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The Newsvendor with Pricing: A Stochastic Elasticity Perspective

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We introduce a measure of elasticity of stochastic demand, called the elasticity of lost sales, which offers a unifying perspective on the well-known newsvendor with pricing problem. This new concept provides a framework to characterize structural results for coordinated and uncoordinated pricing and inventory strategies. Concavity and submodularity of the profit function and monotonicity of the inventory/price policies are characterized by monotonicity conditions, or bounds, on the lost sales elasticity. These elasticity conditions are satisfied by most relevant demand models in the literature. Our results explain, unify and complement previous work on price-setting newsvendor models.

1. Introduction

The literature on stochastic price-inventory problems is vast, but relatively scattered. Existing results and techniques are usually dependent on parametric, additive or multiplicative demand models and/or specific distributions of uncertainty. Even for the simplest newsvendor with pricing problem, a unified solution framework and general understanding of what drives existing results is still lacking. In particular, it would be desirable to understand what stochastic models of demand lead to tractable, unique and structured solutions.

Our goal is to characterize general models of stochastic price-dependent demand which allow a unified solution approach and structural results for price-inventory problems. We achieve this by introducing a new concept of stochastic demand elasticity, and showing that: (1) desirable structural properties of the objective function and optimal solution can be expressed as bounds or monotonicity conditions on elasticity, and (2) most of the relevant demand models used in the literature satisfy these elasticity conditions.

This paper focuses on the simplest static stochastic price-inventory model: the price-setting newsvendor with lost sales. Our elasticity characterization of stochastic price-dependent demand is not limited, however, to the operational model at hand, and we expect our results to be useful for modeling and analyzing other types of inventory-pricing problems as well. In a companion paper, we show that these results extend for more complex operational models, including flexible manufacturing and revenue management (Kocabiyikoglu and Popescu 2007).

Depending on industry and business context, price and quantity decisions are necessarily sequential, or can be coordinated, which motivates us to study both scenarios in this paper. Traditionally, in both industry and academic work, pricing and inventory decisions have been treated separately, with price as a marketing function and inventory (or production), as an operational decision. In various settings, price and inventory decisions are sequential, with marketing setting prices given inventory levels and/or operations setting inventory given list prices. For example, in long lead-time industries such as fashion, inventory decisions are necessarily made way ahead of pricing decisions, whereas production/inventory decisions for staple goods are made based on market prices. Recent advances have demonstrated significant profit benefits from the coordination of price and inventory decisions, in diverse industries such as consumer electronics, publishing and automotive.

We consider a single product newsvendor with pricing (NVP) model with lost sales, where the firm makes one pricing and/or inventory decision, and there is no opportunity to change price or replenish. Decisions are made before demand, modeled as a general stochastic function of price, is observed, and excess demand is lost.

We identify a new measure of demand elasticity, the elasticity of the rate of lost sales, which allows to characterize structural properties of the profit function, and of the optimal price and quantity policies. Specifically, if stochastic sales elasticity is greater than 1/2, the joint pricing and newsvendor problem is concave, hence easy to solve and admits a unique solution. A uniform elasticity lower bound of 1 characterizes submodularity of the profit function, hence monotonicity of the optimal price and quantity policies, in a sequential decision setting. Alternatively, these results are guaranteed by monotonicity of the sales elasticity in its respective arguments. We identify a general class of stochastic demand models that has monotone elasticity, and show that it includes the majority of models studied in the NVP literature. Thus our results unify, generalize and complement this literature.

The vast literature on coordinated pricing and inventory decisions has been reviewed by Yano and Gilbert (2003) and Chan, Shen, Simchi-Levi and Swann (2004), and specifically for the newsvendor problem by Petruzzi and Dada (1999). Some representative NVP works include Karlin and Carr (1962), Lau and Lau (1988), Mills (1959), Nevins (1966), Yao, Chen and Yan (2006), Young (1978) and Zabel (1970). These papers, and most of the literature, provide results for additive, multiplicative or additive-multiplicative demand models; our general stochastic price-dependent demand model encompasses and generalizes these. In a recent paper, Raz and Porteus (2006) argue that using a simplified, additive or multiplicative model can result in substantially lower profits, relative to a general one. They also use a general demand model, which (unlike ours) is specified up to a finite number of empirically estimated fractiles, assumed piecewise linear in price.

Our model falls in the class of static NVP decision models with stochastic demand and lost sales. Both in terms of model and results, our paper is closest to Petruzzi and Dada (1999), Yao, Chen and Yan (2006), Young (1978) and Zabel (1970), and generalizes the latter three. These papers also investigate uniqueness of the optimal coordinated price-quantity solution under a lost sales model with stochastic demand. In addition, we investigate properties of the price/quantity policies in a sequential decision framework; similar properties were studied by Zabel (1970) for pricing policies under multiplicative demand. A detailed comparison of our model and findings with these papers is postponed to § 6, once our main results are established.

Although the setup and results are not comparable, our contribution is similar in spirit to Ziya, Ayhan and Foley (2004) and Lariviere (2006), who discuss and unify important demand assumptions used for pricing and/or inventory problems. Their conditions translate desirable concavity results of a deterministic profit function into failure rate properties of the stochastic willingness to pay distribution underlying a deterministic price-demand function.

Several streams of NVP literature, albeit not comparable to our work, are worth mentioning. NVP models where all demand is served at the set price, and excess demand is backlogged at a price independent cost preclude lost sales as a special case, hence are not comparable to ours (e.g. Federgruen and Heching 1999, Agrawal and Seshadri 2000 and Chen and Simchi Levi 2003). Multi-period models are also beyond our scope (e.g. Karlin and Carr 1962, Mills 1962, Zabel 1972, Thowsen 1975, Federgruen and Heching 1999, Netessine 2006). Additional issues investigated in the NVP literature include the relation between stochastic and deterministic solutions, and comparative statics with respect to cost.

