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We analyze a dynamic pricing problem where consumer's purchase decisions are affected by representative past prices, summarized in a reference price. We propose a new, behaviorally motivated reference price mechanism, based on the peak-end memory model proposed by Fredrickson and Kahneman (1993). Specifically, we assume that consumers' reference price is a weighted average of the lowest and last price. Gain or loss perceptions with respect to this reference price affect consumer purchase decisions in the spirit of prospect theory, resulting in a non-smooth demand function. We investigate how these behavioral patterns in consumer anchoring and decision processes affect the optimal dynamic pricing policy of the firm. In contrast with previous literature, we show that peak-end anchoring leads to a range of optimal constant pricing policies even with loss neutral buyers. This range becomes wider if consumers are loss averse. In general, we show that skimming or penetration strategies are optimal, i.e. the transient pricing policy is monotone, and converges to a steady state, which depends on the initial price perception. The value of the steady state price decreases, the more consumers are sensitive to price changes, and the more they anchor on the lowest price.

Key words: Dynamic Pricing, Deterministic Dynamic Programming, Behavioral Consumer Theory, Prospect Theory, Peak-end Rule.

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1. Introduction

In repeat-purchase markets, consumers form price expectations, also known as reference prices. Actual prices are perceived as discounts or surcharges relative to these reference prices, according to prospect theory (Tversky and Kahneman 1991); this perception affects the demand for a firm's product, and hence its profitability. As a result, for example, the often practised discounts by firms, while usually profitable in the short term, may erode consumers' price expectations, and willingness to pay, and thereby negatively affect long run profitability. It is therefore important for a firm to understand (1) how its pricing policy affects consumers' price expectations and purchase decisions, and (2) how to set prices over time to maximize profitability in this context.

There is a growing body of marketing, economics and operations literature studying how firms should optimally set prices when consumers anchor on past prices (Kopalle et al. 1996, Fibich et al. 2003, Heidhues and Kőszegi 2005, Popescu and Wu 2007). This literature builds on behavioral theories (prospect theory, mental accounting, etc.) to model demand dependence on reference prices, but pays little attention to the anchoring and memory processes, i.e. how reference prices are formed. A common assumption in the literature is that consumers' reference price is a weighted average of past prices. Specifically, the anchoring mechanism follows an exponentially smoothed process (see Mazumdar et al. 2005 for a review).

This paper proposes a different, behaviorally motivated anchoring mechanism, whereby consumers anchor on the lowest and most recent prices observed, and the reference price is a weighted average of the two. This assumption is motivated by the *peak-end rule*, a "snapshot model" of remembered utility, proposed by Fredrickson and Kahneman (1993). The goal of this paper is to understand how this consumer anchoring process influences the optimal pricing strategies of the firm. We investigate the robustness of state of the art results on pricing with reference effects (in particular those obtained by Popescu and Wu, 2007), with respect to this reference price formation mechanism. We seek to understand how behavioral regularities, such as loss aversion and peak-end anchoring, interact in determining the structure of the optimal pricing policy. Finally, we aim to provide insights as to which aspects of consumer behavior the firm should assess in order to optimize pricing decisions.

Behavioral motivation and evidence. The marketing literature finds a strong empirical validation of reference dependence and prospect theory in consumer purchase decisions, reviewed in Kalyanaram and Winer (1995). The role of historic prices in forming price expectations is supported in many empirical studies. Winer (1986), Greenleaf (1995), and Erdem et al. (2001) find evidence for an adaptive expectations framework, where the reference price is an exponentially smoothed average of past prices. Due to its simplicity, this has been the memory model used throughout the dynamic pricing literature. However, as pointed out in Rabin (1996), this model is "not very well grounded in behavioral evidence" (page 8).

Our work is motivated by a different memory model: the peak-end rule, proposed by Fredrickson and Kahneman (1993). This model hypothesizes that the overall evaluation of past experiences is based on representative moments of experience, so called "moment based approach". These authors propose an averaging scheme, whereby most experiences are assigned zero weight, and a few most relevant snapshots, specifically the peak and the end (i.e. the strongest and the most recent experience) receive positive weights. The psychological literature supporting the peak-end rule is reviewed in Kahneman (2000). Peak-end anchoring models have been empirically validated in various psychological contexts, including pain recall, advertisements, social interactions, receipt of money payments.

The moment based approach to remembered utility has also been evidenced in other economic contexts. are affected not only by current income, but also by past income levels among which the highest one is the most relevant. Oest and Paap (2004) propose a model where each household's reference price is an average of past prices recalled by that household. Therefore, each household is hypothesized to anchor on selective moments to form its reference price. They apply the model to estimate the average recall probabilities of specific past prices based on scanner data.

In the pricing context, we posit that the representative peak-end moments in reference price formation are associated with the lowest and the last price, i.e. the highest and the most recent transaction utility. While the empirical validity of the peak-end rule in the pricing context is yet to be investigated, several studies find support for anchoring on most recent and extreme prices. Uhl (1970), Krishnamurti et al. (1992), and Chang et al. (1999) find support for the last price as a reference point. Nwokoye (1975) reports that some consumers anchor on extreme prices, i.e. the lowest and the highest, in their price judgements. The reference price is also an indicator of what consumers may consider as a “fair price”, and consumers’ perception of fairness is typically anchored on the lowest prices (see e.g. Xia et al. 2004).

Contribution and relation with the literature. This paper fits in a growing body of behavioral operations literature, reviewed by Loch and Wu (2007), Gino and Pisano (forthcoming), and Su (forthcoming). Our work contributes to the pricing and revenue management (RM) literature, for which a comprehensive reference is the book by Talluri and van Ryzin (2004); see also Bitran and Caldentey (2003) for a review. A recent review on modeling customer behavior in RM and auctions is due to Shen and Su (2007). Within the RM literature, our work contributes to the behavioral pricing stream, and comes closest to Kopalle et al. (1996), Fibich et al. (2003), and especially Popescu and Wu (2007) reviewed below.

Other work in behavioral operations that incorporates consumer learning models includes Gaur and Park (2007; consumers learn fill rates), Liu and van Ryzin (2007; consumers learn about rationing risk), and Ovchinnikov and Milner (2005; consumers learn about the likelihood of last-minute sales).

There are a few other papers that study the dependence of demand on past prices. Ahn et al. (2007) propose a model where demand in a period also contains the residual demand from previous periods not realized due to higher prices. Fleischmann, Hall and Pyke (2005) provide history dependent, endogenous pricing models that capture stock piling effects on demand.

From the behavioral economics literature, reviewed by Camerer et al. (2004) and Ho et al. (2006), a related paper is Heidhues and Kőszegi (2005). They propose a new model where the reference price is defined as an internal, rational expectations equilibrium.

This paper builds directly on the existing literature on dynamic pricing with reference effects. We briefly review here the papers most closely related to ours. Greenleaf (1995) provides numerical insights into the firm’s optimal pricing policy. The first analytic results are due to Kopalle et al. (1996), and extended by Fibich et al. (2003). These authors prove the monotonicity and convergence of the optimal price paths under a piecewise linear demand model. Popescu and Wu (2007), henceforth PW, extend these findings to general demand functions and reference effects, and provide structural results. Like previous studies, they show that loss aversion leads to a range of steady states, which collapses to a single point for loss neutral buyers. Complementing previous research, they find that monotonicity of the transient pricing policy is a consequence of linear reference effects, and does not hold in general. In their model, it is optimal for the firm to induce a consistent perception of gain or loss, by systematically pricing above, respectively below the reference price. In parallel work, Nasiry and Popescu (2008) extend these results to capture asymmetries in the exponential smoothing memory process, in the spirit of the model proposed by Gaur and Park (2007). Our work explores similar issues, under a different model of consumer memory and behavior.

This paper proposes and investigates a new price anchoring process, based on the peak-end rule. This makes our work different from the current literature on pricing with reference effects, which has focused exclusively on exponentially smoothed memory models. In what follows, we summarize our main findings, and relate them to existing literature.

Consistent with previous results for loss averse consumers, we find that a constant pricing policy is optimal for a range of relatively low initial price expectations, supporting an EDLP policy (every day low price; Lal and Rao 1997, Fibich et al. 2003). In our case, the range of constant optimal prices (called steady states), is due to the kinked anchoring process, and therefore persists when consumers are loss neutral (i.e equally sensitive to discounts and surcharges). This is in contrast with previous literature. The range of steady state prices is wider, the more loss averse consumers are, and the more sensitive to the lowest price anchor.

In general, the more consumers anchor on the minimum price, the lower the steady state prices and the firm's profits. For relatively high initial price expectations, the firm should follow a skimming strategy, converging on the long run to a unique optimal price, which is independent of consumers' initial price perception and their sensitivity to the minimum price. In general, we find that firms should follow either skimming or penetration strategies, i.e. the optimal pricing policy of the firm is monotonic. In contrast with the literature, in particular PW, the result is true in our case under general, non-linear reference effects. This suggests that memory processes are important in determining the optimal pricing policy of the firm. From a methodological standpoint, we develop a non-standard approach to solve a dynamic program with kinked reward and transition functions.

Structure. The rest of the paper is structured as follows. In Section 2, we develop the general demand and reference price formation models, based on behavioral assumptions, and set up the dynamic pricing problem of the firm. Section 3 studies the optimal transient and long term pricing policies for loss averse buyers with linear reference effects. Section 4 compares the optimal pricing policy with that of a myopic firm, which is shown to systematically underprice the product. Section 5 studies the sensitivity of the optimal profits and solution to the lowest price anchor. Section 6 extends our findings to general non-linear reference effects, under behavioral assumptions motivated by prospect theory. Conclusions and future research directions are summarized in Section 7.

2. The Model

This section provides the main setup for the paper. We describe how consumers make purchase decisions based on prices and reference prices, and how this decision affects the demand for a firm's product. We model demand dependence on reference prices based on prospect theory (Tversky and Kahneman 1991), following the general setup in PW.

Mental accounting theory (Thaler 1985) describes the utility from purchase experiences as consisting of two components: acquisition utility and transaction utility. Acquisition utility is the monetary value of the product, captured by the difference between price and consumer's reservation price. Transaction utility corresponds to the psychological value of the deal, determined by the gap between the reference price and the price, $x = R - p$.

At the aggregate level, this motivates a demand model of the general form proposed in PW:

$$d(p,R) = d_0(p) + h(R - p, R). \tag{1}$$

Here $d_0(p) = d(p,p)$ is the base demand, i.e. demand in absence of reference effects, and $h(x,R)$ is the reference effect, which captures the demand dependence on the reference point. In particular,

$h(0,R)=0$. In addition, prospect theory postulates that consumers are loss averse, i.e. they react more strongly to surcharges (perceived losses) than to discounts (perceived gains) of the same magnitude. The next section focuses on a loss averse demand model with linear reference effects, specifically: $h(x,R)=\lambda x$ for $x\leq 0$, and $h(x,R)=\gamma x$ for $x\geq 0$, where $\lambda\geq\gamma>0$ captures loss aversion. Section 6 extends the results to general, non-linear reference effects, satisfying behavioral assumptions motivated by prospect theory, such as diminishing sensitivity.

