac{1}{2}\left(\sigma_p^2 - \sigma_p\sigma_x\rho\right)
\]

It follows that the instantaneous Jensen's alpha differs from the discrete Jensen's alpha by a bias equal to \((\sigma_p^2 - \sigma_p\sigma_x\rho)/2\).

Like for the Sharpe ratios, it is important to recognize that (1) the bias can sizeable relative to the size of alpha itself, (2) while the discrete and instantaneous alphas do depend on whether returns are expressed per day, per month, or per year, the ranking of portfolios that they produce does not, (3) the relative size of the bias does not depend on whether returns are expressed per day, per month, or per year, and (4) if the alphas are estimated from data, then the importance of the bias is independent of the frequency of the data.

Point (1) is illustrated in Section 8, using the same sample of funds as in Section 5. In that sample, the bias ranges from minus 2 percent to plus 367 percent of the discrete alpha.

To make points (2) and (3), we calculate the instantaneous alpha for returns expressed per period of length \( \tau \), and then we express the bias as a fraction of the discrete alpha.

Observe that the definition of the instantaneous alpha as
\[
\alpha_{\text{inst}} = \mu_p - r - \beta_{\text{inst}}(\mu_x - r)
\]
is based on instantaneous returns per period of length one, say one year. The instantaneous alpha corresponding to rates of return per time period of length $\tau$ is

$$(\mu_p - \tau)\tau - \beta_{\text{inst}}(\mu_x - \tau)\tau = \alpha_{\text{inst}}\tau$$

It is clear that the rankings of funds produced by $\alpha_{\text{inst}}\tau$ and $\alpha\tau$ are independent of $\tau$. This illustrates point (2). The size of the bias is

$$\alpha_{\text{inst}}\tau - \alpha\tau = \frac{1}{2} \left( \sigma_p^2 - \sigma_x \sigma_p \rho \right) \tau$$

which goes to zero as the length $\tau$ of the time interval goes to zero. However, expressed as a fraction of the discrete alpha, the bias is

$$\frac{\alpha_{\text{inst}}\tau - \alpha\tau}{\alpha\tau} = \frac{\alpha_{\text{inst}} - \alpha}{\alpha}$$

which is independent of $\tau$. This demonstrates point (3).

Finally, (4) if the alphas are estimated from data, then the same arguments made for the instantaneous and discrete Sharpe ratios apply.

Similarly to the case of the instantaneous Sharpe ratio, the equation

$$\alpha_{\text{inst}} = \alpha + \frac{1}{2} \left( \sigma_p^2 - \sigma_x \sigma_p \rho \right)$$

expresses the instantaneous alpha in terms of discrete time moments of the rates of return. This is useful when estimating it from data.

The relation between the instantaneous and the discrete alpha implies that the two alphas may well be different. This is illustrated by some sample calculations in the following section.

8 Sample Calculations: Alphas

Table 2 lists the results. The table is calculated from the same data as Table 1. Although the data are monthly, the table is based on annualized continuously compounded rates of return.

The column “% Bias” shows the difference between the instantaneous and the discrete alpha, as a percent of the discrete alpha. Observe that for some of the funds, the relative bias is substantial.
Table 2: Discrete and instantaneous alphas

<table>
<thead>
<tr>
<th>Fund Name</th>
<th>$\text{cov}(p, x)$</th>
<th>$\sigma_p^2$</th>
<th>$\alpha$</th>
<th>$\alpha_{\text{inst}}$</th>
<th>$% \text{Bias}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIM Constellation A</td>
<td>1.920</td>
<td>3.828</td>
<td>0.260</td>
<td>1.214</td>
<td>366.9</td>
</tr>
<tr>
<td>20th Century Vista Investors</td>
<td>2.160</td>
<td>5.400</td>
<td>2.787</td>
<td>4.407</td>
<td>58.1</td>
</tr>
<tr>
<td>T. Rowe Price New Horizons</td>
<td>1.788</td>
<td>3.216</td>
<td>0.230</td>
<td>0.944</td>
<td>310.4</td>
</tr>
<tr>
<td>Fidelity Magellan</td>
<td>1.440</td>
<td>1.764</td>
<td>6.241</td>
<td>6.403</td>
<td>2.6</td>
</tr>
<tr>
<td>Vanguard Windsor</td>
<td>1.428</td>
<td>1.896</td>
<td>3.581</td>
<td>3.815</td>
<td>6.5</td>
</tr>
<tr>
<td>Income Fund of America</td>
<td>0.720</td>
<td>0.454</td>
<td>7.088</td>
<td>6.955</td>
<td>-1.9</td>
</tr>
</tbody>
</table>

Annualized continuously compounded rates.
The results are expressed as percent, not pure numbers.

9 Conclusions

This paper has proposed modifications of the Sharpe ratio and Jensen's alpha which make these performance evaluation criteria consistent with the dynamic aspect of asset allocation decisions. Specifically, the modifications take into account the fact that investors may rebalance their portfolios over time.

The instantaneous Sharpe ratio and the instantaneous alpha need not deliver the same rankings of funds as their discrete versions. In fact, the instantaneous Sharpe ratio penalizes a fund less for taking risk than does the discrete ratio. To illustrate this, we performed some calculations for a small sample of mutual funds. Indeed, the instantaneous Sharpe ratio reversed the ranking of one of these funds relative to the index. This happened because the fund had higher volatility than the index.

We derived a precise interpretation of Jensen's alpha in terms of optimal portfolio choice by relating it to the Sharpe ratio. Specifically, a positive alpha of a fund means that an investor who initially holds a benchmark index fund can improve his Sharpe ratio by diverting a small fraction of his wealth into the fund.

The modified performance evaluation criteria proposed in this paper have been derived under the simplest possible assumptions: the log excess returns on each fund are identically and independently normally distributed over time.
(but not across funds). There is scope to explore the modifications required when the fund returns follow more general processes, as well as when fund managers have market timing ability or other kinds of superior information, as in Grinblatt and Titman (1989b). There is also scope for empirical work on the modified performance measures, particularly the instantaneous alpha, comparing the conclusions with the extensive literature on various versions of the discrete Jensen's alpha. Finally, the ideas of this paper have implications for tests of the CAPM. These extensions go beyond the boundaries of this paper, but we hope to address them in future work.
References


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