The rest of the paper is organized as follows. The model and underlying assumptions are presented in § 2. Sequential price and inventory policies are investigated in § 3, and the joint price-inventory model in § 4. Both sections translate properties of the optimal price-inventory policies into bounds and monotonicity conditions on elasticity. Equivalent characterizations of the stochastic sales elasticity and conditions for its monotonicity are presented in § 5. A discussion of our assumptions and detailed comparison with related NVP literature is provided in § 6. Finally, § 7 concludes the paper.

2. The Model

This paper considers a profit maximizing firm seeking to optimize inventory x and/or price p decisions for a single product. These decisions are made either sequentially or simultaneously, before observing an uncertain, price dependent demand, $\mathbf{D}(p)$, and excess demand is lost. For simplicity, we assume a constant unit cost c; all our results extend without loss of generality to increasing and convex cost functions $c(x)$. In a coordinated setting (studied in § 4), the firm jointly optimizes price and quantity decisions in order to maximize expected profit:

$$\max_{p,x} R(p,x), \text{ where } R(p,x) = p\mathbb{E}\left[\min(\mathbf{D}(p),x)\right] - c\,x, \tag{1}$$

where the constrained (or truncated) revenue is denoted by

$$r(p,x) = p\mathbb{E}\left[\min(\mathbf{D}(p),x)\right]. \tag{2}$$

In an uncoordinated environment (studied in § 3), a price-setting firm optimizes the objective in (1) with respect to price p for a given inventory x, whereas a quantity-setting firm optimizes the same objective with respect to x, for a given p.

Let $q(p,x) = 1 - F(p,x) = P(\mathbf{D}(p) \geq x)$ denote the probability of excess demand, for a given inventory x and price p. Prices are confined to a compact interval P so that $q(p,0) = 1$, i.e. $P(\mathbf{D}(p) \leq 0) = 0$ for all $p \in P$. In this model, the firm will not sell the product at a loss (because $R(0,0) = 0$), so we can restrict optimization wlog to $p \in [c,\infty) \cap P$. We also assume that the firm only considers stocking quantities $x \in X$ that can be cleared if sold at marginal cost, i.e. $q(c,x) = 1$ (this assumption is only used for the monotonicity of quantity with respect to price). For simplicity of exposition, our analysis focuses on the optimal unconstrained solution, ignoring boundary constraints on price and capacity; this is typical in the literature. All structural properties, however, are inherited by the corresponding truncated solutions.

The price dependent stochastic demand is modeled as

$$\mathbf{D}(p) = d(p,\mathbf{Z}). \tag{3}$$

The random variable $\mathbf{Z}$, referred to as demand *risk*, captures uncertainty about market conditions and has a continuously differentiable density function ϕ. The demand function $d(p,z)$ is assumed decreasing in price p, strictly increasing in z, and twice differentiable in p and z. Throughout the paper we use the terms increasing/decreasing, positive/negative in their weak sense. Monotonicity of demand in z allows to uniquely define the inverse function $z(p,x)$ such that $d(p,z(p,x)) = x$. Finally, we assume that the riskless (or pathwise) unconstrained revenue $\pi(p,z) = pd(p,z)$ is strictly concave in p for any risk realization z. Denoting partial derivatives by corresponding subscripts, this condition amounts to $2d_p(p,z) + pd_{pp}(p,z) < 0$.

The general demand model (3) encompasses the additive and multiplicative demand models commonly used in the newsvendor literature, as well as the more general additive-multiplicative model (see Young, 1978):

$$\mathbf{D}(p) = d(p, \mathbf{Z}) = \alpha(p)\mathbf{Z} + \beta(p), \tag{4}$$

where $\alpha(p), \beta(p)$ are decreasing functions of p. Note that for $\alpha(p) \equiv 1$ this is the additive model (price influences the location of the demand distribution), whereas for $\beta(p) \equiv 0$ this is the multiplicative model (price influences demand scale). Moreover, our setup (3) allows for more general demand models, e.g. $d(p, z) = log(z - bp)$.

The keystone for our developments is a new elasticity concept corresponding to stochastic demand, referred to as the *elasticity of lost sales*, or *stochastic demand elasticity.* Specifically, this is the elasticity of the rate of lost sales, or excess demand, defined as the percentage change in the rate of lost sales with respect to price:

Definition 1 *The elasticity of lost sales corresponding to a stochastic demand model* $\mathbf{D}(p)$ *is defined for any price* p *and inventory* x, *as*

$$\mathcal{E}(p, x) = -\frac{pq_p(p, x)}{q(p, x)} = \frac{pF_p(p, x)}{1 - F(p, x)}. \tag{5}$$

The elasticity of lost sales $\mathcal{E}(p, x)$ combines the relative sensitivity of sales with respect to its underlying factors, inventory and price. A detailed discussion and equivalent expressions for lost sales elasticity are provided in § 5.

3. Price-Inventory Interactions in a Sequential Decision Process

In most practical settings, price and inventory decisions are made sequentially. Price-setting firms (or marketing divisions) decide price as a function of inventory, whereas quantity-setting firms (or operations divisions) decide inventory levels based on pre-set prices. This amounts to optimizing $R(p, x)$ in (1) with respect to p, respectively x, with the other variable as a parameter. This section investigates the uniqueness and structure of the optimal price, respectively inventory policies. Specifically, we ask how to adjust price in response to a change in inventory levels, and production in response to a price change. Despite the asymmetry of the price and inventory-setting problems, the structure of the obtained results is surprisingly similar. This motivates us to present the two problems in parallel.