The firm's short term profit is denoted $\pi(p,R)=pd(p,R)$, and the base profit is $\pi_0(p)=p\,d_0(p)$. These expressions implicitly assume zero cost; all our results extend for a non-zero marginal cost c. The following assumptions are made throughout the paper, and consistent with PW.

Assumption 1. *(a) Demand $d(p,R)$ is decreasing in p and increasing in R, and the reference effect, $h(x,R)$, is increasing in the perception gap, $x=R-p$. (b) The base demand, $d_0(p)$, is non-negative bounded, continuous and downward sloping in price, p. (c) The base profit $\pi_0(p)$ is non-monotone and strictly concave.*

We assume that consumers' reference price, R_t, at each stage t is formed based on the peak-end rule (Fredrickson and Kahneman 1993), as a weighted average of the minimum price, r_{t-1}, and the last price of the product, p_{t-1}. That is,

$$R_t=\theta r_{t-1}+(1-\theta)p_{t-1}, \tag{2}$$

where $r_{t-1}=\min(r_{t-2},p_{t-1})$, and $0\leq\theta\leq 1$ captures how much consumers anchor on the lowest price. This adaptation model is in contrast with previous models used in the pricing literature (including PW), mainly relying on exponential smoothing.

Given initial conditions r_0 and p_0, the firm maximizes infinite horizon β-discounted revenues :

$$J(r_0,p_0)=\max_{p_t\in\mathbf{P}}\sum_{t=1}^{\infty}\beta^{t-1}\pi(p_t,R_t),\text{ where } R_t=\theta r_{t-1}+(1-\theta)p_{t-1},\quad\text{and}\quad\beta\in(0,1).$$

Throughout the paper we assume that prices are confined to a bounded interval $\mathbf{P}=[0,\bar{p}]$, where, for simplicity, $\bar{p}$ is such that $d_0(\bar{p}) = 0$ (this assumption avoids boundary solutions). Sets are denoted by bold letters throughout. The Bellman Equation for this problem is:

$$J(r_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ \pi(p_t, R_t) + \beta J\Big(\min(p_t, r_{t-1}), p_t \Big) \Big\}, \qquad (3)$$
$$\text{where} \quad R_t = \theta r_{t-1} + (1-\theta) p_{t-1}.$$

Intuitively, we expect that higher reference prices should enable the firm to extract higher profits from the market.

Lemma 1. *The value function, $J(r,p)$, is increasing in both arguments.*

Proof: By Assumption 1a, $d(p_t, R_t)$ is increasing in $R_t = \theta r_{t-1} + (1-\theta) p_{t-1}$. Hence $\pi(p_t, R_t) = p_t d(p_t, R_t)$ is increasing in p_{t-1} and r_{t-1}. Moreover the transition in the Bellman Equation (3) is increasing in r_{t-1} (and independent of p_{t-1}). So the value function is increasing in its arguments (Stokey et al. 1989, Theorem 4.7). □

3. Linear Reference Effects

This Section investigates the solution of Problem (3) when consumers marginal sensitivity to discounts, respectively surcharges, is constant, i.e. reference effects are piecewise linear. We first identify the steady state prices, i.e. the optimal constant price policies, and then describe the transient behavior of the price dynamics. Finally, we specialize our results for the case when consumers are loss-neutral, i.e. exhibit equal sensitivity to discounts and surcharges, and contrast them with previous literature.

According to prospect theory (Tversky and Kahneman 1991), surcharges have a stronger impact on demand than discounts of the same magnitude. This effect, known as loss aversion, is captured by a kinked demand function. The demand model used in this section is:

$$d(p,R) = d(p) - \lambda(p-R)^+ + \gamma(R-p)^+ = \begin{cases} d(p) - \lambda(p-R), & \text{if } p \geq R \\ d(p) - \gamma(p-R), & \text{if } p \leq R \end{cases}, \qquad (4)$$

with $\lambda \geq \gamma$ accounting for loss aversion. The corresponding profit function is:

$$\pi(p,R) = \Big[d(p) - \lambda(p-R)^+ + \gamma(R-p)^+ \Big] p = \begin{cases} \pi_\lambda(p,R), & \text{if } p \geq R \\ \pi_\gamma(p,R), & \text{if } p \leq R \end{cases}. \qquad (5)$$

It is easy to see that, for $k \in \{\lambda, \gamma\}$, the smooth profit functions

$$\pi_k(p,R) = \Big[d(p) + k(R-p)\Big]p = \pi_0(p) + k(R-p)p, \tag{6}$$

are both supermodular in (p,R). A function $f(x,y)$ is said to be supermodular if $f(x^h,y^h) - f(x^l,y^h) \geq f(x^h,y^l) - f(x^l,y^l)$ for all $x^l < x^h$ and $y^l < y^h$ (see Topkis 1998). Together with loss aversion ($\lambda \geq \gamma$), this implies supermodularity of short term profit, π. Moreover, the kinked profit π can be written as the minimum of the smooth profit functions π_λ and π_γ. These results are essential for our future developments, and can be summarized as follows.

Lemma 2. *The short-term profit,* $\pi(p,R) = \min\Big(\pi_\lambda(p,R), \pi_\gamma(p,R)\Big)$, *is supermodular in* (p,R).

Proof: See Appendix. □

By Topkis' Theorem (Topkis 1998, Theorem 2.8.2), this result confirms the intuition that myopic firms, i.e. those focused on short term profits, should charge higher prices when consumers have higher price expectations. Myopic policies are studied and compared to optimal ones in Section 4.

Lemma 2 allows to write the Bellman Equation for this section as follows:

$$J(r_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ \min(\pi_\lambda, \pi_\gamma)(p_t, R_t) + \beta J\Big(\min(p_t, r_{t-1}), p_t\Big)\Big\}. \tag{7}$$

3.1. Steady States

This section characterizes the long term pricing strategy of the firm facing loss averse consumers with demand given by (4). Identifying the steady states of Problem (7) requires a non-standard approach, because, in this problem, both the short-term profit and the transition in the value function (memory structure) are non-smooth. Our analysis is based on a bounding technique, which identifies the steady states of Problem (7) based on those of a series of smooth problems, for which standard methods can be applied.

For $\alpha \in [0,1]$, and $r \in \mathbf{P}$, consider the following smooth problem with one-dimensional state:

$$J_r^\alpha(p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ (1-\alpha)\pi_\lambda(p_t, \theta r + (1-\theta)p_t) + \alpha \pi_\gamma(p_t, p_{t-1}) + \beta J_r^\alpha(p_t)\Big\}. \tag{8}$$

We argue that J_r^α provides an upper bound for J.

Lemma 3. *For any $r \leq p$, we have $J(r,p) \leq J_r^{\alpha}(p)$.*

Proof: Let $r_{t-1} \leq p_{t-1}$. We have:

$$\begin{aligned} J(r_{t-1}, p_{t-1}) &= \max_{p_t \in \mathbf{P}} \Big\{ \pi(p_t, R_t) + \beta J(\min(r_{t-1}, p_t), p_t) \Big\} \\ &\leq \max_{p_t \in \mathbf{P}} \Big\{ (1-\alpha)\pi_\lambda(p_t, R_t) + \alpha \pi_\gamma(p_t, R_t) + \beta J\Big(r_{t-1}, p_t\Big) \Big\} \\ &\leq \max_{p_t \in \mathbf{P}} \Big\{ (1-\alpha)\pi_\lambda(p_t, R_t) + \alpha \pi_\gamma(p_t, p_{t-1}) + \beta J\Big(r_{t-1}, p_t\Big) \Big\} \\ &= J^{\alpha}_{r_{t-1}}(p_{t-1}). \end{aligned}$$

The first inequality holds because $\pi = \min(\pi_\lambda, \pi_\gamma) \leq (1-\alpha)\pi_\lambda + \alpha\pi_\gamma$, and the value function is increasing in its arguments (Lemma 1). The second inequality holds because $p_{t-1} \geq R_t = \theta r_{t-1} + (1-\theta)p_{t-1}$, for $p_{t-1} \geq r_{t-1}$. □

We next identify steady states of Problem (8) which will help characterize those of Problem (7). Define q and s to be the unique respective solutions of the equations:

$$\pi_0'(p) - \lambda(1-\beta(1-\theta))p = 0, \tag{9}$$

$$\pi_0'(p) - \gamma(1-\beta)p = 0. \tag{10}$$

Uniqueness of q and s follows because the above left hand sides (LHS) are strictly decreasing in p, by concavity of π_0 (Assumption 1c). Moreover, it is easy to verify that $q \leq s$, because $1-\beta(1-\theta) \geq 1-\beta$, and $\lambda \geq \gamma$.

Lemma 4. *Problem (8) admits a unique steady state, which solves:*

$$\pi_0'(p) - \Big[\lambda(1-\alpha)(2-(1-\theta)(1+\beta)) + \alpha\gamma(1-\beta)\Big]p + \lambda(1-\alpha)\theta r = 0. \tag{11}$$

(a) For any $r \in [0,q]$, $p_\lambda^{**}(r)$ is a steady state of Problem (8) for $\alpha = 0$, where $p_\lambda^{**}(r)$ solves:

$$\pi_0'(p) - \lambda(2-(1-\theta)(1+\beta))p + \lambda\theta r = 0. \tag{12}$$

(b) For any $r \in [q,s]$, there exists $\alpha \in [0,1]$ such that r is a steady state of Problem (8).

Proof: It is easy to check that, if Problem (8) admits an interior steady state, this solves the Euler Equation (11). Moreover, this equation admits a unique solution, because $\pi_0'(p) < 0$ and the coefficient of p in (11) can be written as $-(1-\beta)((1-\alpha)\lambda(1+\theta)+\alpha\gamma) \leq 0$. It remains to verify that a steady state exists, and must be interior. Existence of a steady state follows from supermodularity of the objective function, because π_λ and π_γ are supermodular in (p, R). By Topkis Theorem (Topkis 1998, Theorem 2.8.2), this implies that the pricing paths of Problem (8) are monotonic on the bounded domain $\mathbf{P}$, hence converge to a steady state p^{**}.

Finally, we argue that a steady state must be interior. First, $p^{**} = 0$ cannot be a steady state because any non-zero pricing strategy achieves positive profits. Second, Assumption 1c insures that $p^{**} < \bar{p}$ for any steady state of Problem (8). Indeed, because $\pi_0(p)$ is non-monotone, its largest maximizer, $\hat{p}$, is interior, i.e. $\hat{p} < \bar{p}$. Moreover, concavity of π_0 implies: $J^\alpha(\hat{p}) \geq \frac{\pi(\hat{p})}{1-\beta} \geq \frac{\pi(p^{**})}{1-\beta} = J^\alpha(p^{**})$. Finally, because J^α is increasing, we conclude that $p^{**} \leq \hat{p} < \bar{p}$, so p^{**} is interior and solves the Euler Equation (11).

(a) By definition, $p_\lambda^{**}(r)$ solves (11) for $\alpha = 0$. This has a unique solution because the LHS is strictly decreasing in p, positive at $p = 0$ and negative for a high enough $p \in \mathbf{P}$.

