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ABSTRACT

How should an investor unwind a portfolio in the face of recurring and uncertain liquidity needs? We propose a model of portfolio liquidation in two periods to investigate this question, initially posed by Myron Scholes following the fall of Long Term Capital Management. We show that when the expectation of future liquidity needs is low, the optimal solution involves selling assets that have low permanent and temporary price impacts of trading. However, when there is a high probability of a large future liquidity need, the optimal solution involves retaining assets that have a small temporary impact of trading. In the face of potential future adversity, there is a high option-value to the temporary component of liquidity. The permanent component of liquidity does not share this feature, so that investors will prefer to sell assets with a low ratio of permanent to temporary price impact in the early stages of a crisis, and to hold on to assets with a high ratio of permanent to temporary price impact to protect themselves against an aggravation of the crisis.

Keywords: Portfolio Choice, Liquidity, Distressed Liquidation, Deleveraging

1 Introduction

Following the crisis that surrounded the downfall of Long-Term Capital Management (LTCM) in 1998, Myron Scholes raised the following problem: *how should an investment manager unwind a portfolio when faced with present and possible future liquidity needs?* Describing the situation where "it is not possible to know the extent of the unfolding crisis," he noted:

> Most market participants respond by liquidating their most liquid investments first to reduce exposures and reduce leverage. ... However, after the liquidation, the remaining portfolio is most likely unhedged and more illiquid. Without new inflows of liquidity, the portfolio becomes even more costly to unwind and manage.
>
> Scholes, 2000

This problem, which we call the Scholes liquidation problem, is prevalent during unstable financial periods. In the recent financial crisis, banks incurred large losses during the forced contraction of their balance sheets as access to short-term financing through repo markets dried up (*e.g.*, Adrian and Shin, 2009; Brunnermeier, 2009). A systemic deleveraging process propagated through the banking sector, in which careful liquidation became crucial to preserving wealth and surviving the crisis.

The key question of interest here is to determine which assets should be sold to meet short-term obligations, keeping in mind the potential for liquidity needs in the future. This problem is distinct from a related problem that has been extensively analyzed in the past. Previous work has focused on the optimal way to liquidate a single asset, either as a monopolist (*e.g.*, Bertsimas and Lo, 1998; Huberman and Stanzl, 2005) or against selling pressure (*e.g.*, Brunnermeier and Pedersen, 2005; Carlin, Lobo, and Viswanathan, 2007).[1] In these papers, a trader needs to sell a particular asset for exogenous reasons. In the problem posed by Scholes, the trader needs to generate cash or reduce leverage, and chooses which assets to sell. We will see that the optimal solution to this problem has fundamentally different economic implications.

We develop a two-period model where, in each period, the net cost of trading and the price impact of trading on the market value of the assets is based on a continuous-time market. A single investor holds a portfolio of assets, each with a market price that depends on how liquid it is. The

[1] See also Vayanos, 1998; Almgren and Chriss, 1999; Almgren, 2000; Fedyk, 2001; DeMarzo and Uroevic, 2004; Oehmke, 2008; and Chu, Lehnert, and Passmore, 2009.

price of each asset is impacted by trading depending on its permanent and temporary components of liquidity (*e.g.*, Sadka, 2005).[2] The investor optimally trades the assets in the portfolio over a finite amount of time to maximize the market value of the resulting positions. We focus exclusively on the case of a risk-neutral investor.

We begin by analyzing a single-period problem in which the trader does not have to consider future needs for liquidity. We first characterize the optimal trading strategy of an investor who is unencumbered by leverage constraints or liquidity needs. We show that the optimal strategy leads the trader to accumulate assets that, all else being equal, have low ratio of temporary to permanent price impact of trading.

We then consider two constrained-trading problems in which the investor either faces a limit on leverage or experiences an urgent need for liquidity (*i.e.*, need for cash). We call the latter scenario *myopic deleveraging* because the investor is not required to consider any future implications of holding a particular portfolio. These two scenarios both yield a simple but non-obvious result: the optimal trading policy is, in general, not monotonic in either of an asset's permanent and temporary components of liquidity. The intuition is as follows. When the price impact of trading (either the permanent or the temporary component) for an asset is high, the investor is required to sell more of it to generate cash. However, when this is the case, the trader will also tend to sell other assets more. How the investor should trade-off between these effects will depend on the parameters of the problem.

We further investigate the issue through comparative statics holding constant the shadow price of the leverage or cash constraint. We find that, for the same severity of the leverage or cash constraint (as measured by its marginal cost), the amount of an asset that an investor sells during distress is monotonically decreasing in both measures of illiquidity. That is, in the one-period problem, the investor optimally sells assets that are more liquid to meet pending obligations.

This result changes, though, when we consider the two-period model for Scholes' problem. In period one, the investor is required to unwind part of the portfolio to reduce leverage. Subsequently, with some probability, the investor may experience another liquidity shock and be required to further unwind the portfolio in the second period. If no further distress occurs, trading ends. However, if the investor suffers further distress, the problem faced in the second period is identical

[2]The permanent component of liquidity is the change in the asset's price that that depends on the cumulative amount traded, and is independent of the rate at which the asset is traded. The temporary component of liquidity measures the instantaneous, reversible price pressure that results from trading. See Carlin, Lobo, and Viswanathan (2007) for further discussion.

to the single-period case. The probability of the future need for liquidity is known to the investor, as is the size of the potential shock.

A central question of interest is *what is the option value of holding liquid assets?* A tradeoff arises in the first period of the two-period problem. Selling the more liquid assets first will limit the immediate loss in value; however, the resulting portfolio will be more vulnerable to a continued shock in future periods. Selling the less liquid assets first will result in a portfolio that is more robust to a continued adverse environment; however, this can result in possibly unnecessary loss in value if there is no subsequent shock.

The solution in the two-period model is qualitatively different from the myopic deleveraging case in several ways. In the case of myopic deleveraging, the investor will only trade just enough to meet the margin constraint; since trading is costly, there is no benefit to trading any more than necessary. This is not the case in the two-period model. When the expected second-period shock is large enough, the investor will always want to trade away from the margin in the first period. In doing so, the investor retains cash to protect against a future shock.

Another, more surprising difference, is that the temporary component of liquidity is central to risk-management behavior in the two-period model. If the expected need for liquidity is small, the investor behaves in a similar way as in the one-period problem. However, when the expected need for liquidity is large, the investor holds on to assets with a low temporary impact of trading and sells relatively illiquid assets. This does not extend, however, to the permanent component of liquidity. No matter how large the expectation for the second-period shock may be, the investor always favors selling off more of assets with a low permanent price impact of trading in the first period.

This sheds light on the nature of the solution to the Scholes liquidation problem. Assets with concentrated ownership or those with a high degree of asymmetric information (*i.e.*, those with high permanent price-impact) will not be prioritized for liquidation when an investor experiences a recurrent need for liquidity. Assets that are heavily traded, where there are many opportunities to access counterparties, may or may not be liquidated early. If the expected need for liquidity is small, the investor optimally sells these securities to meet early obligations. However, if the expected need for liquidity is large, the investor will hold onto these assets, preserving the option to sell them in the future.

The analysis in this paper adds to a rather large literature on optimal liquidation, which has focused on the case of a single asset. One exception is Duffie and Ziegler (2003), who numerically investigate the trade-off between selling off an illiquid asset to keep a "cushion of liquid assets," and selling a liquid asset to maximize short-term portfolio value. Illiquidity is modeled as linear transaction costs, and permanent price impact of trading is not considered (they note, however, that this may be a central concern for large investors). Our paper considers both temporary and permanent price impact and, albeit with a considerably simpler model of uncertainty, provides an analytical derivation of structural properties of the optimal solution. In this sense, and to our knowledge, we are the first to consider the problem of unwinding a portfolio where the choice of assets to be sold is endogenous based on the liquidity characteristics of the assets in the portfolio.