We first remind the well known optimality equations for price, respectively quantity, and show how they can be expressed in terms of lost sales elasticity:

Proposition 1 *(a) For any given quantity x, the profit function $R(p,x)$ is concave in p, and the optimal price $p^*(x)$ solves:*

$$\int_0^x q(p,v)(1-\mathcal{E}(p,v))dv = 0. \tag{6}$$

(b) For any given price p, $R(p,x)$ is concave in x, and the optimal order quantity $x^*(p)$ solves:

$$q(p,x) = c/p, \tag{7}$$

or equivalently $\int_c^p q(v,x)(1-\mathcal{E}(v,x))dv = 0.$

Proof: Proof. Because the minimum of concave functions is concave, $r(p,x) = \mathbb{E}[\min(\pi(p,\mathbf{Z}), p\,x)]$ is concave in p (because $\pi(p,z)$ is concave in p) and in x. Eq. (6) states the first order condition with respect to p. To see this, write (2) as $r(p,x) = p\int_0^x q(p,v)dv$, and hence

$$r_p(p,x) = \int_0^x \left(q(p,v) + pq_p(p,v)\right)dv = \int_0^x q(p,v)(1-\mathcal{E}(p,v))dv. \tag{8}$$

The first order condition with respect to x gives the well known equation (7). As above, because $q(c,x) = 1$, we can write this condition as

$$r_x(p,x) = pq(p,x) - c = \int_c^p \frac{\partial}{\partial v}(vq(v,x) - c)dv = \int_c^p q(v,x)(1-\mathcal{E}(v,x))dv. \tag{9}$$

■

Uniqueness of $x^*(p)$ allows to denote the evaluation of any generic function f along the optimal path $x^*(p)$ as $f^*(p) = f(p,x)|_{x=x^*(p)}$. Similarly, we can write $f^*(x) = f(p,x)|_{p=p^*(x)}$. This is a slight abuse of notation; however, the generic argument of f^* makes the evaluation path unambiguous. We also denote $f^*_x(p) = f_x(p,x)|_{x=x^*(p)}$, the derivative of $f(p,x)$ with respect to x evaluated at the optimal quantity, and $f^*_x(x) = f_x(p,x)|_{p=p^*(x)}$, the derivative evaluated at the optimal price. In this notation, used throughout the paper, the derivative always precedes functional evaluation.

Recall, $z(p,x)$ was defined such that $d(p,z(p,x)) = x$. In particular, for a quantity-setting firm, $z^*(p) = z(p,x^*(p))$, and the optimality condition (7) amounts to $P(\mathbf{Z} \geq z^*(p)) = c/p$. This shows that under the optimal inventory policy, the demand "risk" $z^*(p)$ associated with the optimal inventory level $x^*(p)$ is increasing in price p.

We next investigate sensitivity properties of the optimal price $p^*(x)$, respectively quantity $x^*(p)$ policies; these results will also be useful in characterizing the solution of the coordinated problem in the next section. Economic theory indicates that lower prices lead to higher mean demand, and consequently drive up the preferred inventory levels. On the other hand, stochastic inventory theory predicts that lower prices lead to lower safety stocks (as a lower price reduces the underage cost of shortages relative to the overage cost of leftovers), driving down the preferred inventory levels. These two arguments suggest that under stochastic demand the relationship between price and quantity is ambiguous, as acknowledged by Zabel (1970). Our counterexamples at the end of this section also illustrate this fact.

The next result shows that monotonicity of the optimal price $p^*(x)$, respectively quantity $x^*(p)$ policies can be guaranteed by a uniform lower bound of one on the lost sales elasticity. In general, a strong sufficient condition for comparative statics is submodularity of the objective function, guaranteed by Topkis' theorem (see Topkis 1998, Theorem 2.8.2). A function $g(x,y)$ is said to be submodular if it has monotone decreasing differences, that is $g(x^+,y^+) - g(x^+,y^-) \leq g(x^-,y^+) - g(x^-,y^-)$ for all $x^+ \geq x^-, y^+ \geq y^-$. For differentiable functions, submodularity is equivalent to a negative cross-derivative. Supermodularity is defined by the opposite inequality.

Proposition 2 *The profit (revenue) function is submodular if and only if $\mathcal{E} \geq 1$. In this case the inventory and pricing policies $x^*(p)$ and $p^*(x)$ are decreasing in their respective arguments.*

Proof: Proof. Marginal revenue is $r_x(p,x) = pq(p,x)$, which together with (5) gives:

$$r_{xp} = q(p,x) + pq_p(p,x) = q(p,x)(1 - \mathcal{E}(p,x)) \leq 0, \tag{10}$$

whenever $\mathcal{E} \geq 1$. Sensitivity results follow by Topkis' Theorem. ■

Submodularity of the revenue function is sufficient, but not necessary for monotonicity of the price, respectively inventory policies. Necessary and sufficient conditions can be expressed as bounds on the sales elasticity along the optimal decision path.

Proposition 3 *(a) $p^*(x)$ is decreasing in x if and only if $\mathcal{E}^*(x) \geq 1$.*
(b) $x^*(p)$ is decreasing in p if and only if $\mathcal{E}^*(p) \geq 1$.