(b) Substituting $p = r$ in (11), we have $L(r, \alpha) = \pi_0'(r) - \lambda[(1-\alpha)(1-\beta(1-\theta)) + \alpha(1-\beta)]r = 0$. Equations (9) and (10) translate to $L(q, 0) = 0$ and $L(s, 1) = 0$. Because $L(r, \alpha)$ is decreasing in r, it follows that for all $r \in [q, s]$, $L(r, 0) \leq 0$ and $L(r, 1) \geq 0$. The result follows because $L(r, \alpha)$ is continuous in α. □

The next result characterizes the steady states of Problem (7) based on the steady states of Problem (8) identified in Lemma 4.

Lemma 5. *(a) For $r \in [0, q]$, $(r, p_\lambda^{**}(r))$ is a steady state of Problem (7), where $p_\lambda^{**}(r)$ solves (12). (b) For $r \in [q, s]$, (r, r) is a steady state of Problem (7).*

Proof: We first show that $p_\lambda^{**}(r)$, as defined by (12), is feasible, i.e. $p_\lambda^{**}(r) \geq r$ for $r \in [0, q]$. Note that $p_\lambda^{**}(r)$ is increasing in r and single crosses the identity line from above at q, which is defined by (9). Feasibility follows because, at $r = 0$, (12) has a unique positive solution, $p_\lambda^{**}(0)$.

For $r \in [0, q]$, the constant pricing policy $p_t \equiv p_\lambda^{**}(r)$ is optimal for Problem (8) with $\alpha = 0$, and feasible for Problem (7). Because $r \leq q$, $\min(r, p_\lambda^{**}(r)) = r$ and $R = \theta r + (1-\theta) p_\lambda^{**}(r) \leq p_\lambda^{**}(r)$, which implies $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. This constant pricing policy yields the same value in both problems, hence it is also optimal for Problem (7), and $(r, p_\lambda^{**}(r))$ is a steady state of Problem (7).

For $r \in [q, s]$, the constant pricing policy $p_t \equiv r$ is optimal for Problem (8), feasible for Problem (7) ($\pi_\lambda = \pi_\gamma$ along this path), and yields the same value in both problems. Therefore (r, r) is a steady state of Problem (7). □

The two thresholds q and s, defined in Lemma 4, partition the initial state space into the following regions: $\overline{\mathbf{R}}_{1a} = \{(r,p) \mid p \geq p_\lambda^{**}(r),\, r \leq q\}$, $\overline{\mathbf{R}}_{1b} = \{(r,p) \mid p \leq p_\lambda^{**}(r),\, r \leq q\}$, $\overline{\mathbf{R}}_2 = \{(r,p) \mid p \geq r,\, q \leq r \leq s\}$, and $\overline{\mathbf{R}}_3 = \{(r,p) \mid p \geq r,\, r \geq s\}$, as shown in Figure 1. For convenience, we also define the projections of these regions onto the r axis: $\mathbf{R}_1 = [0, q]$, $\mathbf{R}_2 = [q, s]$, and $\mathbf{R}_3 = [s, \bar{p}]$.

The main result in this section confirms that the steady states identified in Lemma 5 are indeed the only steady states of Problem (7).

Proposition 1. *The set of steady states of Problem (7) is* $\{(r, p_\lambda^{**}(r)) \mid r \in \mathbf{R}_1\} \cup \{(r, r) \mid r \in \mathbf{R}_2\}$. *In particular, the value of the steady state prices is decreasing in* λ*, and increasing in* β*.*

Proof: See Appendix. □

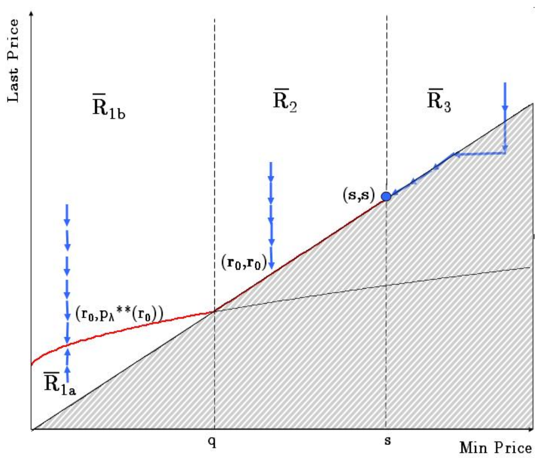

Figure 1 Steady states and optimal price path of Problem (7). The red line depicts the range of steady states, and the blue arrows show the price path and convergence points starting at an arbitrary point in any of the regions.

The result says that value of the steady state is lower, the more sensitive consumers are to deviations from the reference price . Furthermore, a more patient firm (higher β) charges higher steady state prices. These sensitivity results are consistent to those observed in the previous literature on exponentially smoothed memory models (PW, Fibich et al. 2003, Kopalle et al. 1996). Also consistent with the literature is the range of steady states observed under loss aversion.

Interestingly, our predictions are different from the literature in the loss neutral case, i.e. when consumers are equally sensitive to gain and losses, captured here by $\lambda = \gamma$. In our case, a range of steady states still obtains, as specified in Proposition 1 (with γ replaced by λ in (10)). This result is in contrast with the literature on exponentially smoothed memory models, which obtained a unique global steady state for loss neutral buyers. The reason to have a range of steady states for our problem, even under smooth linear reference effects, is the kinked memory structure. Loss aversion (i.e. the demand kink) causes a wider range of steady state prices, which becomes wider the more loss averse consumers are.

We infer that capturing behavioral asymmetries in either consumer anchoring or decision processes leads to a range of steady states. This also suggests that, for a range of initial price expectations, EDLP (everyday low prices) is more likely to be an optimal policy when consumers exhibit such behavioral asymmetries (Lal and Rao 1997, Kopalle et al. 1996, Fibich et al. 2003).

3.2. Optimal Policy and Price Paths

This section investigates the transient pricing policy of the firm. Specifically, we study convergence and monotonicity properties of the price paths of Problem (7), starting at an arbitrary initial state (r_0, p_0), $r_0 \leq p_0$. Denote the optimal pricing policy of the firm by:

$$p^*(r_{t-1}, p_{t-1}) = \arg\max_{p_t} \Big\{ \pi(p_t, R_t) + \beta J(\min(r_{t-1}, p_t), p_t) \Big\}.$$

The optimal price path is then defined by $p_t = p^*(r_{t-1}, p_{t-1})$, $t \geq 1$, and the state path is (r_t, p_t) with $r_t = \min\{r_{t-1}, p_t\}$.

Our first result in this section shows that, if (r_0, p_0) is in any of the three regions $\mathbf{R}_i$, $i = 1, 2, 3$, defined in Section 3.1, the state path remains in that region.

Proposition 2. *(a) If* $r_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$*, then* $p_t \geq r_0$ *for all* t*. (b) If* $r_0 \in \mathbf{R}_3$*, then* $r_{t-1} \in \mathbf{R}_3$ *for all* t*.*

Proof: See Appendix. □

The first part of this proposition shows that, in regions $\mathbf{R}_1$ and $\mathbf{R}_2$, the optimal price path is always above r_0, i.e. the minimum price does not change over time, and so the state path remains within the region. For $\mathbf{R}_3$, we prove that, although the minimum price changes over time, it never drops below the threshold s defined in Lemma 4, so the state path remains in the same region. This result leads us to identify the possible convergence points of the optimal price paths, starting at any initial state.

Proposition 2 implies that, if the optimal price path of Problem (7) converges, it converges to a steady state in the same region as the initial state (r_0, p_0). These steady states are identified in Proposition 1. Specifically, they are: (1) $(r_0, p^{**}(r_0))$ for $r_0 \in \mathbf{R}_1$, (2) (r_0, r_0) for $r_0 \in \mathbf{R}_2$, and (3) (s, s) for $r_0 \in \mathbf{R}_3$.

Based on these results, we now turn to characterize the optimal price paths of Problem (7). Proposition 2 shows that, for $r_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$, $r_t = r_0$, so Problem (7) can be written (with r_0 as a parameter) as follows:

$$J_{r_0}(p_{t-1}) = \max_{p_t \geq r_0} \Big\{ \pi(p_t, R_t) + \beta J_{r_0}(p_t) \Big\}, \tag{13}$$

where $R_t = \theta r_0 + (1-\theta) p_{t-1}$. That is, $J(r,p) = J_r(p)$ for $r \in \mathbf{R}_1 \cup \mathbf{R}_2$, and $r \leq p$. Because π is supermodular (Lemma 2), the optimal policy in Problem (13) is monotone, so $p_t^*(r_0, p_{t-1})$ is increasing in p_{t-1}. Therefore, the optimal price path is monotonic in a bounded interval, and hence converges to a steady state. This must be $(r_0, p^{**}(r_0))$, by Proposition 1.

In region $\mathbf{R}_3$, i.e. for $r_0 \geq s$, we show in the Appendix that the firm decreases prices in each period, approaching s. This is done by observing that prices must eventually fall below r_0 (but not below s, by Proposition 2), at a certain time T. Until that time, T, a finite horizon version of Problem (13) is solved (and the same structural results hold). After time T, we show that optimal prices $p_t = r_t$ solve:

$$\tilde{J}(p_{t-1}) = \max_{p_t \in \mathbf{P}} \{ \pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t) \}. \tag{14}$$

Again, supermodularity of π insures that the optimal path is decreasing to s, starting at $p_T = r_T \geq s$.

The following proposition characterizes the optimal price paths for Problem (7).

Proposition 3. *For an initial (r_0, p_0), the optimal price path $\{p_t\}$ of Problem (7) converges monotonically to a steady state, which is (a) $p_\lambda^{**}(r_0)$, if $r_0 \in \mathbf{R}_1$, (b) r_0, if $r_0 \in \mathbf{R}_2$, and (c) s, if $r_0 \in \mathbf{R}_3$.*

Proof: See Appendix. □

Corollary 1. *The optimal pricing policy, $p^*(r, p)$, is increasing in both r and p.*

The corollary follows from the proof of Proposition 3 (see Appendix).

Proposition 3 insures that, starting at (r_0, p_0), the price path monotonically converges to a unique steady state. Monotonicity is driven by supermodularity of short-term profits, $\pi(p, R)$ (Lemma 2), within the relevant regions, via Proposition 2. The steady state of Problem (7) depends only on r_0; p_0 influences the price path, but not the steady state.

Proposition 3 shows that there exists a threshold, s (defined by (10)), such that for high initial minimum prices, $r_0 \geq s$, the firm will decrease prices down to s. Otherwise, the firm will always charge prices above r_0. Prices converge to r_0 for intermediate values of the minimum price, $q \leq r_0 \leq s$, and to a higher steady state, $p_\lambda^{**}(r_0)$, for low values, $r_0 < q$. If the initial price, p_0, is below the steady state (Region $\overline{\mathbf{R}}_{1a}$), the optimal price is increasing and thus induces a consistent perception of loss. Otherwise, the gain/loss perception may alternate (see e.g. Figure 2); this is in contrast with previous insights obtained in the literature (PW, Theorem 4). A constant steady state price path either stays equal to the reference price (for $r \geq q$), or induces a consistent perception of loss ($p_\lambda^{**}(r) > r$ for $r < q$). This is because the steady state price either equals the minimum price, or stays above it (see Proposition 3), and thus also above the reference price.

Global monotonicity of the price path for linear reference effects is consistent with the results obtained in the literature, under exponential smoothing memory models (PW, Proposition 4). In contrast with this literature, however, Section 6 shows that, under a peak-end memory structure, price monotonicity is preserved for general non-smooth reference effects.