The analysis in this paper also adds to the literature on window dressing (*e.g.*, Carhart, Kaniel, Musto, and Reed, 2002). Portfolio managers have been shown to trade assets at the end of repeating periods to make their earnings or holdings look better to others. There is risk in doing this, since there may be a need to unwind the new positions in subsequent periods. The model proposed in this paper relates to this question as well, and some of our conclusions may be readily applied to it.

The remainder of the paper is organized as follows. In Section 2, we present our one-period trading model, and consider scenarios in which the investor is unconstrained, in which the investor has a constraint on leverage, and in which the investor needs to generate cash through liquidation. In Section 3, we explore the Scholes problem by solving a two-period portfolio management problem. Section 4 concludes. All proofs are in appendix.

2 One Period: Window-Dressing and Myopic Deleveraging

2.1 Price and Trading Model and Unconstrained Solution

Consider a single risk-neutral investor who trades a portfolio of n assets in continuous-time over a finite horizon. At any time $t \in [0, \tau]$, $Y_t \in \mathbb{R}^n$ is the rate at which the investor trades the assets. The investor's holdings are denoted by $X_t \in \mathbb{R}^n$, where $X_t = x_0 + \int_0^t Y_s ds$. We will generally assume $x_0 > 0$. We assume that Y_t is an L^2-function.

The prices of the assets at time t are given by $P_t \in \mathbb{R}^n$, which is determined by

$$P_t = q + \Gamma X_t + \Lambda Y_t. \tag{1}$$

This is a multi-dimensional version of the pricing equation used in Carlin, Lobo, and Viswanathan (2007).[3] The expression has three parts. The first term $q \in \mathbb{R}^n$ specifies the intercept of the linear model, that is, the equilibrium prices that arise when the investor does not hold any assets and

[3] A similar pricing relationship for a single asset, where Γ and Λ are scalars, was previously derived by Gennotte

is not trading.[4] The second and third terms partition the price impact of trading into permanent and temporary components. The permanent component measures the change in the price that is independent of the rate at which any of the assets are traded. This impact is likely to be high when the amount of asymmetric information associated with an asset is high or ownership of the asset in the market is concentrated. The temporary component measures the instantaneous, reversible price pressure that results from trading. This component is likely to be high when the asset is thinly traded or there is a paucity of readily-available counterparties in the market. Both $\Gamma \in \mathbb{R}^{n \times n}$ and $\Lambda \in \mathbb{R}^{n \times n}$ are matrices in which each diagonal entry is an asset's own price impact and each off-diagonal entry is the cross-price impact between two assets. We assume that Γ and Λ are symmetric ($\Gamma' = \Gamma$ and $\Lambda' = \Lambda$), non-negative ($\Gamma_{i,j} \geq 0$ and $\Lambda_{i,j} \geq 0$ for all i, j), and positive definite ($x'\Gamma x > 0$ and $x'\Lambda x > 0$ for all $x \in \mathbb{R}^n$ such that $x \neq 0$).

We denote the initial and final positions by $x_0 = X_0$ and $x_1 = X_\tau$, and the cumulative trade by $y_1 = x_1 - x_0$. Prior to trading, the asset prices are $p_0 = P_{0_-} = q + \Gamma x_0$. After trading is complete, the price is

$$p_1 = P_{\tau_+} = q + \Gamma x_1 = q + \Gamma(x_0 + y_1) = p_0 + \Gamma y_1.$$

Using the prime to denote the transpose operator, the end-of-period assets are

$$a_1 = p_1' x_1 = (p_0 + \Gamma y_1)' (x_0 + y_1) = a_0 + (p_0 + \Gamma x_0)' y_1 + y_1' \Gamma y_1,$$

a quadratic function of y_1.

The cash that is generated from trading over $[0, \tau]$ is

$$\kappa_1 = \int_0^\tau -P_t' Y_t dt = \int_0^\tau - \left(p_0 + \Gamma \int_0^t Y_s ds + \Lambda Y_t \right)' Y_t dt.$$

We assume that cash is counted directly against liabilities for risk-management purposes or for the satisfaction of margin constraints. Denoting the initial liabilities by l_0, the liabilities at time τ can then be written as $l_1 = l_0 - \kappa_1$.

The optimal trading schedule is obtained from the following lemma.

and Kyle (1991), and Vayanos (1998). Empirical support for partitioning the price impact of trading into permanent and temporary components is found in Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1990), Cheng and Madhavan (1997), and Sadka (2005). See Carlin, Lobo, and Viswanathan (2007) for further discussion.

[4] We keep the within-period price path deterministic to be able to focus on the consequences of uncertainty about future liquidity shocks, while keeping the problem analytically tractable. For small τ and when considering the possibility of a large shock, we believe this to be a reasonable assumption. Further, as will be noted below, our model is easily extended to consider a uniform relative price movement over all assets.

Lemma 1. *Consider an investor that wishes to maximize some function $f(a_1, l_1)$, increasing in a_1 and decreasing in l_1. The optimal execution schedule is a constant trading rate, that is*

$$Y_t^* = \frac{1}{\tau} y_1, \ t \in [0, \tau].$$

The intuition for Lemma 1 is that, for any given set of trades y_1 and resulting final prices p_1 and assets a_1, the concavity of the integrand in κ_1 leads the trader to smooth trades over time to minimize the transaction costs due to the temporary price impact of trading. Lemma 1 allows us to simply focus on how much to liquidate each asset.

Since trading occurs at a constant rate for each asset, the end-of-period liabilities may be computed as

$$l_1 = l_0 + \left(p_0 + \Lambda y_1 + \frac{1}{2}\Gamma y_1\right)' y_1 = l_0 + p_0' y_1 + y_1' \left(\Lambda + \frac{1}{2}\Gamma\right) y_1.$$

By simple accounting, e_1, the investor's equity at the end of trading, equals $a_1 - l_1$. We can use this to express the equity as:

$$e_1 = a_1 - l_1 = e_0 + x_0' \Gamma y_1 - y_1' \left(\Lambda - \frac{1}{2}\Gamma\right) y_1. \tag{2}$$

The end-of-period equity is strictly concave in the trade vector y_1 if and only if $\Lambda - \frac{1}{2}\Gamma$ is positive definite. We assume

$$\Lambda \succ \frac{1}{2}\Gamma, \tag{3}$$

ensuring that the trader's problem is well-posed. Note that, without this restriction, the trader may embark on trades of infinite size and, in doing so, obtain arbitrarily large equity. This suggests that it is a reasonable condition to ensure economic soundness of the model.

The following immediate result characterizes the solution to the trader's unconstrained trading problem, that is, when the trader faces no leverage constraints and may buy and sell assets as he wishes.

Result 1. *The unconstrained optimal trades that maximize the end-of-period equity e_1 are*

$$y_1^* = \left(2\Gamma^{-1}\Lambda - I\right)^{-1} x_0.$$

For diagonal price impact matrices Γ and Λ with diagonal entries γ_i and λ_i, this is

$$y_{1,i}^* = \frac{x_{0,i}}{2\frac{\lambda_i}{\gamma_i} - 1},$$

in which case the optimal trade of asset i is increasing in the initial position in that asset $x_{0,i}$ in its permanent price impact γ_i, and in the ratio γ_i/λ_i, and decreasing in its temporary price impact λ_i.