Proof: Proof. Optimality of $p^*(x)$ and the implicit function theorem imply that $p^*(x)$ is decreasing in x whenever:

$$R^*_{xp}(x) = q^*(x)(1 - \mathcal{E}^*(x)) \leq 0, \tag{11}$$

i.e. whenever $\mathcal{E}^*(x) \geq 1$. The second part is analogous. In particular, by the implicit function theorem, we obtain $x^*(p) = d^*_p(p)(1 - 1/\mathcal{E}^*(p))$; an expression for $p^*(x)$ can also be obtained, but is less compact. ■

Proposition 3 provides necessary and sufficient conditions characterizing the sensitivity of price, respectively inventory decisions in uncoordinated settings. These conditions may not always be easy to check, as they require evaluating/optimizing the lost sales elasticity along the optimal policy path. Sufficient conditions that are usually easier to verify can be expressed in terms of monotonicity of $\mathcal{E}(p,x)$ in x, respectively p. Stochastic demand models with monotone lost sales elasticity are presented in § 5.

Proposition 4 *(a) If $\mathcal{E}(p,x)$ is increasing in x, then $p^*(x)$ is decreasing in x.*

(b) If $\mathcal{E}(p,x)$ is increasing in p, then $x^*(p)$ is decreasing in p.

Proof: Proof. (a) Write $r_p(p,x) = \int_0^x Q(p,v)dv$, where $Q(p,x) = q(p,x)(1 - \mathcal{E}(p,x))$. Because $\mathcal{E}(p,x)$ is increasing in x and $q(p,x) \geq 0$, for any p, $Q(p,x)$ crosses zero at most once, and from above. Therefore the first order condition $0 = \int_0^x Q(p^*(x),v)dv$, implies that $Q^*(x) = Q(p^*(x),x) \leq 0$, that is $\mathcal{E}^*(x) \geq 1$. The result follows by Proposition 3.

(b) Similarly, $\mathcal{E}(p,x)$ increasing in p implies $Q(p,x)$ crosses zero at most once, and from above as p increases. The result follows again by Proposition 3, as the first order condition $0 = \int_c^p Q(v, x^*(p))dv$ implies that $Q^*(p) = Q(p, x^*(p)) \leq 0$, i.e. $\mathcal{E}^*(p) \geq 1$. ■

Counterexamples. Figures 1 and 2 illustrate non-monotonicity of the optimal quantity, respectively price policies, for simple demand models that violate the conditions of Proposition 4.

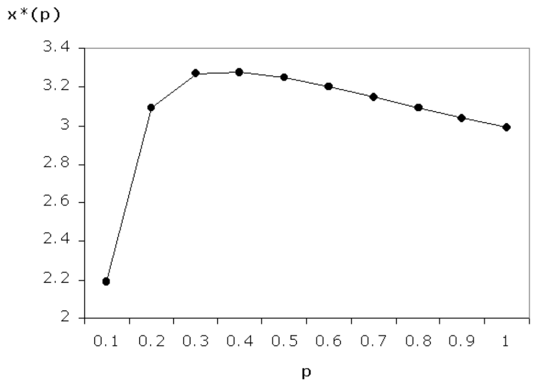

Figure 1: Non-decreasing optimal quantity path $x^*(p)$ for $\mathbf{D}(p) = p^{-2}\mathbf{Z}$, $\mathbf{Z} \sim Uniform(0,1)$.

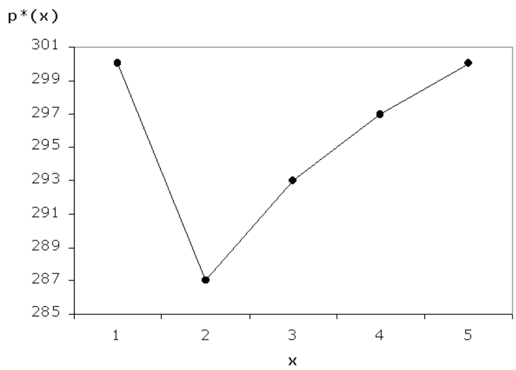

Figure 2: Non-decreasing optimal price path $p^*(x)$ for $\mathbf{D}(p) = 20 - 0.5p + \mathbf{Z}, \phi(z) = 0.5/\sqrt{z}$.

4. Simultaneous Price-Inventory Optimization

This section focuses on jointly optimized price and quantity decisions. We present three alternative conditions for Problem (1) to admit a unique solution (p^{**}, x^{**}). First, a uniform elasticity (lower) bound of 1/2 is proved sufficient for joint concavity of the profit function, and hence uniqueness of the optimal price-quantity solution.

Proposition 5 *If $\mathcal{E} \geq 1/2$, then $R(p,x)$ is jointly concave.*

Proof: Proof. To show that the Hessian matrix of $R(p,x)$ is negative semi-definite, by Proposition 1, it remains to check that its determinant $\Delta(p,x)$ is positive. The second order derivatives are:

$$R_{xx}(p,x) = -pf(p,x) \tag{12}$$

$$R_{xp}(p,x) = q(p,x) + pf(p,x)d_p(p,z) \tag{13}$$

$$R_{pp}(p,x) = \mathbb{E}\left[\pi_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] - pf(p,x)d_p^2(p,z(p,x)), \tag{14}$$

where $\mathbf{\Omega} = \mathbf{\Omega}(p,x) = (\mathbf{D}(p) \leq x)$ defines the event of no lost sales (arguments are omitted for notational convenience). Equation (14) is obtained by differentiating twice, with respect to p, the objective in (1), written as:

$$R(p,x) = pxq(p,x) + \mathbb{E}\left[\pi(p,\mathbf{Z});\mathbf{\Omega}\right] - cx. \tag{15}$$