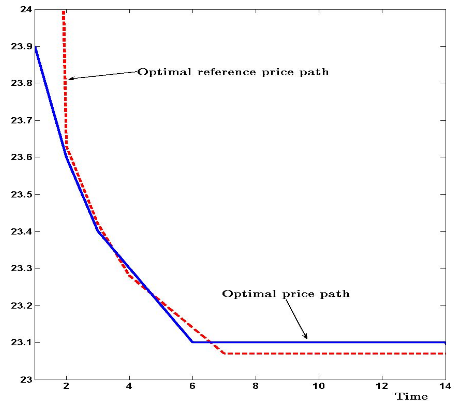

Figure 2 Alternating price perception. $d(p,R) = 500 - 10p - 3(p-R)^+ + 2(R-p)^+, \theta = 0.3, \lambda = 3, \gamma = 2, \beta = 0.5, q = 22.78, s = 23.81, r_0 = 19, p_0 = 28$.

4. Comparison with Myopic Solution

This section compares the optimal policy of the firm, as derived in the previous section, with than of a so-called myopic firm. A myopic firm is one that does not take into account the effect of its current pricing policy on future demand, and profits, and focuses on short term profit maximization. We next analyze the behavior of a myopic firm over time, as loss averse consumers update their reference price in response to the firm's pricing policy.

We first characterize the myopic pricing policy, $p^M(R) = \arg\max_{p\in\mathbf{P}} \pi(p,R)$. For $k \in \{\lambda,\gamma\}$, let

$$p_k(R) = \text{argmax}_{p\in P}\, \pi_k(p,R), \tag{15}$$

where π_λ and π_γ are as defined in Section 3 (see equation (6)). The functions $p_\lambda(\cdot)$ and $p_\gamma(\cdot)$ are increasing, because π_λ and π_γ are supermodular in (p,R). We show that $p_\lambda(\cdot)$ and $p_\gamma(\cdot)$ single cross the identity line from above, and characterize their fixed points.

Lemma 6. *Let $k \in \{\lambda,\gamma\}$. (a) There exists a unique fixed point R_k of $p_k(\cdot)$, i.e. $R_k = p_k(R_k)$, and R_k solves $\pi_0'(R_k) = kR_k$. Moreover, $R_\lambda \le R_\gamma$. (b) For $R \le R_k$, we have $R \le p_k(R) \le R_k$, and for $R \ge R_k$, we have $R \ge p_k(R) \ge R_k$.*

Proof: See Appendix. □

Lemma 6 implies the following structure of the myopic pricing policy.

Proposition 4. *The optimal pricing policy of the myopic firm is:*

$$p^M(R) = \begin{cases} p_\lambda(R), & if \quad R \le R_\lambda \\ R, & if \quad R_\lambda \le R \le R_\gamma \\ p_\gamma(R), & if \quad R \ge R_\gamma \end{cases} .$$

Proof: See Appendix. □

Figure 3 illustrates the optimal myopic pricing policy.

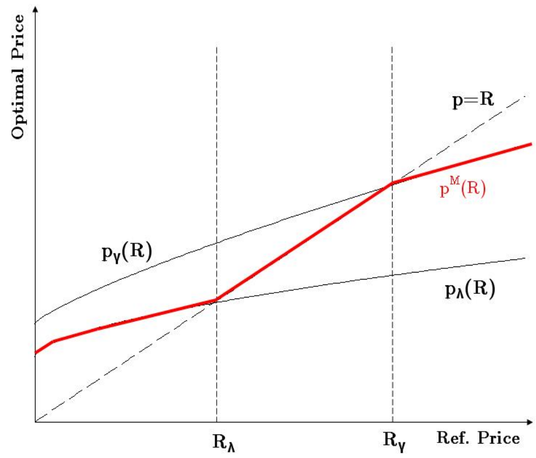

Figure 3 Optimal one stage pricing policy.

Intuitively, we expect the myopic firm to charge higher prices when consumers have higher reference prices. Supermodularity of $\pi(p, R)$ (Lemma 2) insures this via Topkis Theorem (Topkis 1998, Theorem 2.8.2). This property also suffices to conclude monotonicity of the price path followed by a myopic firm. Over multiple periods, the myopic firm solves: $p_t^M = p_t^M(r_{t-1}, p_{t-1}^M) = \arg\max\{\pi(p_t^M, R_t)\}$, where $R_t = \theta r_{t-1} + (1-\theta)p_{t-1}$.

Proposition 3 implies that, under loss aversion, there is a range of steady states for the myopic firm. Formally, the myopic problem is equivalent to setting $\beta = 0$ in Problem (7). The myopic steady states are thus specified by setting $\beta = 0$ in Proposition 3. In particular, we have $q = R_\lambda$ and $s = R_\gamma$, as defined in Lemma 6.

Consider a myopic firm and a strategic firm applying their optimal strategies in each period, while consumers update their reference prices according to equation (2). In each period, the state for a myopic firm is (r_t^M, p_{t-1}^M), resulting in $R_t^M = \theta r_t^M + (1-\theta)p_{t-1}^M$, and the state for a strategic firm is (r^*_{t-1}, p^*_{t-1}), resulting in $R^*_t = \theta r^*_t + (1-\theta)p^*_{t-1}$. The following proposition shows that the myopic firm underprices the product and follows a monotonic pricing strategy.

Proposition 5. *For any initial state (r_0, p_0), the price charged by a myopic firm at any point in time is less than the optimal price, i.e. $p^M_t \le p_t$, for all t. Furthermore, the myopic price paths converge monotonically to a constant price, which is: (a) $(r_0, p^{M*}_\lambda(r_0))$ for $r_0 \le R_\lambda$, where $p^{M*}_\lambda(r_0)$ solves: $\pi_0'(p) - \lambda(1+\theta)p + \lambda\theta r_0 = 0$, (b) (r_0, r_0) for $R_\lambda \le r_0 \le R_\gamma$, and (c) (R_γ, R_γ) for $r_0 \ge R_\gamma$.*

Proof: See Appendix. □

A myopic firm erodes future demand by charging low prices and thus lowering the future reference price. In contrast, by charging relatively higher prices, a strategic firm benefits, on the long term, from higher reference prices.

The results in this section imply that the myopic firm charges lower steady state prices than a strategic firm, starting at the same initial state. A constant myopic pricing policy can be optimal for a range of initial price expectations, whenever the firm's discount factor satisfies $\beta \le \frac{1-\frac{\gamma}{\lambda}}{1-\theta}$. In particular, this is always true if consumers anchor sufficiently on low prices, i.e. if $\theta \ge \gamma/\lambda$; practical evidence suggests that $\gamma/\lambda \simeq 0.5$ (Kahneman et al. 1990, Tversky and Kahneman 1991, Ho and Zhang 2008). More specifically, if $\theta \ge 1 - \frac{1}{\beta}(1-\frac{\gamma}{\lambda})$, then the myopic constant pricing policy is optimal for all initial states $p_0 = r_0 \in [q, R_\gamma]$. Otherwise, a constant myopic pricing policy can never be optimal. It is therefore important for a firm to understand the magnitude of behavioral parameters, such as θ, λ and γ, and their impact on profits and decisions.

5. Sensitivity to Memory Anchoring Effects

This section analyzes the sensitivity of the optimal policy structure and of the firm's profits with respect to the parameter θ, which measures how strongly consumers anchor on the lowest price. Because $R = \theta r + (1-\theta)p$, as θ increases, consumers anchor more on the lowest price, and hence have lower price expectations. In the extreme case when $\theta = 1$, consumers anchor only on the minimum price, $R = r$. For $\theta = 0$, consumers have short term memory, $R = p$, i.e. the last price paid for the product is their reference point, recovering a special case of the exponentially smoothed model in PW with $\alpha = 0$.

From (10), s is independent of θ, because it solves $\pi_0'(p) = \gamma(1-\beta)p$. Thus there exists a global threshold, s, such that for large enough initial low-price anchors ($r_0 \geq s$), and independent of how much weight θ consumers put on the minimum price, the price path decreases to a global steady state, s. Therefore, as long as the initial price perception is sufficiently high, r_0 and θ do not affect the long run optimal policy of the firm.

In contrast, a minimum price below s, affects the long term strategy (steady state price) of the firm. If r_0 is sufficiently low ($r_0 \leq q$), the firm's optimal long run policy is to charge a price above $r_0 \leq p_\lambda^{**}(r_0) = p_\lambda^{**}(r_0, \theta)$ which solves (12). As consumers anchor more on minimum prices, we see lower optimal long run prices, i.e. $p_\lambda^{**}(r;\theta)$ is decreasing in θ. This is because the LHS in (12) is decreasing in θ for $r \leq p$.

From (9), $q = q(\theta)$ is decreasing in θ. As θ varies in $[0,1]$, $q(\theta)$ decreases from p_λ^{**} to R_λ (defined in Lemma 6), where p_λ^{**} solves the equation:

$$\pi_0'(p) - \lambda(1-\beta)p = 0. \tag{16}$$

Equation (16) is the same as equation (10) defining s, with γ replaced by λ.

As $\theta \to 0$, i.e. as consumers anchor more on the last price paid, $p_\lambda^{**}(r_0;\theta)$ converges to p_λ^{**}, which is unique and independent of r_0. In the extreme case when $\theta = 0$, p_λ^{**} is a global steady state. All the price paths starting at an initial reference price below p_λ^{**} converge to this steady state.

For an intermediary range of initial minimum prices, $q(\theta) \leq r_0 \leq s$, the firm's long run optimal price is r_0. The range $[q(\theta), s]$ of steady states (r_0, r_0) is wider, the more consumers pay attention to the minimum price. In particular, this range always includes $[p_\lambda^{**}, s]$, the range of steady states obtained for $\theta = 0$.

Proposition 6 summarizes the above results, and shows that the optimal prices and profits decrease the more consumers account for lowest prices.

Proposition 6. *(a) The optimal prices,* $p^*(p,r;\theta)$*, and profits,* $J(r,p;\theta)$*, in Problem (7) are decreasing in* θ*. (b)* $q(\theta)$ *and* $p^{**}_\lambda(r;\theta)$ *are decreasing in* θ*, and* $s(\theta)\equiv s$ *is independent of* θ*.*

Proof: $\pi(p,R)$ is supermodular in (p,R) by Lemma 2, and $R=R(\theta)=p+\theta(r-p)$ is decreasing in θ for $r\leq p$. Therefore π is submodular in (p,θ) and $p^*(p,r;\theta)$ is decreasing θ. Moreover, because π is increasing in R, and R is decreasing in θ, we conclude that the value function, $J(r,p;\theta)$, is decreasing in θ (Stokey and Lucas, Theorem 4.7). □

Our results so far suggest the following sequential estimation procedure for determining the optimal long run policy of the firm:

1. Compute the global threshold s based on (10). Only consumers' sensitivity to discounts, γ, needs to be estimated for this.

2. Assess if lower prices than s were charged in the past (or recalled by consumers). If not, s is the optimal long term price.

3. If the lowest price is $r\in[p^{**}_\lambda,s]$, where p^{**}_λ is given by (12), then this price is the optimal long term price. Consumers' sensitivity to surcharges, λ, needs to be estimated for this.