If an investor were to repeatedly follow this policy without constraints on leverage, the size of both assets and liabilities increases at a geometric rate. The composition of the portfolio is impacted by the initial positions, and is concentrated on assets that are illiquid in terms of their permanent price impact. In fact, the rate of growth of asset i is

$$1 + \frac{\gamma_i}{2\lambda_i - \gamma_i} = \left(1 - \frac{\gamma_i}{2\lambda_i}\right)^{-1},$$

so that, if the investor is singularly concerned with maximizing equity and unlimited leverage is allowed, the portfolio will become concentrated in assets with a high ratio of permanent to temporary price impact. If all assets have the same ratio λ_i/γ_i, the portfolio weights remain constant. This also holds for the non-diagonal case where $\Gamma_{i,j}/\Lambda_{i,j} = r$ for all i and j, in which case the optimal trades are proportional to the existing positions, $y_1^* = \frac{r}{2-r} x_0$.

For some brief additional insight into the effect of cross-asset price impacts, consider a problem with two assets. Define

$$\Theta = \Lambda - \frac{1}{2}\Gamma = \begin{bmatrix} \theta_1 & \theta_c \\ \theta_c & \theta_2 \end{bmatrix}, \quad \text{and} \quad \Gamma = \begin{bmatrix} \gamma_1 & \gamma_c \\ \gamma_c & \gamma_2 \end{bmatrix}.$$

The optimal trades are then $y_1^* = \frac{1}{2}\Theta^{-1}\Gamma x_0$, which can be written as

$$y_1^* = \frac{1}{2(\theta_1\theta_2 - \theta_c^2)} \begin{bmatrix} \theta_2\gamma_1 - \theta_c\gamma_c & \theta_2\gamma_c - \theta_c\gamma_2 \\ \theta_1\gamma_c - \theta_c\gamma_1 & \theta_1\gamma_2 - \theta_c\gamma_c \end{bmatrix} x_0.$$

If there is no permanent cross-asset price impact, $\Gamma_{1,2} = 0$, we have

$$y_1^* = \frac{1}{2(\theta_1\theta_2 - \theta_c^2)} \begin{bmatrix} \theta_2\gamma_1 x_{0,1} - \theta_c\gamma_2 x_{0,2} \\ \theta_1\gamma_2 x_{0,2} - \theta_c\gamma_1 x_{0,1} \end{bmatrix},$$

and the optimal trades are decreasing in the temporary cross-asset price impact. If $\theta_c = 0$ ($\Lambda_{1,2} = \frac{1}{2}\Gamma_{1,2}$), we have

$$y_1^* = \frac{1}{2} \begin{bmatrix} \frac{\gamma_1}{\theta_1} x_{0,1} + \frac{\gamma_c}{\theta_1} x_{0,2} \\ \frac{\gamma_2}{\theta_2} x_{0,2} + \frac{\gamma_c}{\theta_2} x_{0,1} \end{bmatrix},$$

and the optimal trades are increasing in the permanent cross-asset price impact. This suggests that cross-asset temporary price impact leads to a less aggressive build-up of leverage, while the opposite is true for permanent cross-asset price impact. That is, while they change the optimal solution, the cross-asset price impact terms have qualitatively similar effects as the same-asset price impact terms.

Taking a large position concentrated in assets with a high ratio of permanent to temporary price impact is essentially a strategy of cornering the market for actively-traded assets with small issues and subject to asymmetric information. In general, however, this is not a good strategy for a large player with liquidity constraints, especially in a multi-period setting. We consider these issues in turn. In the next two subsections, we analyze the investor's problem with liquidity constraints in a one-period setting, and then study the two-period problem in Section 3.

2.2 Optimal Trades with Margin Constraint

Due to either margin requirements imposed by lenders or to regulatory or risk-management constraints there is, under normal circumstances, a limit on the financial leverage that an investor can incur. Different ratios quantifying the degree to which an investor is leveraged can be found in the literature. Three commonly-used ratios are liabilities over assets, assets over equity, and liabilities over equity. All three ratios are increasing in the degree of financial leverage, and are readily related to each other by $l/a = \frac{l/e}{l/e+1}$ and $a/e = l/e + 1$. We specify limits on financial leverage via a bound ρ on the ratio of debt to equity, that is

$$\frac{l_1}{e_1} \leq \rho.$$

This inequality can be written as a quadratic constraint on y_1,

$$\rho e_0 - l_0 + (\rho \Gamma x_0 - p_0)' y_1 - y_1' \left(\rho \left(\Lambda - \frac{1}{2}\Gamma \right) + \Lambda + \frac{1}{2}\Gamma \right) y_1 \geq 0. \tag{4}$$

If no leverage is allowed after the trading period ($\rho = 0$), this constraint is $-p_0' y_1 - y_1' \left(\Lambda + \frac{1}{2}\Gamma\right) y_1 \geq l_0$, which states that, after accounting for transaction costs, the trades must generate enough cash to cover all liabilities. If arbitrarily large leverage is permitted ($\rho \to +\infty$), the constraint becomes the solvency constraint $e_1 \geq 0$.

The condition for the constraint on leverage to be convex and bounded is that $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. (Note that if $\rho \leq 1$ the constraint is convex for any Λ and Γ such that $\Lambda \succ 0$ and $\Gamma \succ 0$.) This condition is implied by (3) and is therefore automatically ensured in our framework. It is a less restrictive assumption than (3) so that the problem may be bounded for some objective functions that are not concave.[5]

[5] Though maximization of a non-concave function over a convex set is in general an intractable problem, in such a case we can actually still solve the problem and obtain a solution quite similar to that in Result 2 using a result from convex analysis known as the $\mathcal{S}$-*lemma* (see, *e.g.*, Pólik, Terlaky (2007)), a quadratic analog to the Farkas lemma. If condition (3) is not satisfied, however, an unrestricted trader may still improve equity arbitrarily simply through the act of trading.

Result 2. *Consider the optimal window-dressing problem where the investor chooses trades to maximize equity subject to a constraint on leverage,*

$$\begin{aligned} &\textit{maximize} \quad e_1 \\ &\textit{subject to} \quad l_1 \leq \rho e_1. \end{aligned}$$

There is a $z \geq 0$ *such that the optimal trades are given by*

$$y_1^* = \frac{1}{2}\left((1+z\rho)\left(\Lambda - \frac{1}{2}\Gamma\right) + z\left(\Lambda + \frac{1}{2}\Gamma\right)\right)^{-1}\left((1+z\rho)\,\Gamma x_0 - z p_0\right).$$

In the case where the price impact matrices Γ *and* Λ *have diagonal structure, with diagonal entries* γ_i *and* λ_i*, the optimal trades can be written as*

$$y_{1,i}^* = \frac{1}{2} \cdot \frac{(1+z\rho)\,\gamma_i x_{0,i} - z p_{0,i}}{(1+z\rho)\left(\lambda_i - \frac{1}{2}\gamma_i\right) + z\left(\lambda_i + \frac{1}{2}\gamma_i\right)}.$$

The optimal trade of asset i, $y_{1,i}^*$, *is increasing in* ρ *and decreasing in* l_0. *The optimal trade of asset* i, $y_{1,i}^*$ *is not, in general, monotonic in* λ_i, γ_i, *or the ratio* γ_i/λ_i.

The optimal trades for the diagonal case can alternatively be presented in the following form, which makes the effects of the price impact parameters and shadow price more clear:

$$y_{1,i}^* = \frac{\tilde{\gamma}_i x_{0,i} - \tilde{z}}{(1+\tilde{z})2\tilde{\lambda}_i - (1-\tilde{z})\tilde{\gamma}_i},$$

where $\tilde{z} = z/(1+z\rho) \in [0, 1/\rho]$ is monotonic in the shadow price, and $\tilde{\gamma}_i = \gamma_i/p_i$ and $\tilde{\lambda}_i = \lambda_i/p_i$ are the relative price impacts.

According to Result 2, the investor trades more of an asset when the margin constraint is less restrictive (higher ρ) and when the investor is more leveraged initially (higher l_0). Trades are non-monotonic in the price-impact parameters due to two opposing effects, which can be appreciated as follows. Consider an asset for which the price impact of trading increases, and how this changes the optimal trades. On the one hand it will be comparatively more costly to deleverage, requiring the investor to liquidate a larger share of the portfolio. On the other hand, the investor will prefer to sell less of this particular asset, and more of others. Which effect dominates as to the amount that is liquidated of the asset in question is determined by how quickly the optimal trades shift away from that asset, versus how quickly the fraction of the portfolio that needs to be liquidated increases.