Pairing up terms, we obtain:

$$\Delta(p,x) = R_{xx}(p,x)R_{pp}(p,x) - R_{xp}^2(p,x) \tag{16}$$

$$= -pf(p,x)\mathbb{E}\left[\pi_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] - q(p,x)\left[q(p,x) + 2pf(p,x)d_p(p,z(p,x))\right] \tag{17}$$

At this point, we use the following equivalent expression for lost sales elasticity which obtains from writing $F_p(p,x) = \frac{\partial}{\partial p}P(d(p,\mathbf{Z}) \geq x) = -f(p,x)d_p(p,z(p,x))$ in the definition:

$$\mathcal{E}(p,x) = \frac{-pf(p,x)d_p(p,z(p,x))}{q(p,x)}. \tag{18}$$

This allows to rewrite (17) as:

$$\Delta(p,x) = -pf(p,x)\mathbb{E}\left[\pi_{pp}(p,\mathbf{Z});\mathbf{\Omega}\right] + q(p,x)^2(2\mathcal{E}(p,x) - 1). \tag{19}$$

The first term is positive by concavity of π and the second because $\mathcal{E} \geq 1/2$. ■

To appreciate the unexpected strength of the result, remark that it does not hold for deterministic demand. Indeed, the deterministic revenue function $p\min(d(p,z(p,x)),x)$ is not jointly concave in (p,x), even when $\pi(p,z) = pd(p,z)$ is concave. With sufficient variability in excess demand, guaranteed by the elasticity bound, the extreme effect of the deterministic case can be smoothed out. For example, suppose that $\mathbf{Z}$ has an exponential distribution with cdf $1 - e^{-\lambda z}$ (i.e. constant failure rate λ). If $\mathbf{D}(p) = a - bp + \mathbf{Z}$, then $\mathcal{E} \geq 1/2$ holds for sufficiently large prices $p > 1/(2\lambda b)$. If $\mathbf{D}(p) = \mathbf{Z} - log(p)$, then $\mathcal{E} \equiv \lambda$, so the price-inventory solution is unique for $\lambda \geq 1/2$.

Joint concavity of $R(p, x)$, while sufficient, is not necessary for existence of a unique optimal solution for Problem (1). Weaker sufficient conditions include concavity of the optimal revenue along one of the optimal paths $x^*(p)$ or $p^*(x)$, denoted $R^*(p) = R(p, x^*(p))$, respectively $R^*(x) = R(p^*(x), x)$. This is guaranteed by an elasticity bound of 1/2 along the optimal path.

Proposition 6 *(a) If $\mathcal{E}^*(x) \geq 1/2$, then profit along the optimal price path, $R^*(x)$ is concave in x.*

(b) If $\mathcal{E}^*(p) \geq 1/2$, then profit along the optimal quantity path, $R^*(p)$ is concave in p.

Proof: Proof. By the envelope theorem, we have $\frac{\partial^2}{\partial x^2} R^*(x) = R_{xx}(p, x) - \left. \frac{R_{xp}^2(p, x)}{R_{pp}(p, x)} \right|_{p=p^*(x)}$. This is negative because $R_{pp}^*(x) < 0$ by optimality of $p^*(x)$, and $\Delta^*(x) \geq 0$ by (16), (19) and the elasticity bound. The second part is analogous. ■

These results allow to solve Problem (1) numerically as a one dimensional concave optimization problem, and guarantee uniqueness of the optimal solution. Alternatively, monotonicity of elasticity $\mathcal{E}(p, x)$ in p or x guarantees existence of a unique optimum for this problem.

Proposition 7 *If $\mathcal{E}(p, x)$ is increasing in x, or alternatively, in p, then Problem (1) has a unique price-allocation solution (p^{**}, x^{**}).*

Proof: Proof. By Propositions 3 and 4, monotonicity of $\mathcal{E}(p, x)$ in p or x guarantees the elasticity bound required by Proposition 6, which completes the proof. ■

5. Stochastic Demand Models and Elasticity Conditions

Our findings so far lead us to conclude that the relevant results concerning the NVP problem are driven by monotonicity of lost sales elasticity. It remains to characterize which stochastic price-demand models satisfy this important property.

We begin by providing several alternative characterizations of the lost sales elasticity $\mathcal{E}(p, x)$. These can be cast in terms of the riskless elasticity of demand and the failure rate of the demand distribution. Table 1 gives the notation and relationship between the distributions of $\mathbf{Z}$ and $\mathbf{D}(p) = d(p, \mathbf{Z})$, as well as their failure (or hazard) rates (FR) and generalized failure rates (GFR). A distribution has IFR, respectively IGFR if it has increasing FR, respectively increasing GFR. We define the (absolute) elasticity of riskless demand with respect to (1) price, as $\tilde{\epsilon}_P(p,z) = -\frac{pd_p(p,z)}{d(p,z)}$ and (2) risk, as $\tilde{\epsilon}_Z(p,z) = \frac{zd_z(p,z)}{d(p,z)}$. For simplicity, elasticity here always refers to its absolute value.

Table 1: Notation and relationships for $\mathbf{Z}$ and $\mathbf{D}(p)$.