4. If $r_0<p^{**}_\lambda$, it is necessary to assess the anchoring parameter θ, and calculate $q=q(\theta)$, based on (9). Comparing the minimum price with q determines the equilibrium price, via Proposition 3.

6. General Reference Effects

This section extends the results obtained in Section 3 under linear reference effects, to general, non-linear, reference effects, under behavioral assumptions motivated by prospect theory (Tversky and Kahneman 1991). The demand function with general reference effects is given by:

$$d(p,R)=d(p)+h^K(R-p,R). \tag{17}$$

The kinked reference effect is defined as $h^K(x,R)=\mathbf{1}_{x\geq 0}h^G(x,R)+\mathbf{1}_{x\leq 0}h^L(x,R)$, where $x=R-p$ is the reference gap, i.e. the perceived discount or surcharge relative to the reference point. The smooth functions h^G and h^L satisfy Assumption 1, i.e. for $k\in\{L,G\}$, $h^k(0,R)=0$, $h^k(x,R)$ is increasing in x and $h^k(R-p,R)$ is increasing in R (because $d(p,R)$ is increasing in R). The following additional assumptions on the reference effect are supported in part by behavioral considerations:

Assumption 2. *(a)* $(h^L - h^G)(x,R)$ *is single crossing in* x. *(b)* $(h^L - h^G)(x,R)$ *is increasing in* x*, for* $x \leq 0$. *(c)* $(h^L - h^G)(R-p,R)$ *is increasing in* R *for* $x \geq 0$. *(d)* $h_1^L(0,R) > h_1^G(0,R)$ *for all* R. *(e)* $h^G(R-p,R)$ *is supermodular in* (p,R).

The first three assumptions restrict the extensions of the gain and loss parts of the reference effect over the entire domain. Assumption 2d captures loss aversion. Assumption 2e can be formalized as $h_{11}^G + h_{12}^G \leq 0$. Over gains ($x \geq 0$), this is implied by two behavioral assumptions of prospect theory: diminishing sensitivity, in particular concavity of the value function on the gain domain (implying $h_{11}^G \leq 0$), and decreasing curvature (implying $h_{12}^G \leq 0$). The assumption further restricts the degree of convexity of the extension of h^G over the loss domain. For details on prospect theory assumptions, see Section 2.1 in PW.

The single-crossing property (Assumption 2a) allows to write the kinked reference effect as: $h^K(x,R) = \min\Big(h^L(x,R), h^G(x,R)\Big)$, and the short-term profit function as:

$$\pi^K(p,R) = \min\Big(\pi^L(x,R), \pi^G(x,R)\Big). \tag{18}$$

The following technical assumption is made on the smooth profit functions π^L and π^G.

Assumption 3. *(a)* $\pi^k(p,R)$ *is strictly concave in* p*, for* $k \in \{L,G\}$. *(b)* $\pi^L(p,R)$ *is supermodular in* (p,R). *(c)* $\pi^L(p, \theta r + (1-\theta)p)$ *is strictly concave in* p. *(d)* $\pi^L(p, \theta r + (1-\theta)p)$ *is supermodular in* (r,p). *(e)* $\pi_1^k(p,p) = \pi_0'^k(p) - p\, h_1^k(0,p)$ *is strictly decreasing in* p*, for* $k \in \{L,G\}$

Concavity of short term profits (Assumption 3a) is a typical economic assumption. Supermodularity (Assumption 3b) supports the intuition that a higher reference price enables the myopic firm to charge higher prices. Supermodularity of π^G immediately follows from the supermodularity of h^G (Assumption 2e), and the fact that $h^G(R-p,R)$ is increasing in R (Assumption 1a). Supermodularity of π^L restricts the degree of convexity of h^L in the reference gap, x. Assumptions 3c and 3d insure the uniqueness of steady states and the intuition that higher minimum prices in the price history enable the firm to charge higher prices in the long run. Assumption 3c essentially means that the marginal profit is more sensitive to changes in price than changes in the reference price in the direction of $(p, \theta r + (1-\theta)p)$. Defining $\tilde{\pi}^L(x,R) = \pi^L(p,R)$, for $x = R - p$, Assumption 3c is equivalent to $\tilde{\pi}^L(x,R)$ being strictly concave in R, i.e. keeping the reference gap, x, constant, profit exhibits decreasing marginals with respect to R. Under Assumption 3a, Assumption 3d holds trivially if the reference effect, h^L (i.e. demand) is convex in R. If h^L is concave in R, then Assumption 3d restricts its concavity in R. Assumption 3e, means that if the firm charges the reference price, profit is more sensitive to changes in price at lower reference prices. That is, marginal profit diminishes as price is set to the reference price (see PW, Assumption 3b).

Section 6.2 provides conditions on common absolute and relative difference reference effect models to satisfy Assumptions 2 and 3.

6.1. General Results

Throughout this section, we impose Assumptions 1,2 and 3 on the demand model. We first obtain the following result, which extends Lemma 2 for general reference effects:

Lemma 7. *The short-term profit,* $\pi(p,R) = \min\Big(\pi^L(p,R), \pi^G(p,R)\Big)$*, is supermodular in* (p,R).

Proof: See Appendix. □

Therefore, the Bellman Equation for this section can be written as:

$$J(r_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \Big\{ \min(\pi^L, \pi^G) + \beta J\Big(\min(p_t, r_{t-1}), p_t \Big) \Big\}, \tag{19}$$

where $R_t = \theta r_{t-1} + (1-\theta)p_{t-1}$, and $\pi(p,R) = p\, d(p,R)$, with $d(p,R)$ defined by (17).

We use the same approach as in Section 3 to identify the steady states of Problem (19). The auxiliary upper bound problem, J_r^α, used to identify the steady states of Problem (19) has precisely the same structure as in (8). The Euler Equation characterizing the steady states of J_r^α becomes:

$$(1-\alpha)(\pi_1^L + \beta(1-\theta)\pi_2^L)(p, \theta r + (1-\theta)p) + \alpha(\pi_1^G + \beta \pi_2^G)(p,p) = 0. \tag{20}$$

For simplicity, we use the same notation as in Section 3 to define q and s as the unique solutions of the respective equations:

$$\pi_0'(p) - (1 - \beta(1-\theta))\, p\, h_1^L(0,p) \,=\, 0, \tag{21}$$

$$\pi_0'(p) - (1-\beta)\, p\, h_1^G(0,p) = 0. \tag{22}$$

In particular, for linear reference effects $h^L(x,R) = \lambda x$ and $h^G(x,R) = \gamma x$, we recover equations (9) and (10). As in Section 3, these thresholds define the regions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. The following lemma identifies steady states of J_r^α, based on which we characterize the steady states of Problem (19).

Lemma 8. *(a) For $r \in \mathbf{R}_1$, $p_L^{**}(r)$ is a steady state of J_r^α for $\alpha = 0$, where $p_L^{**}(r)$ solves:*

$$\pi_1^L(p, \theta r + (1-\theta)p) + \beta(1-\theta)\pi_2^L(p, \theta r + (1-\theta)p) = 0. \tag{23}$$

In this case $(r, p_L^{**}(r))$ is a steady state of Problem (19).

(b) There exists $\alpha \in [0,1]$ such that, for $r \in \mathbf{R}_2$, r is a steady state for J_r^α. In this case (r,r) is a steady state of Problem (19).

Proof: See Appendix. □

Assumption 3c insures the uniqueness of $p_L^{**}(r)$ as the solution to equation (23). Assumption 3d guarantees that $p_L^{**}(r)$ is increasing in r, i.e. a higher initial minimum price enables the firm to charge higher long run prices.

A similar proof as for Proposition 2 shows that, if the initial state is in any of the regions $\mathbf{R}_i$, $i = 1,2,3$, then the state path remains in that region over time. Thus, if the price path converges, it must converge to a point in the same region. The following result extends Proposition 3, insuring that the only steady states of Problem (7) are the ones specified in Lemma 8, and all optimal price paths are monotonic.

Proposition 7. *The set of steady states of Problem (19) is: $\{(r, p_L^{**}(r)) \mid r \in \mathbf{R}_1\} \cup \{(r,r) \mid r \in \mathbf{R}_2\}$. Starting at an initial state (r_0, p_0), the optimal price path $\{p_t\}$ converges monotonically to a steady which equals: (a) $p_L^{**}(r_0)$, if $r_0 \in \mathbf{R}_1$, (b) r_0, if $r_0 \in \mathbf{R}_2$, and (c) s, if $r_0 \in \mathbf{R}_3$. Finally, the optimal pricing policy, $p^*(r,p)$, for Problem (19) is increasing in both r and p.*

Proof: See Appendix. □

Therefore, under Assumptions 2, and 3, the results in Section 3 extend to general reference effects. Specifically, the memory structure leads to a range of steady states. This range is affected by the slopes of h^L and h^G at $x=0$ (see equations (21) and (22)).

Interestingly, price monotonicity is a robust effect in our model, holding under general reference effects. Under the adaptive expectations framework, the optimal price paths are generally not monotonic (see e.g. Figure 4 in PW). This suggests that high-low pricing may be triggered by memory of intermediate prices, in an adaptive expectation framework. Overall, we conclude that the memory process is an important factor affecting the structure of optimal pricing policies.

6.2. Special Demand Models

Assumptions 2 and 3 impose conditions on the reference effects and thus implicitly on the demand function. These assumptions involve appropriately extending the loss and gain components of the kinked reference effect over the entire domain, so as to satisfy equation (18). This section provides explicit modeling conditions on the reference effect, which are sufficient to validate these assumptions. We focus on most common absolute and relative difference reference effect models (see Section 2 in PW for a discussion), and linear extensions of the reference effect.

Absolute and relative difference models (AD, respectively RD) are defined by a reference effect h where $h(x,R)=f(x)$, $x=R-p$ for AD models and $x=\frac{R-p}{R}$ for RD models. In both cases f is increasing in x and $f(0)=0$. Furthermore, following prospect theory, $f(x)$ is concave for $x\geq 0$ and convex for $x\leq 0$, and further $\lambda=f'(0^-)\geq f'(0^+)=\gamma$ (loss aversion). For both models, we extend the gain and loss components of the reference effect $f(x)$ linearly, by defining smooth functions g, and l as follows: (1) $g(x)=f(x)$ for $x\geq 0$ and $g(x)=\gamma x$ for $x\leq 0$, and (2) $l(x)=f(x)$ for $x\leq 0$ and $l(x)=\lambda x$ for $x\geq 0$.

Proposition 8. *Proposition 7 holds for (a) AD models satisfying (i) $f'(-x)\geq\gamma$ for $x\in\mathbf{P}$, (ii) $f^{''}(0)=0$, and (iii) $\frac{f^{''}(x)}{f^{'}(x)}\leq\frac{1}{\bar{p}}$ for all $|x|\in\mathbf{P}$, and for (b) RD models satisfying (i) $f'(x)\geq\gamma$ for $x\leq 0$, (ii) $f^{''}(0)=0$, and (iii) $(1-x)\frac{f^{''}(x)}{f^{'}(x)}\leq 2$ for all x.*

Proof: (a) The proof follows by verifying the Assumptions 1, 2, and 3. Convexity of $f(x)$ and part (i) insure the single crossing property (Assumption 2a) over the loss domain. Part (iii) insures the supermodularity of π^L over the entire domain (Assumption 3b). The Other assumptions are easily verified.