One can find examples of problems where, over a reasonable range for the price impact parameters, $y_{1,i}^*$ exhibits non-monotonic behavior. Figure 1 presents two examples where the optimal trades are not monotonic in the price impact parameters. In both cases there are two assets with $x_0 = [1\ 1]'$ and $p_0 = [1\ 1]'$, and the investor is required to deleverage from a ratio of 19 ($l_0 = 1.9$, $e_0 = 0.1$) to meet $\rho = 10$. In the first example, $\gamma_1 = \gamma_2 = 0$, $\lambda_2 = 0.05$, and λ_1 ranges from 0.001 to 0.05. In the second example, $\lambda_1 = \lambda_2 = \gamma_2 = 0.026$, and γ_1 ranges from 0 to 0.05.

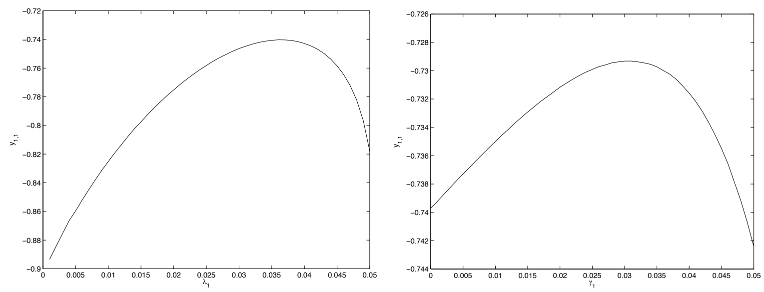

Figure 1: Examples where the optimal trades are not monotonic in the price impact parameters.

In this paper, we only consider constraints on leverage and, as we will see next, constraints on the size the trade in each asset in the diagonal case. In practice, an investor may want to incorporate other constraints into the problem. From the point of view of computational tractability, any modification that preserves the convexity of the problem can be easily handled. This includes constraints on position size or on trade size, or any number of risk constraints in a mean-variance framework (see, *e.g.*, Lobo, Fazel and Boyd (2007)). However, in most cases we lose the ability to provide an analytical description of the optimal policy, or to provide structural insights into it. We present next an important exception to this, with a formulation that captures the major features of the problem of deleveraging under distress, and for which we can provide substantial insight into the solution.

2.3 Forced Deleveraging in the Diagonal Case

We are especially interested in modeling liquidity shocks which force an investor to quickly sell assets to reduce leverage. These shocks may arise from a number of reasons, such as a decrease in the value of assets, unexpectedly large investor withdrawals, or margin calls due to a transition to a more risk-averse environment (which in our model translates to requiring a lower ρ). When doing a fire sale to mitigate risk, the investor is not allowed to increase positions nor to short-sell, which corresponds to the box constraints

$$-x_{0,i} \leq y_{1,i} \leq 0.$$

We remind the reader that, in everything that follows, we will be considering sales of assets, which correspond to negative values in the trades vector. Thus, $y_{1,i} < y_{1,j}$ should be interpreted as meaning we liquidate *more* of asset i than asset j.

A recurring assumption in our analysis of the diagonal case in both the single-period and two period models is the assumption that, for all i, $\rho\gamma_i x_{0,i} \leq p_{0,i}$. If this condition is violated for a particular asset i, selling any amount of it hurts both net equity and the leverage ratio. Such an asset can therefore be excluded from the problem without loss of generality.

The case of diagonal structure with box constraints can be analyzed as follows.

Result 3. *Consider the single-period deleveraging problem*

$$\begin{array}{ll} \textit{maximize} & e_1 \\ \textit{subject to} & l_1 \leq \rho e_1 \\ & -x_0 \leq y_1 \leq 0, \end{array}$$

where Γ *and* Λ *have diagonal structure, with diagonal entries* γ_i *and* λ_i*, and deleveraging is required* ($\frac{l_0}{e_0} > \rho$)*. The optimal solution satisfies* $\frac{l_1}{e_1} = \rho$ *and there exists a* $z > 0$ *such that the optimal trades are given by*

$$y^*_{1,i} = \max\left(-x_{0,i},\ \min\left(0,\ \frac{1}{2}\cdot\frac{(1+z\rho)\,\gamma_i x_{0,i} - z p_{0,i}}{(1+z\rho)\left(\lambda_i - \frac{1}{2}\gamma_i\right) + z\left(\lambda_i + \frac{1}{2}\gamma_i\right)}\right)\right).$$

The optimal trade of asset i*,* $y^*_{1,i}$*, is increasing in* ρ *and decreasing in* l_0*, but is not, in general, monotonic in* $x_{0,i}$*, in* λ_i*, nor in* γ_i*.*

As in the previous result, we have the alternative formulation

$$y^*_{1,i} = \max\left(-x_{0,i},\ \min\left(0,\ \frac{\tilde{\gamma}_i x_{0,i} - \tilde{z}}{(1+\tilde{z})2\tilde{\lambda}_i - (1-\tilde{z})\tilde{\gamma}_i}\right)\right),$$

for some $\tilde{z} \in [0, 1/\rho]$, where $\tilde{\gamma}_i = \gamma_i/p_i$ and $\tilde{\lambda}_i = \lambda_i/p_i$.

Not surprisingly, the trader will sell-off so that the margin constraint binds. This maximizes value in the single period case but, as we will see in the next section, may not hold in the multi-period setting. Consistent with the findings in Section 2.2, trading in any asset i is not, in general, monotonic in its price impact parameters.

To gain more insight into the relationship between asset sales during distress and the price impact parameters, we can analyze the problem given a constant shadow price. That is, we can consider portfolio modifications such that the marginal penalty in equity for increasing the margin requirement is constant. This allows us to isolate the effect of shifting sales from one asset to another from the price impact on the marginal cost of partial liquidation of the portfolio.

The next result characterizes some important relationships, and will allow us to illuminate the fundamental changes in the structure of the optimal trades in the two-period case.

Result 4. *For the single-period problem with diagonal structure, consider directional derivatives of the problem parameters $(p_0, \gamma, \lambda, x_0, l_0, \rho)$ such that the shadow price z is constant. That is, consider modifications of the portfolio such that the marginal penalty in equity for increasing the margin requirement is constant. Along such directions, $y^*_{1,i}$ is increasing in γ_i, in λ_i, and in $x_{0,i}$.*

From this we derive the two following corollary results:

i) Assets with low price impact are prioritized for liquidation. If two assets i and j are such that $p_{0,i} = p_{0,j}$, $\gamma_i \leq \gamma_j$, $\lambda_i \leq \lambda_j$ and $x_{0,i} = x_{0,j}$, then $y^*_{1,i} \leq y^*_{1,j}$.

ii) Assets with smaller holdings are prioritized for liquidation. If two assets i and j are such that $p_{0,i} = p_{0,j}$, $\gamma_i = \gamma_j$, $\lambda_i = \lambda_j$, and $x_{0,i} < x_{0,j}$, then $y^*_{1,i} \leq y^*_{1,j}$ (unless asset i is fully liquidated, such that $y^*_{1,i} = -x_{0,i} > y^*_{1,j}$).

Result 4 allows us to establish an ordering for myopic distressed sales. Over assets that are otherwise identical and of which the investor has similar holdings, the more liquid assets are sold first. Likewise, over assets that are equally liquid, assets of which the investor has smaller holdings are sold first. Note that, while the proofs of the monotonicity with respect to λ_i and $x_{0,i}$ are trivial from the partial derivatives of $y^*_{1,i}$ and hold without the distressed-deleveraging box constraints, this is not the case for γ_i (for which the monotonicity does not hold without the no-shorting constraint).