	$\mathbf{D}(p) = d(p,\mathbf{Z})$	$\mathbf{Z}$	Relationship for $z = z(p,x)$
Density	$f(p,x)$	$\phi(z)$	$\phi(z) = f(p,x)d_z(p,z)$
CDF	$F(p,x)$	$\Phi(z)$	$F(p,x) = \Phi(z)$
FR	$h_D(p,x) = \frac{f(p,x)}{1-F(p,x)}$	$h_Z(z) = \frac{\phi(z)}{1-\Phi(z)}$	$h_Z(z) = h_D(p,x)d_z(p,z)$
GFR	$g_D(p,x) = xh_D(p,x)$	$g_Z(z) = zh_Z(z)$	$g_Z(z) = g_D(p,x)\tilde{\epsilon}_Z(p,z)$

Let $\epsilon_P(p,x) = \tilde{\epsilon}_P(p,z(p,x)) = -pd_p(p,z(p,x))/x$, and similarly we define $\epsilon_Z(p,x)$. From (18), an alternative expression for lost sales elasticity can be obtained as the product of the generalized failure rate and the price elasticity of riskless demand:

$$\mathcal{E}(p,x) = g_D(p,x)\epsilon_P(p,x). \tag{20}$$

Lost sales elasticity $\mathcal{E}(p,x)$ combines the relative sensitivity of demand with respect to its underlying factors, risk (z) and price (p), at a given inventory level x. There is a natural correspondence between elasticity and GFR (see Lariviere 2006). GFR is the percentage change in the excess demand rate with respect to stocking quantity, which can be interpreted as the elasticity of the survival function (excess demand rate) with respect to quantity. Riskless price elasticity is the percentage change in riskless demand with respect to price, which is precisely the GFR of the willingness to pay distribution. Indeed, if $\mathbf{W}(z)$ denotes the willingness to pay distribution when market conditions are given by z, and corresponding market size is $N(z)$, then $d(p,z) = N(z)P(\mathbf{W}(z) \geq p)$ and hence $\epsilon_P(p,x) = g_W(p,x)$.

Let $\tilde{\mathcal{E}}(p,z) = \mathcal{E}(p,d(p,z))$ denote the elasticity for a given risk level z, i.e. $\tilde{\mathcal{E}}(p,z)$ is such that $\mathcal{E}(p,x) = \tilde{\mathcal{E}}(p,z(p,x))$. Furthermore, denote $\tilde{\epsilon}_{PZ}(p,z) = \frac{\tilde{\epsilon}_P(p,z)}{\tilde{\epsilon}_Z(p,z)}$ the cross elasticity of riskless demand with respect to price (p) and risk (z), and $\delta(p,z) = -p\frac{d_p(p,z)}{d_z(p,z)}$. From Table 1, an alternative expression for $\tilde{\mathcal{E}}(p,z)$ obtains in terms of the GFR of $\mathbf{Z}$, $\tilde{\mathcal{E}}(p,z) = g_Z(z)\tilde{\epsilon}_{PZ}(p,z)$.

5.1 Demand Models with Monotone Elasticity

We next present conditions for monotonicity of elasticity $\mathcal{E}(p,x)$ in quantity x and price p, respectively, which are the key drivers of concavity and sensitivity results obtained in the previous sections.

The following observation is repeatedly used for our results, and obtains by differentiating $d(p,z(p,x)) = x$ on both sides with respect to p.

Lemma 1 *The amount of risk z that induces a given demand level x is increasing in price:*

$$z_p(p,x) = \frac{-d_p(p,z(p,x))}{d_z(p,z(p,x))} \geq 0. \tag{21}$$

We next provide results relating monotone elasticity and failure rate of demand $\mathbf{D}(p)$. First, lost sales elasticity $\mathcal{E}(p,x)$ is higher for higher quantities if and only if demand failure rate $h_D(p,x)$ is increasing in price. Alternative sufficient conditions are cast in terms of the GFR of $\mathbf{D}(p)$ and riskless price elasticity.

Proposition 8 *(a) $\mathcal{E}(p,x)$ is increasing in x if and only if $\mathbf{D}(p)$ has IFR, that is, whenever $\mathbf{D}(p)$ is decreasing in p in the hazard rate order.*[1]

(b) If $\mathbf{D}(p)$ has IGFR, and $\tilde{\epsilon}_P(p,z)$ is increasing in z, then $\mathcal{E}(p,x)$ is increasing in x. If moreover $\tilde{\epsilon}_P(p,z)$ is increasing in p, then $\mathcal{E}(p,x)$ is also increasing in p.

Proof: Proof. (a) Because $\mathcal{E}(p,x) = p\frac{-q_p(p,x)}{q(p,x)} = -p\frac{\partial}{\partial p}\log q(p,x)$, we obtain

$$\frac{\partial}{\partial x}\frac{\mathcal{E}(p,x)}{p} = -\frac{\partial}{\partial p}\frac{\partial}{\partial x}\log q(p,x) = \frac{\partial}{\partial p}\frac{f(p,x)}{q(p,x)} = \frac{\partial}{\partial p}h_D(p,x).$$

Therefore monotonicity of elasticity in x is equivalent to monotonicity of h_D in p.

(b) The first part obtains from the elasticity expression (20) and Lemma 1. The second part follows by observing that $\tilde{\epsilon}_P(p,z)$ increasing in p and z implies $\epsilon_P(p,x)$ is increasing in p. This obtains by differentiating $\epsilon_P(p,x) = \tilde{\epsilon}_P(p,z(p,x))$ with respect to p, and using again Lemma 1. ■

We next provide sufficient conditions in terms of the riskless demand d and risk distribution $\mathbf{Z}$ for sales elasticity to be increasing in x and p. These conditions are general, and easy to check in an application context.