(b) The proof follows by verifying the Assumptions 1, 2, and 3. Supermodularity of $g(x)$ is immediate because g is increasing in R and $-g'(x) + (1-x)g''(x) \leq 0$. Part (iii) insures the supermodularity of π^L over the entire domain. Further, this assumption guarantees that Assumptions 3c and 3d hold. Others are also simply verified. $\square$

Proposition 8 provides relatively simple conditions on common AD and RD models for the results of Section 6 to hold. In particular, these conditions are satisfied for piecewise linear AD and RD reference effects. Condition (iii) in Proposition 8, essentially limits the convexity of reference effects over losses; this is supported by empirical evidence which finds the reference effect to be nearly linear on the loss domain (see e.g. Abdellaoui et al. 2007 for a review). Our conditions (i) and (ii) are obtained from constructing linear extensions of the reference effect on the loss, respectively gain domains. Weaker conditions can be obtained, at the cost of mathematical and expository complexity, by constructing non-linear extensions of the reference effect, which satisfy Assumptions 1, 2, and 3.

7. Conclusions

The literature on dynamic pricing with reference effects, generally assumes that price anchors are formed by an exponential smoothing process, as a weighted average of all past prices. In this paper, we motivated the peak-end rule, as a consumer anchoring and memory mechanism. First proposed by Fredrickson and Kahneman (1993), the peak-end rule suggests (in the pricing context) that consumers' price judgements are based on two representative past prices, the lowest and the most recent price. The reference price is formed as a weighted average of these two prices. In this context, we studied the optimal pricing strategies of a monopolist, in response to consumers that exhibit behavioral regularities. Specifically, consumers anchor on past prices, by forming a reference price based on the peak-end rule, and their purchase decisions are influenced by these anchors, in the spirit of prospect theory.

Our results showed how the peak-end anchoring process interacts with loss aversion to affect the structure of the optimal pricing strategies. Specifically, we showed that this memory structure leads to a range of constant optimal pricing strategies even with loss neutral buyers, unlike the current literature. The range of steady states is wider in the loss averse case, and becomes wider the more loss averse consumers are, and the more they anchor on lowest prices. Consistent with the literature, the value of the steady state prices decreases with consumers' sensitivity to gains and losses. In addition, the more consumers anchor on the minimum price, the lower the optimal prices, and the firm's profits. In contrast with previous literature, we found that optimal price paths are monotonic, following a traditional skimming or penetration strategy even under general reference effects. Finally, our findings also identify which behavioral parameters are important for a firm to measure, and in which sequence, in order to determine its optimal pricing strategy.

The interaction between the peak-end anchoring process and consumers' loss attitude has other consequences for the firm's optimal pricing strategy. Specifically, the general conception in the reference pricing literature is that loss seeking behavior leads to a cycling optimal pricing policy, i.e. no constant pricing policy can be optimal (PW and references therein). However, numerical investigation on the peak-end model shows that the optimal policy converges to a steady state under some conditions; in particular it always converges in the extreme case where consumers anchor only on the lowest price.

We conclude by highlighting limitations of our work, and providing some suggestions for further research in behavioral dynamic pricing. (1) Our results motivate the moment-based approach to remembered utility as a new reference price formation model. Depending on the context, it is interesting to explore which other prices may affect consumers' decision making process. This may include an expiry and updating scheme for the minimum price. (2) Similarly, it would be interesting to investigate how other types of memory asymmetries, such as the kinked exponential smoothing model proposed in Gaur and Park (2007), affect the optimal pricing policy of the firm. Preliminary results (Nasiry and Popescu 2008) indicate that our insights remain valid in that context. (3) Current empirical research in the pricing context has focused on validating exponentially smoothed reference price models. Our work highlights the importance of testing alternative memory models, such as the peak-end rule, and determining the relevant parameters, in particular θ, in a practical setting. (4) Like previous work in this area, our paper ignores the heterogeneity in consumer preferences and reference price formation, as well as the possibility of having multiple reference prices. It is an open challenge to explore the optimal pricing structure under such heterogeneities. (5) The current model ignores the possible strategic reaction of consumers to the pricing strategies of the firm. For example, consumers may postpone their purchase in response to a skimming type strategy. Future research should guide firms to respond to consumer strategic behavior, combined with reference dependence, and ask whether any of these behavioral effects is dominant.

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Appendix: Proofs

Proof of Lemma 2: For $p^l \leq p^h$ and $R^l \leq R^h$, we need to show that:

$$\pi(p^h, R^h) - \pi(p^l, R^h) \geq \pi(p^h, R^l) - \pi(p^l, R^l). \tag{24}$$

We show this by considering all possible cases: (1) $p^l \leq p^h \leq R^l \leq R^h$, (2) $p^l \leq R^l \leq p^h \leq R^h$, (3) $p^l \leq R^l \leq R^h \leq p^h$, (4) $R^l \leq p^l \leq p^h \leq R^h$, (5) $R^l \leq p^l \leq R^h \leq p^h$, (6) $R^l \leq R^h \leq p^l \leq p^h$. Cases 1 and 6 follow immediately because all (p, R) pairs fall on π_γ or π_λ, respectively, which have increasing differences in (p, R).

Case 2: For $p^l \leq R^l \leq p^h \leq R^h$, (24) simplifies to:

$$\gamma(R^h - p^h)p^h - \gamma(R^h - p^l)p^l \geq -\lambda(p^h - R^l)p^h - \gamma(R^l - p^l)p^l, \text{ or}$$

$$\gamma(R^h - p^h)p^h + \lambda(p^h - R^l)p^h \geq \gamma(R^h - R^l)p^l.$$

Because $\lambda > \gamma$ and $p^h \geq R^l$, it is sufficient to show $\gamma(R^h - p^h)p^h + \gamma(p^h - R^l)p^h \geq \gamma(R^h - R^l)p^l$, which holds because $\gamma p^h(R^h - R^l) \geq \gamma p^l(R^h - R^l)$.

Case 3: For $p^l \leq R^l \leq R^h \leq p^h$, (24) simplifies to:

$$-\lambda(p^h - R^h)p^h - \gamma(R^h - p^l)p^l \geq -\lambda(p^h - R^l)p^h - \gamma(R^l - p^l)p^l,$$

or, $\lambda p^h(R^h - R^l) \geq \gamma p^l(R^h - R^l)$, which is true.

Case 4: For $R^l \leq p^l \leq p^h \leq R^h$, (24) simplifies to:

$$\gamma(R^h - p^h)p^h - \gamma(R^h - p^l)p^l \geq -\lambda(p^h - R^l)p^h + \lambda(p^l - R^l)p^l, \text{ or}$$

$$p^h\Big(\gamma(R^h - p^h) + \lambda(p^h - R^l)\Big) \geq p^l\Big(\lambda(p^l - R^l) + \gamma(R^h - p^l)\Big).$$

It is enough to show that $\gamma(R^h - p^h) + \lambda(p^h - R^l) \geq \lambda(p^l - R^l) + \gamma(R^h - p^l)$, which holds because $\lambda(p^h - p^l) \geq \gamma(p^h - p^l)$.

Case 5: For $R^l \leq p^l \leq R^h \leq p^h$, (24) simplifies to:

$$-\lambda(p^h - R^h)p^h - \gamma(R^h - p^l)p^l \geq -\lambda(p^h - R^l)p^h + \lambda(p^l - R^l)p^l, \text{ or}$$

$$\lambda p^h(R^h - R^l) \geq \lambda p^l(p^l - R^l) + \gamma(R^h - p^l)p^l.$$

Because $\lambda \geq \gamma$ and $R^h \geq p^l$, it is sufficient to show that:

$$p^h(R^h - R^l) \geq p^l(p^l - R^l) + (R^h - p^l)p^l = p^l(R^h - R^l),$$

which obviously holds.

Proof of Proposition 1: For any steady state (r, p), two cases are possible, either $r = p$, or $r < p$. In the second case, $R < p$ and starting at (r, p), the price path gives a consistent perception of loss. Therefore the steady state price must be the same as the steady state of Problem (8), with $\alpha = 0$, i.e. $p = p_\lambda^{**}(r)$. Thus Problem (7) has only two types of steady states. It remains to identify the regions where each type of steady state is relevant.

First assume $r < q$. We show by contradiction that (r, r) cannot be a steady state of Problem (7). If (r, r) is a steady state, the profit from charging a constant price $p_t \equiv r$ exceeds the profit on path $p_t = r + \delta, \forall t$, i.e.:

$$\frac{\pi_0(r)}{1-\beta} \geq \pi_0(r+\delta) - \lambda\delta(r+\delta) + \frac{\beta}{1-\beta}\Big(\pi_0(r+\delta) - \lambda(r+\delta-R)(r+\delta)\Big),$$

where $R = \theta r + (1-\theta)(r+\delta)$. This reduces to: $\pi_0(r+\delta) - \pi_0(r) \leq \lambda(r+\delta)(1-\beta(1-\theta))$. Dividing both sides by δ and letting δ go to zero, we have:

$$\pi_0'(r) \leq \lambda(1-\beta(1-\theta))r. \tag{25}$$

This holds with equality for $r = q$ (see (9)), and because the LHS is strictly decreasing in r, (25) implies $r \geq q$, a contradiction. We conclude that, for $r < q$, the only possible steady state for Problem (7) is $(r, p_\lambda^{**}(r))$.

Moreover, because $p^{**}(r) < r$ for $r > q$, it follows that, for $r \geq q$, the only possible steady state is (r, r). We prove by contradiction that (r, r) cannot be a steady state for $r > s$. If (r, r) is a steady state, the profit from charging a constant price $p_t \equiv r$ exceeds the profit along the alternative path $p_t = r - \delta$, $\forall t$, i.e.:

$$\frac{\pi_0(r)}{1-\beta} \geq \pi_0(r-\delta) + \gamma\delta(r-\delta) + \frac{\beta}{1-\beta}\pi_0(r-\delta).$$

This reduces to $\pi_0(r) - \pi_0(r-\delta) \geq \gamma\delta(1-\beta)(r-\delta)$. Dividing by δ and letting δ go to zero, we have:

$$\pi_0'(r) \geq \gamma(1-\beta)r, \tag{26}$$

which holds with equality for $r = s$ (see (10)). Because $\pi_0'(r)$ is strictly decreasing in r, (26) implies that $r \leq s$, a contradiction. We conclude that steady states of the form (r, r) can only be relevant when $q \leq r \leq s$.

Proof of Proposition 2: (a) We prove this in two parts, depending if $r_0 \in \mathbf{R}_1$, or $r_0 \in \mathbf{R}_2$. For $r_0 \in \mathbf{R}_1$, we consider two cases: $(r_0, p_0) \in \overline{\mathbf{R}}_{1a}$ and $(r_0, p_0) \in \overline{\mathbf{R}}_{1b}$.

Claim 1. For $(r_0, p_0) \in \overline{\mathbf{R}}_{1a}$, then $p_t \geq r_0$ for any t.