The monotonicities in the price impact parameters imply that in distressed sales due to short-lived shocks, traders should deleverage as described by Scholes. Specifically, traders sell-off their most liquid holdings to generate cash or decrease their liabilities. The problem changes, however, when a future further need for liquidity may arise. In some cases, it may still be optimal to prioritize the selling-off of more liquid securities. However, we will see that in other cases risk management may lead to different optimal strategies. To investigate this, we turn to a two-period setting next.

3 Two Periods: Optimal Deleveraging with Recurring Shock

3.1 Model and Preemptive Deleveraging

Consider now a single investor who trades in n assets over two periods. Each period is a discrete amount of time $[0, \tau]$, in which trading occurs continuously as before. Prices arise from the process in (1) and the investor is restricted to satisfy constraints on leverage at the end of each time period $j \in \{1, 2\}$,

$$\frac{l_j}{e_j} \leq \rho.$$

Since we wish to study policies regarding deleveraging under distress, we restrict the investor's trades to be reductions in positions and disallow shorting.

The key difference now is that there is uncertainty during period one about whether the investor will face the need for further liquidity during period two. This uncertainty may arise because of unforseen equity withdrawals, higher cash requirement to fund other areas of the business, less favorable funding conditions (*e.g.*, tighter margin constraints), or a uniform (*i.e.*, systematic) drop in asset prices. The uncertainty is resolved between the periods. We model the shock as an early equity withdrawal, which generalizes to all the situations mentioned. Specifically, the amount withdrawn is a Bernoulli random variable Δ such that

$$\Delta = \begin{cases} \delta, & \text{with probability } \pi \\ 0, & \text{with probability } 1 - \pi. \end{cases} \tag{5}$$

If there is a second-period shock, liabilities increase by δ, resulting in a more leveraged balance sheet. If this is the case, the investor is required to liquidate assets in the second trading period to deleverage to within allowed limits.

Following the notation of the previous section, the equilibrium price after the second period is

$$\begin{aligned} p_2 &= q + \Gamma x_2 \\ &= p_0 + \Gamma \left(y_1 + y_2\right), \end{aligned}$$

and the assets at the end of the second period are

$$\begin{aligned} a_2 &= p_2' x_2 \\ &= \left(p_0 + \Gamma \left(y_1 + y_2\right)\right)' \left(x_0 + y_1 + y_2\right) \\ &= a_0 + \begin{bmatrix} p_0 + \Gamma x_0 \\ p_0 + \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Gamma & \Gamma \\ \Gamma & \Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

Using the same price model as for the single-period problem, the investor trades y_2 for an average price of $p_1 + \Lambda y_2 + \frac{1}{2}\Gamma y_2$. After withdrawals Δ, the investor is then left with liabilities

$$\begin{aligned} l_2 &= l_1 + \Delta + \left(p_1 + \Lambda y_2 + \frac{1}{2}\Gamma y_2\right)' y_2 \\ &= l_0 + \Delta + \begin{bmatrix} p_0 \\ p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \end{aligned}$$

and the second-period equity, $e_2 = a_2 - l_2$, is

$$e_2 = e_0 - \Delta + \begin{bmatrix} \Gamma x_0 \\ \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\frac{1}{2}\Gamma \\ -\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The second period leverage constraint

$$\frac{l_2}{e_2} \le \rho$$

can be written as a quadratic constraint on the vector of first- and second-period trades,

$$\begin{aligned} &\rho e_0 - l_0 - (1+\rho)\Delta + \begin{bmatrix} \rho\Gamma x_0 - p_0 \\ \rho\Gamma x_0 - p_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &- \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \rho(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma & \frac{1}{2}(1-\rho)\Gamma \\ \frac{1}{2}(1-\rho)\Gamma & \rho(\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ge 0. \end{aligned} \tag{6}$$

Now, we proceed to characterize optimal trading in this multi-period setting. The first important question that we address is whether the leverage constraint

$$\frac{l_1}{e_1} \le \rho$$

binds in the first period. That is, we consider whether the investor deleverages preemptively (*i.e.*, more than is immediately required) in the first period when there is a potential need for liquidity in the future. The following result addresses this question.

Result 5. *Suppose that the investor's initial holdings are such that $l_0/e_0 > \rho$ and that the trade $y_1 = -x_0/2$ strictly satisfies the first-period margin constraint. Further, assume that*

$$((1+\rho)\Lambda + \Gamma)\, x_0 \le p_0. \tag{7}$$

Then, there exists a threshold shock level $\hat{\delta} \ge 0$ such that the optimal two-period solution satisfies $l_1/e_1 = \rho$ for all $\delta \in [0, \hat{\delta}]$ and $l_l/e_l < \rho$ for all feasible $\delta > \hat{\delta}$.

Result 5 says that, under mild conditions, the optimal two-period liquidator may in fact deleverage beyond what is required in the first period. When the potential need for future liquidity is large, the margin constraint does not bind in the first period. This result arises even though the investor is risk neutral. Due to the convexity of the penalty incurred in a large fire sale, the investor manages future liquidity risk by over-liquidating the portfolio early on.

Result 5 also implies that if the future need for liquidity is high enough, the investor substitutes liquid assets for illiquid ones. If we interpret cash to be the $(n+1)^{th}$ asset in the portfolio, the investor overweighs this liquid asset to the detriment of other securities when the future need for liquidity is sufficiently large.

The conditions in Result 5 are mild and can be appreciated as follows. First, there is a bound on the first period shock so that a liquidation of exactly one-half the portfolio is sufficient to generate enough cash to meet the first-period margin constraint. Given that such a trade is very extreme, this assumption is rather weak. The condition in (7) is simply an upper bound on the temporary impact costs associated with a complete liquidation of the portfolio. If the temporary impact costs associated with trading are so high that this condition is violated, then these transaction costs dominate and risk mitigation behavior as demonstrated in the above result may not occur.

For the remainder of this section, we impose the requirement that the first-period trades be such that the second-period constraint on leverage can be met under any realization of Δ. The two-period problem of the expected-equity maximizing investor is then

$$\begin{array}{ll} \text{maximize} & \mathbf{E}_\Delta\, e_2 \\ \text{subject to} & l_1 \leq \rho e_1,\ l_2 \leq \rho e_2, \forall \Delta \\ & -x_0 \leq y_1 \leq 0,\ \ -x_1 \leq y_2 \leq 0, \forall \Delta, \end{array}$$

where the optimization is over y_1 and y_2, where y_1 is in $\mathbb{R}^n$ and y_2 is $\{0, \delta\} \mapsto \mathbb{R}^n$ (or, equivalently, a random variable in $\mathbb{R}^n$ measurable in the sigma-algebra generated by Δ).

Note that, with this problem specification, the restrictions on trades that prevent the investor from increasing positions and from short-selling in the second period are assumed to hold even if there is no second-period shock. This restriction seems appropriate since, around a crisis event for the investor, or around a period of heightened market uncertainty, more strict risk management prevents the investor from hastily increasing risk exposure. (A multi-period extension of this model might naturally lead to a constraint of this nature.)[6]

[6]If we relax this constraint, the problem is less tractable. The expressions become far more complicated due to the necessity of including the solution to a myopic single-period problem in the second-period objective, weighted by

This constraint simplifies the problem because, when $\Delta = 0$, the optimal second-period trade is then $y_2 = 0$. This is shown by computing the gradient of the objective with respect to y_2 at $y_2 = 0$, which is Γx_1. Under assumptions of no shorting and $\gamma_{i,j} \geq 0$, all entries of the gradient are non-negative. This, together with the concavity of the objective in y_2 (guaranteed by $\Lambda \succ \frac{1}{2}\Gamma$) and the convexity of the box constraints, ensures that, if the constraint on leverage is not binding, the optimum is achieved at $y_2 = 0$. The first-period leverage constraint and first-period trades in turn ensure that, absent a second-period shock, the second-period leverage constraint is not binding.