Proposition 9 *Suppose that $\mathbf{Z}$ has IFR. If $d_{zz} \leq 0$ and $d_{pz} \leq 0$, then $\mathcal{E}(p,x)$ is increasing in x. If moreover $\epsilon_P(p,x)$ is increasing in p, in particular if $d_p + pd_{pp} \leq 0$, then $\mathcal{E}(p,x)$ increases in p.*

Proof: Proof. We show that h_D is increasing in p, which implies, via Proposition 8, the desired result. From Table 1,

$$h_D(p,x) = \frac{h_Z(z(p,x))}{d_z(p,z(p,x))}. \tag{22}$$

Because $\mathbf{Z}$ has IFR, using Lemma 1 we obtain that the nominator is increasing in p. Moreover, from Lemma 1 and the assumptions on d we obtain:

$$\frac{\partial}{\partial p}d_z(p,z(p,x)) = d_{zp}(p,z(p,x)) - d_{zz}(p,z(p,x))\frac{d_p(p,z(p,x))}{d_z(p,z(p,x))} \leq 0, \tag{23}$$

[1] For two nonnegative random variables, $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the hazard rate order ($\mathbf{Y} \succeq_{HR} \mathbf{X}$) if and only if their respective hazard rates satisfy $h_{\mathbf{X}}(z) \geq h_{\mathbf{Y}}(z)$ (Müller and Stoyan, 2002).

so the (positive) denominator in (22) is decreasing in p. This concludes the first part.

For the second part, we write $\mathcal{E}(p,x) = g_D(p,x)\epsilon_P(p,x) = xh_D(p,x)\epsilon_P(p,x)$. From the first part h_D is increasing in p. Hence monotonicity of $\epsilon_P(p,x)$ in p is preserved by $\mathcal{E}(p,x)$. In particular, this is guaranteed if d_{pz} and $d_p + pd_{pp}$ are negative. ■

Proposition 9 assumes that **Z** has IFR. IFR distributions are those with logconcave survival functions, including Gamma, Uniform, Exponential, Normal and truncated Normal. For more on IFR distributions, see Barlow and Proschan (1996), Lariviere (2006) or Lariviere and Porteus (2001).

The sufficient conditions on riskless demand assume that (1) risk has diminishing marginal impact on riskless demand ($d_{zz} \leq 0$) and (2) the marginal demand stimulation from lowering price is larger at higher risk levels ($d_{zp} \leq 0$). The additional condition required for monotonicity of $\mathcal{E}(p,x)$ in p imposes monotonicity in p of the riskless elasticity counterpart, and holds for example if d is concave in p. In particular, the condition $d_p + pd_{pp} \leq 0$ insures concavity of π, assumed in previous sections.

Some simple demand models that violate Proposition 9 and have non-monotone elasticity were presented in the counterexamples in § 3 (Figures 1 and 2). Indeed, in the isoelastic multiplicative demand model used in Figure 1, **Z** is uniform (0,1) so it has IFR, and $d_{pz} = -2p^{-3} \leq 0$, $d_{zz} = 0$. However, $pd_p = -2p^{-2}z$ is increasing in p, thus violating the condition $d_p + pd_{pp} \leq 0$ of Proposition 9. The linear additive demand model used for Figure 2 satisfies $d_{zz}, d_{pz} \leq 0$, but **Z** is not IFR (power distribution).

The next result presents some common demand models that have monotone sales elasticity, without necessarily satisfying the conditions of Proposition 9. Part (a) is simply a special case of Proposition 9 for additive-multiplicative models. Part (b) provides conditions where **Z** may not have IFR. The last part gives an example of a model that is not additive-multiplicative. Note, $d_{zp}(p,z) \geq 0$ in (c), violating the assumptions of Proposition 9.

Corollary 1 *(a) If* **Z** *has IFR and* $d(p,z) = \alpha(p)z + \beta(p)$, *then* $\mathcal{E}(p,x)$ *is increasing in* x. *If* $p\alpha'(p)$ *and* $p\beta'(p)$ *are decreasing in* p *then* $\mathcal{E}(p,x)$ *is also increasing in* p.

(b) If **Z** *has IGFR,* $d(p,z) = \alpha(p)z$ *then* $\mathcal{E}(p,x)$ *is increasing in* x. *If moreover* $\alpha(p)$ *has increasing price-elasticity then* $\mathcal{E}(p,x)$ *is also increasing in* p.

(c) If **Z** *has IFR and* $d(p,z) = log(z - bp)$, *then* $\mathcal{E}(p,x)$ *is increasing in both* x *and* p.

Proof: Proof. Part (b) follows from Proposition 8 (b) because under the multiplicative model

$\tilde{\epsilon}_P(p,z)$ is the elasticity of $\alpha(p)$, which in particular is independent of z. For part (c), we write:

$$\mathcal{E}(p,x) = \tilde{\mathcal{E}}(p, z(p,x)) = h_Z(z(p,x))\delta(p, z(p,x)), \tag{24}$$

where recall that $\delta(p,z) = -p\frac{d_p(p,z)}{d_z(p,z)}$. Because $z(p,x)$ is increasing in p (by Lemma 1) and x, if $\mathbf{Z}$ has IFR, it is sufficient to verify $\delta(p, z(p,x))$ is increasing in x (i.e. $\delta(p,z)$ increasing in z), respectively in p. This is a simple exercise left to the reader. ∎

6. Discussion on Assumptions and Relation with the Literature

We next discuss how our assumptions, model and results relate to the literature. As discussed in the introduction, the NVP papers that come closest to ours are Petruzzi and Dada (1999), Yao et al. (2005), Young (1978) and Zabel (1970). Their underlying demand models are either additive or multiplicative, except Young (1978) who studies a joint additive-multiplicative model. These papers provide conditions for a unique optimal coordinated solution for the NVP problem. Their results rely on specific parametric demand models and/or distributions. In contrast, we provide a general approach and unifying conditions for a larger class of demand models (with increasing stochastic elasticity).