Proof: Denote $J^{\alpha=0}$ the objective function in Problem (8), with $\alpha = 0$. $J^{\alpha=0}$ is supermodular in (p, R), and thus the price path converges monotonically to the steady state price, $p^{**}_\lambda(r_0)$. Because $p_0 < p^{**}_\lambda(r_0)$, the optimal price path for $J^{\alpha=0}$, increases to this steady state, and $p^*_t(R_t) \leq p^{**}(r_0)$ for all t. This implies that $r_t = r_0$ along this path (the minimum price does not change over time), and thus $R_t = \theta r_0 + (1-\theta)p_{t-1} \leq p_{t-1} \leq p_t$. Therefore, $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$, and this path is feasible for (7), and yields the same value which leads us to conclude that the same path is also optimal for Problem (7).

This result is stronger than stated in the claim, because it guarantees also the existence of the steady state, and the monotonicity of the price path.

Claim 2. Given $(r_0, p_0) \in \overline{\mathbf{R}}_{1b}$, then $p_t \geq p^{**}(r_0) \geq r_0$ for any t.

Proof: For $r_0 \leqslant q$, $(r_0, p^{**}_\lambda(r_0))$ is a steady state of Problem (7) (Proposition 1). We show that if at any time it is optimal for the price to be below $p^{**}_\lambda(r_0)$, then $(r_0, p^{**}_\lambda(r_0))$ cannot be a steady state of Problem (7) which is a contradiction.

Let $R^*_1 = \theta r_0 + (1-\theta)p^{**}_\lambda(r_0)$. We show that $p_1 = p^*(r_0, p_0) \geq p^{**}_\lambda(r_0)$, and then by induction we conclude that $p_t \geq p^{**}_\lambda(r_0)$. Assume by contradiction, $p_1 < p^{**}_\lambda(r_0)$. Then:

$$\pi(p_1, R_1) + \beta J(\min(r_0, p_1), p_1) > \pi(p^{**}_\lambda(r_0), R_1) + \beta J(r_0, \pi^{**}_\lambda(r_0)),$$

or equivalently by defining $\Delta J = J(r_0, p^{**}_\lambda(r_0)) - J(\min(p_1, r_0), p_1)$,

$$\pi(p_1, R_1) - \pi(p^{**}_\lambda(r_0), R_1) > \beta \Delta J. \tag{27}$$

Because $p_0 > p^{**}_\lambda(r_0) > p_1$, we have $R_1 > R^*_1$. Supermodularity of $\pi(p, R)$ (Lemma 2), then implies:

$$\pi(p_1, R^*_1) - \pi(p^{**}_\lambda(r_0), R^*_1) \geq \pi(p_1, R_1) - \pi(p^{**}_\lambda(r_0), R_1). \tag{28}$$

Because $p^{**}_\lambda(r_0) > R^*_1$, it follows that $\pi_\lambda(p^{**}_\lambda(r_0), R^*_1) \leq \pi_\gamma(p^{**}_\lambda(r_0), R^*_1)$, and $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. Therefore equation (28) can be written as:

$$\pi(p_1, R^*_1) - \pi_\lambda(p^{**}_\lambda(r_0), R^*_1) \geq \pi(p_1, R_1) - \pi(p^{**}_\lambda(r_0), R_1).$$

Combining with equation (27), we have: $\pi(p_1, R^*_1) - \pi_\lambda(p^{**}_\lambda(r_0), R^*_1) > \beta \Delta J$. Or equivalently:

$$\pi_\lambda(p^{**}_\lambda(r_0), R^*_1) + \beta J(r_0, p^{**}_\lambda(r_0)) < \pi(p_1, R^*_1) + \beta J(\min(p_1, r_0), p_1).$$

This contradicts the fact that $(r_0, p^{**}(r_0))$ is a steady state of Problem (7). We conclude that $p_t \geq p^{**}(r_0)$ for all t.

Claim 3. For $r_0 \in \mathbf{R}_2$, then $p_t \geq r_0$ for any t.

Proof: We show that $p_1 = p^*(r_0, p_0) \geq r_0$. By induction, this shows that $p_t^* \geq r_0$, $\forall t$. Suppose by contradiction, $p_1 < r_0$. Then, because $p_1 \leq \theta r_0 + (1-\theta)p_0 = R_1$, we have $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. Now, (7) can be written as:

$$J(r_0, p_0) = \max_{p_1 < r_0} \Big\{ \pi_\gamma(p_1, R_1) + \beta J(p_1, p_1) \Big\}. \tag{29}$$

We show that, in this case, (r_0, r_0) cannot be a steady state of Problem (7), a contradiction. Because $p_1 < r_0$, we have $\pi_\gamma(p_1, R_1) + \beta J(p_1, p_1) > \pi_\gamma(r_0, R_1) + \beta J(r_0, r_0)$. Equivalently, by defining $\Delta J = J(r_0, r_0) - J(p_1, p_1)$, we obtain:

$$\pi(p_1, R_1) - \pi(r_0, R_1) > \beta \Delta J. \tag{30}$$

Because $\pi_\gamma(p, R)$ is supermodular, and $R_1 > r_0$, it follows that:

$$\pi_\gamma(p_1, R_1) - \pi_\gamma(r_0, R_1) \leq \pi_\gamma(p_1, r_0) - \pi_\gamma(r_0, r_0). \tag{31}$$

Combining equations (31) and (30), we have: $\pi_\gamma(p_1, r_0) - \pi_0(r_0) > \beta \Delta J$, or equivalently: $\pi_0(r_0) - \beta J(r_0, r_0) < \pi_\gamma(p_1, r_0) + \beta J(p_1, p_1)$. This contradicts the fact that (r_0, r_0) is a steady state of Problem (7). We conclude if the initial state is such that $q \leq r_0 \leq s$, then $p_t \geq r_0$ for all t.

(b) From Proposition 1, we know that (s, s) is a steady state of Problem (7). We consider two cases: $p_0 = r_0$ and $p_0 > r_0$.

Assume $r_0 = p_0$. Consider the problem:

$$J^\phi(p_{t-1}, p_{t-1}) = \max_{p_t} \Big\{ \pi(p_t, p_{t-1}) + \beta J^\phi(p_t, p_t) \Big\}. \tag{32}$$

Because the value function in Problem (7) is increasing in its arguments (Lemma 1), it follows that: $J(r_{t-1}, p_{t-1}) \leq J^\phi(p_{t-1}, p_{t-1})$. Equality happens if $r_{t-1} = p_{t-1}$ for all t, i.e. starting from $r_0 = p_0$, the price path is decreasing. By construction, the steady state of Problem (32) is the same as the steady state of the following problem:

$$\tilde{J}(p_{t-1}) = \max_{p_t} \Big\{ \pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t) \Big\}, \tag{33}$$

which is s. Therefore (s, s) is the unique steady state of Problem (32). Because $\pi(p_t, p_{t-1})$ is supermodular, and starting at $p_0 > s$, the optimal price path of Problem (32) is decreasing and converges to s. Starting at an initial state (r_0, p_0) such that $p_0 = r_0 > s$, the optimal path of Problem (32) is feasible for Problem (7) and yields the same value. This is because at each stage $r_{t-1} = p_{t-1}$, which implies $\min(p_{t-1}, p_t) = p_t$ and $R_t = p_{t-1}$. Therefore for such initial states, this price path is optimal for Problem (7) and converges to s. This also implies $r_{t-1} \geq s$, as desired.

Now assume that $p_0 > r_0$. The following claim proves the desired result.

Claim 4. For $p_0 > r_0 > s$, if the optimal price p_t is such that $p_t \leq r_0$, then $p_t \geq s$.

Proof: We show that if p_1 is such that $p_1 \leq r_0$, then $p_1 > s$. By induction, this implies $p_t \geq s$.

Suppose by contradiction that $p_1 = p^*(r_0, p_0) < s$. Then $p_1 \leq \theta r_0 + (1-\theta)p_0 = R_1$, and hence $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. This allows to write (7) as:

$$J(r_0, p_0) = \max_{p_1} \Big\{ \pi_\gamma(p_1, R_1) + \beta J(p_1, p_1) \Big\}.$$

Because $p_1 < s$, we have $\pi_\gamma(p_1, R_1) + \beta J(p_1, p_1) > \pi_\gamma(s, R_1) + \beta J(s, s)$. Equivalently, defining $\Delta J = J(s, s) - J(p_1, p_1)$, we have:

$$\pi_\gamma(p_1, R_1) - \pi_\gamma(s, R_1) > \beta \Delta J. \tag{34}$$

Because $\theta r_0 + (1-\theta)p_0 = R_1 > s$ and $p_1 < s$, and $\pi_\gamma(p, R)$ is supermodular, it follows that:

$$\pi_\gamma(p_1, R_1) - \pi_\gamma(s, R_1) \leq \pi_\gamma(p_1, s) - \pi_\gamma(s, s). \tag{35}$$

Combining equations (35) and (34), we have $\pi_\gamma(p_1, s) - \pi_\gamma(s, s) > \beta \Delta J$, or equivalently, $\pi_0(s) + \beta J(s, s) < \pi_\gamma(p_1, s) + \beta J(p_1, p_1)$. This implies (s, s) cannot be a steady state of Problem (7), a contradiction. We conclude that if the initial state (p_0, r_0) is such that $r_0 > s$, we have $p_t \geq s$.

Proof of Proposition 3: (a) Consider two possible cases: 1) $(r_0, p_0) \in \overline{\mathbf{R}}_{1a}$, and 2) $(r_0, p_0) \in \overline{\mathbf{R}}_{1b}$. The first case is proved in Proposition 2. For the second case the argument in Section 3.2, shows that the price path is monotonic. Moreover, because $p_0 \geq p_\lambda^{**}(r_0)$, it follows that, in this case, the price path is decreasing and converges to $p_\lambda^{**}(r_0)$.

(b) The argument in Section 3.2, shows that the price path in this region is monotonic. Because $p_0 \geq r_0$, the price path is decreasing to its steady state r_0.

(c) In the proof of Proposition 2b, we showed that for initial states $p_0 = r_0 \geq s$, the price path decreases monotonically to the steady state (s, s). Now we focus on the case where the initial state is such that $p_0 > r_0$. The next claim which insures the price path eventually falls below r_0.

Claim 1. *Starting at (r_0, p_0), where $p_0 > r_0 \geq s$, at some point in time, T, the optimal price falls below r_0, i.e $p_T \leq r_0$.*

Proof: Suppose by contradiction that the optimal price path is such that $\{p_t\} > r_0$. Thus the value function in Problem (7), with r_0 as a parameter, can be written as:

$$J_{r_0}(p_0) = \max_{p_1} \Big\{ \pi(p, R) + \beta J_{r_0}(p_1) \Big\}.$$

The objective function is supermodular in (p, R) (Lemma 2), so the price path is monotonic, and converges to a steady state. By Proposition 1, this must be (s, s), which contradicts $p_t > r_0 \geq s$. We conclude that at some point in time, T, the optimal price is such that $p_T \leq r_0$. $\square$

Let T be the first time that the optimal price falls below r_0. Thus at time T, the value function in Problem (7) can be written as:

$$J(r_0, p_{T-1}) = \max_{p_T \le r_0} \Big\{ \pi(p,R) + \beta J(p_T, p_T) \Big\}.$$

$p_T > s$ (proof of Proposition 2, Claim 4). For an initial steady state (r_0, p_0) such that $p_0 = r_0 > s$, the price path decreases monotonically to s, and (s,s) is the corresponding steady state. The value function for $t < T$ is given by the finite horizon model:

$$J_{t-1}(p_{t-1}) = \max_{p_t} \Big\{ \pi(p,R) + \beta J_t(p_t) \Big\}, \quad t < T,$$

where $J_T(p_T) = J(r_0, p_T)$. Because the objective function is supermodular in (p,R), the price path is monotonic, and decreases to p_{T-1}. Note that it cannot be increasing, because then it would have to converge to a steady state above r_0, which is impossible (Proposition 1). In summary, starting at an initial state (r_0, p_0) such that $p_0 > r_0 > s$, the price path decreases until it falls below r_0 and then converges decreasingly to s.