With a slight abuse of notation, we now use $y_2 \in \mathbb{R}^n$ to denote the second period trades associated with the realization $\Delta = \delta$. Likewise, we refer to l_2 and e_2 as the liabilities and equity when $\Delta = \delta$. Noting that when $\Delta = 0$, the optimal second-period equity is $e_2 = e_1$, we can now write the investor's expected-equity-maximization problem as

$$\begin{array}{ll} \text{maximize} & (1-\pi)e_1 + \pi e_2 \\ \text{subject to} & l_1 \leq \rho e_1, \ l_2 \leq \rho e_2 \\ & -x_0 \leq y_1 \leq 0, \ -x_1 \leq y_2 \leq 0. \end{array} \tag{8}$$

The program variables are $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^n$, and the objective is a quadratic functional in R^{2n},

$$\begin{aligned} \mathbf{E}_\Delta\, e_2 &= (1-\pi)e_1 + \pi e_2 \\ &= e_0 - \pi\delta + \begin{bmatrix} \Gamma x_0 \\ \pi \Gamma x_0 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\pi\frac{1}{2}\Gamma \\ -\pi\frac{1}{2}\Gamma & \pi(\Lambda - \frac{1}{2}\Gamma) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

To assure that the problem is well-posed, we assume that $\mathbf{E}_\Delta e_2$ is strictly concave jointly in y_1 and y_2, and that the second-period constraint on leverage is strictly convex jointly in y_1 and y_2. We prove in appendix that this is ensured by the condition

$$\Lambda \succ \max\left(\frac{1+\sqrt{\pi}}{2}, \frac{\rho-1}{\rho+1}\right)\Gamma, \tag{9}$$

which is not much more restrictive that the condition for the single-period case.[7]

the probability $1-\pi$. However, the analysis is identical if we restrict δ to be such that the margin constraint binds in the first period (*i.e.*, $\delta \leq \hat{\delta}$). Further, additional analysis indicates that solutions to this modified problem are qualitatively similar, in that results regarding both preemptive deleveraging and change in preferred order in which assets are liquidated apply for large-enough second-period shocks.

[7]Note, however, that the two-period problem is guaranteed to be bounded using only the weaker assumption that we introduced for the single-period problem, $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. This assumption ensures strict convexity of the first-period leverage constraint, and therefore boundedness of y_1. With y_1 bounded by the first-period constraint on leverage, we only need convexity in y_2 of the second-period constraint on leverage to ensure boundedness of y_2, which is also ensured by $\Lambda \succ \frac{\rho-1}{2(\rho+1)}\Gamma$. This assumption is therefore sufficient to ensure that the optimal trades are finite. The same caveats as in the single-period case apply for this weaker constraint.

Characterizing the solution to problem (8) for the diagonal case is the subject of the next subsection.

3.2 Diagonal Case and Monotonicity

Consider the two-period problem with the two leverage constraints dualized,

$$\begin{array}{ll} \text{maximize} & (1-\pi)e_1 + \pi e_2 + z_1\left(\rho e_1 - l_1\right) + z_2\left(\rho e_2 - l_2\right) \\ \text{subject to} & -x_0 \le y_1 \le 0, \quad -x_1 \le y_2 \le 0. \end{array} \tag{10}$$

If Λ and Γ have diagonal structure and we fix the values of z_1 and z_2, the problem can be decoupled in the assets in that the optimal solution can be obtained by the independent maximization of $(y_{1,i}, y_{2,i})$ for each asset i. We can then independently derive the solution for each asset i as a function of the first- and second-period shadow prices. The objective can be written as a sum of terms associated with each asset,

$$\mathbf{E}_\Delta\, e_2 = c + \sum_{i=1}^{n}\left(b_i' y_i - y_i' A_i y_i\right),$$

where $y_i = \begin{bmatrix} y_{1,i} \\ y_{2,i} \end{bmatrix}$. The constant term c and the linear and quadratic terms b_i and A_i depend on the first- and second-period shadow prices z_1 and z_2, and are as follows,

$$c = e_0 - \pi\delta + (\rho e_0 - l_0)(z_1 + z_2) - (1+\rho)\delta z_2$$

$$b_i = \begin{bmatrix} \rho\gamma_i x_{0,i} + (\rho\gamma_i x_{0,i} - p_{0,i})(z_1 + z_2) \\ \pi\rho\gamma_i x_{0,i} + (\rho\gamma_i x_{0,i} - p_{0,i}) z_2 \end{bmatrix}$$

$$A_i = \begin{bmatrix} \rho(\lambda_i - \frac{1}{2}\gamma_i) + ((\rho+1)\lambda_i - \frac{1}{2}(\rho-1)\gamma_i)(z_1+z_2) & -\frac{1}{2}\pi\rho\gamma_i - \frac{1}{2}(\rho-1)\gamma_i z_2 \\ -\frac{1}{2}\pi\rho\gamma_i - \frac{1}{2}(\rho-1)\gamma_i z_2 & \pi\rho(\lambda_i - \frac{1}{2}\gamma_i) + ((\rho+1)\lambda_i - \frac{1}{2}(\rho-1)\gamma_i) z_2 \end{bmatrix}.$$

The constraints disallowing position increases and short sales can be equivalently stated as

$$y_{1,i} \le 0, \quad y_{2,i} \le 0, \quad \text{and} \quad y_{1,i} + y_{2,i} \ge -x_{0,i}.$$

This defines a triangular feasible set for the first- and second-period trades in each asset. We graph these cases in Figure 2 and derive their optimal trades in Table 3.2. Depending on which constraints are active and inactive, there are seven different cases to consider. Which case occurs depends on

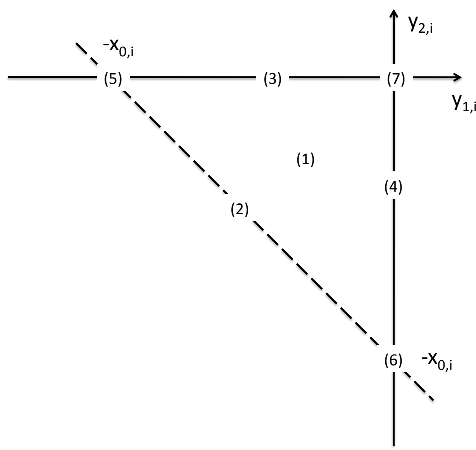

Figure 2: Enumeration of cases for the linear constraints on each asset.

Case	$y^*_{1,i}$	$y^*_{2,i}$
1	$\frac{A_{i,2,2}b_{i,1}-A_{i,1,2}b_{i,2}}{2\left(A_{i,1,1}A_{i,2,2}-A^2_{i,1,2}\right)}$	$\frac{-A_{i,1,2}b_{i,1}+A_{i,1,1}b_{i,2}}{2\left(A_{i,1,1}A_{i,2,2}-A^2_{i,1,2}\right)}$
2	$\frac{b_{i,1}-b_{i,2}+2x_{0,i}(A_{i,1,2}-A_{i,2,2})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$	$\frac{b_{i,2}-b_{i,1}+2x_{0,i}(A_{i,1,2}-A_{i,1,1})}{2(A_{i,1,1}+A_{i,2,2}-2A_{i,1,2})}$
3	$\frac{b_{i,1}}{2A_{i,1,1}}$	0
4	0	$\frac{b_{i,2}}{2A_{i,2,2}}$
5	$-x_{0,i}$	0
6	0	$-x_{0,i}$
7	0	0

Table 1: Enumeration of cases for the linear constraints on each asset.

the amount of deleveraging required immediately, the size and likelihood of a potential shock in the second period, and the liquidity parameters and holdings of a particular asset.