Two assumptions enable us to obtain uniqueness and sensitivity results under general demand conditions: (1) monotonicity of stochastic sales elasticity, $\mathcal{E}(p,x)$ in p and x, and (2) concavity of the riskless profit function, $\pi(p)$. Monotonicity of $\mathcal{E}(p,x)$ insures lower bounds on $\mathcal{E}(p,x)$ along the corresponding optimal policy paths. These are used to obtain uniqueness of the optimal price-inventory solution (p^{**}, x^{**}), and, respectively, monotonicity of $p^*(x)$ and $x^*(p)$ in their respective arguments.

Concavity of π is not necessary for all our results (in particular it is not used for monotonicity of the price and inventory policies), but makes the analysis of a general demand model simpler, and more elegant. This assumption is used in Proposition 1 to obtain uniqueness of $p^*(x)$, and further in Propositions 5 and 6 in order to obtain positivity of the determinant, and hence the desired concavity properties used further to obtain uniqueness of the price-inventory solution. Some specific forms of additive or multiplicative demand allow for weaker concavity assumptions on riskless profit $\pi(p)$. For example, Yao et al. (2006) and Petruzzi and Dada (1999) only use quasiconcavity of riskless profit. Their respective parametric demand models allow to obtain optimal quantity $x^*(p)$, respectively price $p^*(x)$ policies in closed form. These expressions are further substituted into the profit function, and the resulting univariate functions are proved unimodal with respect to price (Yao et al.), respectively quantity (Petruzzi and Dada).

Petruzzi and Dada (1999) study linear-additive ($d(p,z) = a - bp + z,\ a > 0, b > 0$) and isoelastic-multiplicative ($d(p,z) = ap^{-b}z$, $a > 0, b < 1$) demand models, and assume that $\mathbf{Z}$ satisfies $2h_Z^2 + h_Z' \geq 0$. This condition is satisfied by all IFR distributions, but not comparable to IGFR. On the other hand, IGFR of $\mathbf{Z}$ is a sufficient condition for both models, according to our results. Indeed: (1) in the isoelastic-multiplicative case, elasticity $\mathcal{E} = bg_{\mathbf{Z}}(p^b x/a)$ is monotone in x if and only if $\mathbf{Z}$ has IGFR, and (2) in the linear-additive case, elasticity $\mathcal{E} = bph_{\mathbf{Z}}(x - a + bp)$ is monotone in p if and only if $\mathbf{Z}$ has IGFR (monotonicity in x amounts to IFR). Therefore, Petruzzi and Dada's results are generally not comparable to ours. Their results are stronger than some of the special conditions in our paper involving IFR of $\mathbf{Z}$, including Proposition 9 and monotonicity of elasticity in x for the the linear-additive case.

Yao et al. (2006) set up the NVP problem with a general demand model, but their analysis and results are limited to additive ($\mathbf{D}(p) = \beta(p) + \mathbf{Z}$) and multiplicative ($\mathbf{D}(p) = \alpha(p)\mathbf{Z}$) models. They make two additional assumptions on demand: (1) price elasticity of $\alpha(p)$, respectively $\beta(p)$, is increasing for the multiplicative, respectively additive demand model, and (2) $\mathbf{Z}$ has IFR. From Proposition 1, under these assumptions, $\mathcal{E}(p,x)$ is monotone in both p and x. In particular, part (a) of our Corollary 1 encompasses the additive model of Yao et al. (2006), whereas their multiplicative model is captured by part (b). In terms of approach, the optimal price-inventory policy is obtained by optimizing quantity with respect to price first, a sequence also used by Whitin (1955), under a deterministic demand model.

Our results also fully generalize those of Zabel (1970) and Young (1978). These papers optimize price first, with quantity as a parameter, and then solve the resulting univariate problem with respect to quantity along the optimal price path. For multiplicative demand, Zabel (1970) also presents conditions under which the optimal price is monotone in quantity in a sequential decision process. Their analysis relies on specific assumptions on the underlying demand distribution (uniform/exponential in Zabel 1970; distributions with logconcave densities in Young 1978), whereas our approach is general, resting on the general properties of stochastic elasticity.

7. Conclusion

We introduce a new elasticity concept that provides a framework to study pricing and inventory decisions, in the context of the single product price setting newsvendor model. Our key contribution is to characterize general models of stochastic price dependent demand, that guarantee relevant structural properties of the optimal price and inventory policies. These properties include uniqueness of the joint price-inventory solution in a coordinated system, as well as uniqueness and monotonicity properties of the optimal pricing, respectively quantity policies in a sequential decision framework. We achieve this by identifying a new measure of stochastic demand elasticity, the elasticity of (the rate of) lost sales, whose properties drive the desired results, as summarized on Table 2. We further characterize general classes of demand models that satisfy these elasticity conditions. These conditions unify, generalize and complement assumptions commonly made in the literature, such as additive-multiplicative models and failure rate conditions. We expect these results to be useful in modeling stochastic, price-dependent demand and solving other types of inventory-pricing problems. For example, Kocabiyikoglu and Popescu (2007) extend the current results in more complex operational contexts, including flexible manufacturing and revenue management.

Table 2: Summary of our main results.

$\mathcal{E}(p,x)$	$R(p,x)$	$p^*(x)$	$x^*(p)$	(p^{**},x^{**})
$\geq \frac{1}{2}$	joint concave	-	-	unique
≥ 1	submodular	$\downarrow$	$\downarrow$	unique
$\uparrow x$	-	$\downarrow$	-	unique
$\uparrow p$	-	-	$\downarrow$	unique

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