Proof of Lemma 6: (a) By definition, $p_\lambda(R)$ solves the first order condition:

$$\frac{\partial \pi_\lambda(p,R)}{\partial p} = \pi_0'(p) - 2\lambda p + \lambda R = 0.$$

The equation $p_\lambda(R) = R$ results in:

$$\pi_0'(R) - \lambda R = 0. \tag{36}$$

The LHS of (36) is strictly decreasing in R, strictly positive for $R = 0$ and negative for a sufficiently large $R \in \mathbf{P}$. Therefore the above equation has a unique solution, R_λ.

The same argument guarantees the existence of R_γ, the unique solution of the equation:

$$\pi_0'(R_\gamma) - \gamma R_\gamma = 0. \tag{37}$$

Because R_λ and R_γ solve (36) and (37) respectively, and $\lambda \ge \gamma$, if follows that $R_\lambda \le R_\gamma$.

(b) The result follows because $p_\gamma(0) \ge p_\lambda(0) > 0$ and $p_\lambda(\cdot)$ and $p_\gamma(\cdot)$ are increasing functions of R, and thus single cross the identity line from above. These crossing points exist and are unique as shown in part (a) above.

Proof of Proposition 4: First observe that $p_M(R) \in \{p_\lambda(R), p_\gamma(R), R\}$. By (5), $p_\lambda(R)$ is only feasible for $R \le p_\lambda(R)$, i.e. $R \le R_\lambda$ (by Lemma 6b). Also $p_\gamma(R)$ is only feasible for $R \ge p_\gamma(R)$, i.e. $R \ge R_\gamma$ (by Lemma 6b). Hence $p_\lambda(\cdot)$ is optimal for $R \le R_\lambda$ because $\pi = \pi_\lambda$ and $p_\gamma(\cdot)$ is optimal for $R \ge R_\gamma$ because $\pi = \pi_\gamma$. For $R_\lambda \le R \le R_\gamma$, $p^M(R) = R$.

Proof of Proposition 5: The value function is increasing, which implies:

$$p^*(r,p) = \arg\max_{p\in P}\{\pi(p,R) + \beta J(\min(r,p),p)\} \geq \arg\max_{p\in P}\pi(p,R) = p^M(R).$$

Forward induction proves the other part. Starting at the same initial state, R_1, we have $p_1^* = p^*(R_1) \geq p_1^M(R_1)$. This implies that $r_1^* \geq r_1^M$, i.e. the minimum price for the strategic firm in the next period is larger than the one for the myopic firm. Now as the induction assumption, suppose that $p_{t-1}^* \geq p_{t-1}^M$ and $r_{t-1}^* \geq r_{t-1}^M$. This implies $\min(p_{t-1}^*, r_{t-1}^*) \geq \min(p_{t-1}^M, r_{t-1}^M)$. Then the reference price in period t is such that $R_{t-1} \geq R_{t-1}^M$. By Lemma 6, $p^M(R)$ is increasing in R and thus $p_t^M(R_t^M) \leq p_t^M(R_t^*) \leq p^*(R_t^*)$. Therefore the myopic policy underprices the product.

Moreover, because π is supermodular (Lemma 2), it follows that all the price paths are monotonic and converge to a constant price, which depends on the initial state.

Proof of Lemma 7: For $p^l \leq p^h$ and $R^l \leq R^h$ we show that:

$$\pi(p^h,R^h) - \pi(p^l,R^h) \geq \pi(p^h,R^l) - \pi(p^l,R^l). \tag{38}$$

Similar to the case of linear reference effects (Lemma 2), we consider the following exhaustive cases: (1) $p^l \leq p^h \leq R^l \leq R^h$, (2) $p^l \leq R^l \leq p^h \leq R^h$, (3) $p^l \leq R^l \leq R^h \leq p^h$, (4) $R^l \leq p^l \leq p^h \leq R^h$, (5) $R^l \leq p^l \leq R^h \leq p^h$, (6) $R^l \leq R^h \leq p^l \leq p^h$.

Cases (1) and (6) follow because all (p,R) fall on either loss or gain domains, and π^G and π^L are supermodular (Assumption 3b).

Case 2: For $p^l \leq R^l \leq p^h \leq R^h$, (38) becomes:

$$p^h h^G(R^h - p^h, R^h) - p^l h^G(R^h - p^l, R^h) \geq p^h h^L(R^l - p^h, R^l) - p^l h^G(R^l - p^l, R^l).$$

Because $h^L(R^l - p^h, R^l) < h^G(R^l - p^h, R^l)$, a sufficient condition for the above inequality to hold is:

$$p^h h^G(R^h - p^h, R^h) - p^l h^G(R^h - p^l, R^h) \geq p^h h^G(R^l - p^h, R^l) - p^l h^G(R^l - p^l, R^l),$$

which follows because π^G is supermodular in (p,R).

Case 3: For $p^l \leq R^l \leq R^h \leq p^h$, (38) becomes:

$$p^h h^L(R^h - p^h, R^h) - p^l h^G(R^h - p^l, R^h) \geq p^h h^L(R^l - p^h, R^l) - p^l h^G(R^l - p^l, R^l). \tag{39}$$

Supermodularity of $\pi^L(p,R) = p h^L(R-p,R)$ implies:

$$p^h[h^L(R^h - p^h, R^h) - h^L(R^l - p^h, R^l)] \geq p^l[h^L(R^h - p^l, R^h) - h^L(R^l - p^l, R^l)]. \tag{40}$$

On the other hand, Assumption 2c implies that

$$h^L(R^h-p^l,R^h)-h^L(R^l-p^l,R^l)\geq h^G(R^h-p^l,R^h)-h^G(R^l-p^l,R^l), \tag{41}$$

which together with (40) yield the desired result (39).

Case 4: For $R^l \leq p^l \leq p^h \leq R^h$, (38) becomes:

$$p^h h^G(R^h-p^h,R^h)-p^l h^G(R^h-p^l,R^h)\geq p^h h^L(R^l-p^h,R^l)-p^l h^L(R^l-p^l,R^l). \tag{42}$$

Supermodularity of $h^G(R-p,R)$ (Assumption 2e) implies that:

$$h^G(R^h-p^h,R^h)-h^G(R^h-p^l,R^h)\geq h^G(R^l-p^h,R^l)-h^G(R^l-p^l,R^l).$$

On the other hand, Assumption 2b implies

$$h^G(R^l-p^h,R^l)-h^G(R^l-p^l,R^l)\geq h^L(R^l-p^h,R^l)-h^L(R^l-p^l,R^l).$$

Putting these two together, we obtain:

$$h^G(R^h-p^h,R^h)-h^G(R^h-p^l,R^h)\geq h^L(R^l-p^h,R^l)-h^L(R^l-p^l,R^l).$$

Because $h^G(R^h-p^h,R^h)\geq 0$ and $h^L(R^l-p^h,R^l)<0$, we obtain the desired result (42).

Case 5: This follows the same approach as Case (2), driven in this case by supermodularity of π^L.

Proof of Lemma 8: Assumptions 1c and 3e insure that (21) and (22) have unique solutions. This is because the LHS of both equations are strictly decreasing, positive at $p=0$ and negative for a high enough $p\in\mathbf{P}$. Moreover, the solutions of these equations are such that $q\leq s$ (because $1-\beta(1-\theta)\geq 1-\beta$), and $h_1^L(0,p)\geq h_1^G(0,p)$.

(a) By definition (equation (23)), p_L^{**} solves (20) for $\alpha=0$. This has a unique solution because the LHS of (23) is strictly decreasing in p. The derivative of LHS with respect to p is:

$$\pi_{11}^L+(1-\theta)\pi_{12}^L+\beta(1-\theta)(\pi_{12}^L+(1-\theta)\pi_{22}^L), \tag{43}$$

which is negative because of Assumption 3c. Moreover the LHS of (23) is positive for $p=0$, and negative for a high enough $p\in\mathbf{P}$, so this equation has a unique solution.

(b) Substituting $p=r$ in (20), we obtain:

$$\pi_0'(r)-[1-\beta(1-\theta(1-\alpha))]\,r\,h_1(0,r)=0. \tag{44}$$

The LHS of (44) is strictly decreasing in r (Assumptions 3e). For $\alpha = 0$, $r = q$ solves (44). It follows that for $r > q$ and $\alpha = 0$ the LHS is negative. On the other hand, for $\alpha = 1$, $r = s$ solves (44). Therefore for $r < s$ and $\alpha = 1$, the LHS is positive. The result follows because the LHS is strictly increasing in α, negative at $\alpha = 0$ and positive at $\alpha = 1$.

(c) The proof is similar to that of Lemma 5.

Proof of Proposition 7: The proof follows the same lines as Proposition 1. For the same alternative price paths, equations (25) and (26) become $\pi_0'(r) \leq (1-\beta(1-\theta))rh_1^L(0,r)$, respectively $\pi_0'(r) \geq (1-\beta)rh_1^G(0,r)$. The same reasoning as in Proposition 1 shows that steady states of the form $(r_0, p_L^{**}(r_0))$, respectively (r_0, r_0), are relevant only if $r_0 \in \mathbf{R}_1$, respectively $r_0 \in \mathbf{R}_2$. Moreover, an analogous proof to that of Lemma 5 shows that these are steady states of Problem (19). Feasibility of the constant pricing policy, $p_L^{**}(r)$ for $r \in \mathbf{R}_1$ follows because $p_L^{**}(r)$ is increasing in r. To see this, write:

$$\frac{dp_L^{**}(r)}{dr} = -\frac{\theta\pi_{12}^L + \beta\theta(1-\theta)\pi_{22}^L}{\pi_{11}^L + (1-\theta)(1+\beta)\pi_{12}^L + \beta(1-\theta)^2\pi_{22}^L}.$$

This is non-negative because $\pi_{11}^L + 2(1-\theta)\pi_{12}^L + (1-\theta)^2\pi_{22}^L < 0$ (Assumption 3c), $\theta\pi_{12}^L + \theta(1-\theta)\pi_{22}^L \geq 0$ (Assumption 3d), and $0 \leq \beta \leq 1$. Moreover p_L^{**} single crosses the identity line from above, because at $r = 0$ the LHS of (23) is positive. Indeed, the crossing point solves (21), i.e. it is given by q, as defined in Section 6. Moreover, at $r = 0$, equation (23) has a unique positive solution. Therefore for all $r \in \mathbf{R}_1$, we have $p_L^{**}(r) \geq r$.

The proof for the price path is analogous to that of Proposition 3.

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