The first two cases are the most interesting, as the remaining five correspond to more extreme situations. In Case 1, the optimal solution is strictly in the interior of the triangle. If an asset is in this region, it has favorable enough liquidity properties that is optimal to sell some of it in the first period, and some of it in the second period if a subsequent need for liquidity arises. However, even if a shock does occur in the second period, the investor will not fully liquidate the asset. Case 2 is similar to Case 1, except that if the shock occurs, the investor is forced to liquidate the entire stake in the asset. The investor would prefer to sell more of the asset, but the no-short-sales constraint binds and changes the character of the optimal solution.

In Case 3, the investor chooses to liquidate some of the asset in period one but nothing further in period two if a shock occurs. This may occur, for instance, if a large amount of deleveraging is required immediately but the subsequent size of Δ is small. Case 5 is similar except that the investor liquidates all of the asset immediately. This may occur if the first-period deleveraging is large and the asset is very liquid.

In Case 4, the investor does not sell the asset in the first period, but does sell some of it in the second period. This might occur if the asset is relatively illiquid but, due to limitations on positions of the other assets, the investor has no choice but to sell some of it if a second-period shock occurs. Case 6 is similar to Case 4 in that the asset is not sold in the first period, but is completely liquidated in the second period. This might occur if Δ is very large. Finally, in case 7, the investor does not liquidate any of the asset. This might happen if the asset is highly illiquid or if Δ is very small.

In practice, the solution to the investor's trading problem can be computed by several approaches. One alternative is as follows. First, for some given shadow prices (z_1, z_2) and for each asset i, compute all seven cases of $(y_{1,i}, y_{2,i})$ and the associated value of the objective. From these, exclude the cases that violate the conditions $y_{1,i} \leq 0$, $y_{2,i} \leq 0$, and $y_{1,i} + y_{1,i} \geq -x_{0,i}$, and select the valid case with the highest objective value. Then, with the optimal (y_1, y_2) for the dualized problem, compute the value of the investor's objective function that is associated with that particular (z_1, z_2) and the slack in the leverage constraints. Then, update (z_1, z_2) and repeat the procedure to converge to the optimal value that satisfies both leverage constraints. A wide array of updating rules for (z_1, z_2) that will ensure convergence to the optimum are available in the mathematical programming literature. Given that this is a two-dimensional problem, any simple rule will be effective.

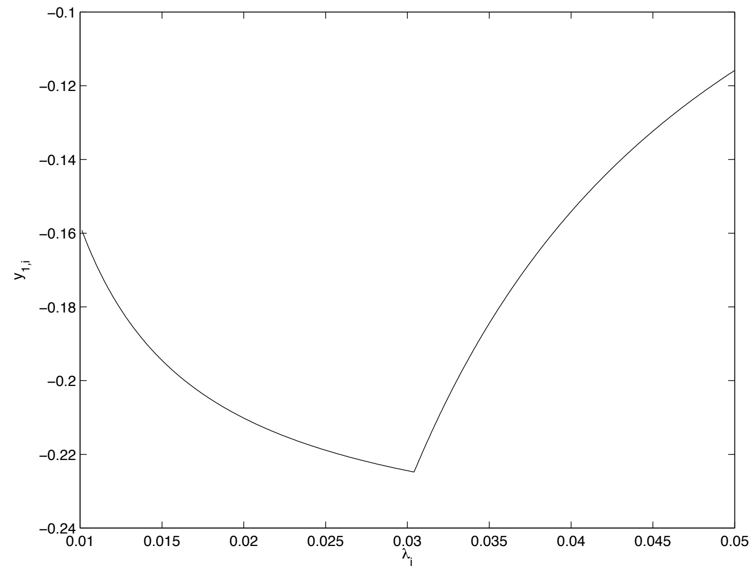

Figure 3: Optimal first-period trade as a function of λ_i.

We now characterize the solution to the investor's two-period problem. Before formally presenting our results regarding monotonicity of the optimal trades in the price impact parameters, it is instructive to consider the following motivating example. For some asset i, we have $x_{0,i} = 1$, $p_{0,i} = 1$, and $\gamma_i = 0.01$. The probability of a second-period shock is $\pi = 0.2$, and the allowed leverage ratio is $\rho = 10$. The other assets in the investor's portfolio, the initial liabilities l_0, and the shock magnitude δ are assumed to be such that the shadow prices are $z_1 = 0.04$ and $z_2 = 0.2$. Figure 3 plots the optimal first-period trade as a function of λ_i. If λ_i is small (less than approximately 0.03), the asset will be fully liquidated in the case of a second-period shock. In this range, for a more liquid asset a larger proportion is held over for the second period (as we will see, this is the consequence of a high second-period shadow price, which arises from the expectation of a severe second-period shock). On the other hand, if λ_i is large (more than approximately 0.03), the asset is not fully liquidated, even in the event that a second-period shock occurs. In this range, comparatively less is sold in both periods when the asset is less liquid, as this particular asset becomes costly to dispose of quickly. The kink in $y^*_{1,i}(\lambda_i)$ comes from the transition from case 2 to case 1 (the constraint $y_{1,i} + y_{2,i} \geq -x_{0,i}$ switches from binding to non-binding). We will more formally explore the structure of the two-period problem, but this motivating example illustrates an important feature of the solution. The monotonicities that held for the one-period problem (Result 4) do not always hold for the two-period problem and there are situations where the investor optimally sells comparatively more of a less liquid asset than of a more liquid asset.

Result 6. *Let i and j be any two assets with equal initial prices and equal initial holdings such that $\lambda_i < \lambda_j$, $\gamma_i < \gamma_j$, and $\gamma_i/\lambda_i < \gamma_j/\lambda_j$. Then, for any δ such that the problem is strictly feasible and any $\pi \in [0,1)$, $y^*_{1,i} < y^*_{1,j}$.*

According to Result 6, if an asset is *a)* more liquid than another asset in terms of its temporary and permanent price impacts, and *b)* has a lower ratio of permanent to temporary price impact, then the investor optimally sells more of the liquid asset in the first period, no matter how extreme the expected second-period shock. The second condition implies that the investor prefers to sell assets that have a relatively low permanent impact of trading compared to the transaction costs of trading (*i.e.*, the temporary component).

So far, this result is consistent with the optimal trading behavior derived in Section 2.3. The next result, however, explores what happens when one asset is more liquid than another, but the ratio points in the opposite direction.

Result 7. *Consider the two-period liquidation problem, and assume that the trade $y_1 = -\frac{1}{2}x_0$ strictly satisfies the first-period margin constraint and that $((1+\rho)\,\lambda_k + \gamma_k)\,x_{0,k} \leq p_{0,k}$ for all k. Let i and j be two assets with equal initial prices and equal initial holdings such that $\gamma_i/\lambda_i < \gamma_j/\lambda_j$. Then, for any $\rho > 0$, $\pi \in [0,1)$, there exists a δ such that $y^*_{1,i} < y^*_{1,j}$.*

According to Result 7, for a large enough expected second-period shock, the ratio of permanent to temporary price is more important than the price impacts considered individually. In this case, the investor may favor selling less liquid assets in the first period to hedge against the future need for liquidity. Selling assets with a low ratio of permanent to temporary price impact is optimal, and may result in keeping more liquid securities on hand for the eventuality of a second shock.

Result 7 has a non-obvious implication. If the expected second-period shock is sufficiently large, the investor may wish to retain more of a liquid asset j that has both lower temporary and lower permanent price impacts of trading than asset i. We can rewrite the ratio condition in Result 7 as

$$\frac{\lambda_i}{\lambda_j} > \frac{\gamma_i}{\gamma_j},$$

from which we see that if the option value of keeping the liquid asset is sufficiently high due to its temporary component, then the investor will prefer to retain the liquid asset and incur the short-term penalty from selling an asset with a higher permanent price impact. This suggests that the temporary price impact is the main determinant of policies to manage the risk that a shock may worsen.

Our final result follows directly from the previous two results. It directly addresses the problem posed by Scholes, and presents a striking departure from the results in the one-period problem.

Result 8. *Consider a two-period liquidation problem where m assets have equal holdings and equal initial prices (which we denote by $i = 1, \ldots, m$, $m \leq n$). Then the following hold:*

(a) If $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m$, then for any δ for which the problem is strictly feasible, we have $y^*_{1,1} < y^*_{1,2} < \cdots < y^*_{1,m}$.

(b) If $\gamma_1 = \gamma_2 = \cdots = \gamma_m$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ and, further, $((1+\rho)\,\lambda_k + \gamma_k)\,x_{0,k} \leq p_{0,k}$ for all k and $y_1 = -\frac{1}{2}x_0$ strictly satisfies the first-period margin constraint, then for any $\rho > 0$, $\pi \in [0, 1)$ there exists a δ such that $y^*_{1,1} < y^*_{1,2} < \cdots < y^*_{1,m}$.

The first statement says that the investor will always want to trade assets that have a low permanent price impact, no matter how great the expected need for future liquidity may be. However, the second statement says that this does not hold for the temporary component of liquidity: if the future shock is sufficiently large, then the investor optimally holds on to assets that have a small temporary component of liquidity in the first period in preparation for the possibility of future distress.

This finding has important consequences and empirical import. Specifically, when we decouple the two determinants of liquidity, it is the size of the expected shock and the temporary component that seem most important in determining which assets are liquidated. Differentials in the permanent impact of trading do not change the qualitative solution to the Scholes liquidation problem. This implies that securities that differ in the amount of asymmetric information, or differ in how concentrated their ownership is, will be liquidated in amounts that preserve their relative order. However, the transaction costs that arise due to limited immediate access to counterparties or to a high-volume market, may affect the order in which assets are liquidated in a multi-period framework. In this sense, our analysis indicates that the magnitude of δ and the matrix Λ drive the qualitative nature of the solution to the Scholes liquidation problem.

4 Conclusion

The question raised by Scholes is central to investors in financial markets during crises, when the option value of holding liquid assets is poorly understood. We have developed a model of distressed liquidation in two periods that has allowed us to begin investigating this question.

We find that when the expected future need for liquidity is high, the option value of liquidity is larger, leading to a shift in preferences towards holding more liquid assets. In a multi-period setting where investors need to consider recurring liquidity shocks, the temporary and permanent components of liquidity do not effect the solution equally. The temporary price impact only has an immediate effect on the investor's objective. On the other hand, the permanent price impact not only has an immediate effect, it also changes the 'state' of the system so that its effect extends into the future. Its effect is magnified relative to the effect of the temporary price impact and, no matter how high the option value of holding liquid assets, the liquidation order never reverses simply due to the permanent price impact.

On a more fundamental level there is, in the first period, a trade-off between the immediate value from liquidation and the option value of holding an asset since, in periods of crisis or instability, an investor cannot quickly rebuild liquidated positions. As Scholes noted, "since it is not possible to know the extent of the unfolding crisis, holding and not selling the less liquid instruments is similar to buying an option to hold a position." Our analysis indicates that the precise form that this trade-off in value takes, that is how immediate value and option value are balanced, is tied in a fundamental way to the ratio between an asset's permanent and temporary price impacts.

A Appendix: Proofs

Proof of Lemma 1.

Proof. For any given trading target y_1, the final assets a_1 does not depend on the execution schedule. A constant trading rate will then maximize κ_1 and minimize the liabilities, a straightforward result in calculus of variations for which we refer the reader to Carlin, Lobo, and Viswanathan (2007), Bertsimas and Lo (1998) or Huberman and Stanzl (2005). $\square$

Proof of Result 1.

Proof. From the gradient of e_1, the first-order conditions are $\Gamma x_0 - 2\left(\Lambda - \frac{1}{2}\Gamma\right) y_1 = 0$ which, for Γ and $\Lambda - \frac{1}{2}\Gamma$ positive definite, has a single solution as stated. From concavity in y_1 of e_1 and continuity of its gradient, the first-order conditions are necessary and sufficient. The monotonicities for the diagonal case follow from differentiation and from the positivity assumptions on x_0, Γ and Λ ($\Gamma \succ 0$ and $\Lambda \succ 0$ imply $\gamma_i > 0$ and $\lambda_i > 0$ for all i). $\square$

Proof of Result 2.

Proof. The condition $\Lambda \succ \frac{1}{2}\Gamma$ that ensures concavity of e_1 also implies convexity of the constraint on leverage, so that the problem is strictly convex and has a unique solution, which must satisfy the first-order conditions,

$$\Gamma x_0 - 2\left(\Lambda - \frac{1}{2}\Gamma\right) y_1^* + z\left((\rho\Gamma x_0 - p_0) - 2\left(\rho\left(\Lambda - \frac{1}{2}\Gamma\right) + \left(\Lambda + \frac{1}{2}\Gamma\right)\right) y_1^*\right) = 0,$$

for some $z \geq 0$. This is equivalent to

$$2\left((1+z\rho)\left(\Lambda - \frac{1}{2}\Gamma\right) + z\left(\Lambda + \frac{1}{2}\Gamma\right)\right) y_1^* = \left((1+z\rho)\,\Gamma x_0 - z p_0\right),$$

and the condition on Γ and Λ, combined with $z > 0$, ensure that these equations are uniquely invertible. The alternative expression for $y_{1,i}^*$ in the diagonal case presented immediately after the result follows from algebraic manipulation, with $\tilde{z} = z/(1+z\rho)$. Increasing ρ (or decreasing l_0) relaxes the margin constraint, which implies a smaller shadow price z. The optimal trade is monotonic in decreasing in $\tilde{z}$, which is increasing in z. The non-monotonicity in λ_i and γ_i follows from any counter-example (see below). $\square$

Proof of Result 3.

Proof. Consider the problem with the leverage constraint dualized,

$$\begin{array}{ll}\text{maximize} & e_1 + z(l_1 - \rho e_1) \\ \text{subject to} & -x_0 \leq y_1 \leq 0,\end{array}$$

With diagonal structure the assets are decoupled, in that the solution can be obtained by separately optimizing the trades in each asset. If the box constraint in asset i is not active, its solution is as stated in Result 2. The monotonicities follow by the same arguments as in Result 2. □

Proof of Result 4.

Proof. The proofs for λ_i and $x_{0,i}$ are immediate from the partial derivative of $y^*_{1,i}$. Consider now γ_i. Since the proof only looks at asset i (the constancy of z ensures that the assets remain decoupled), we can without loss of generality normalize $p_{0,i} = 1$ and $x_{0,i} = 1$. The numerator of the partial derivative $\frac{\partial y^*_{1,i}}{\partial \gamma_i}$ can then be seen to simplify to

$$2\lambda_i + (2\lambda_i - 1)z + z^2.$$

Therefore, the partial derivative is positive for any $z < z_0$ where

$$z_0 = \frac{1}{2}\left(1 - 2\lambda_i - \sqrt{4\lambda_i^2 - 12\lambda_i + 1}\right),$$

if $\lambda_i \leq 3/2 - \sqrt{2}$, and for all z otherwise. From the constraint on short sales $y^*_{1,i} \geq -1$, the dual variable z must satisfy

$$z \leq \frac{2\lambda_i}{1 - 2\lambda_i - \gamma_i} \leq \frac{2\lambda_i}{1 - 4\lambda_i}.$$

The proof is completed by showing that this is less than z_0 for any $\lambda_i > 0$. The inequality can be written as

$$8\lambda_i^2 - 10\lambda_i + 1 > (1 - 4\lambda_i)\sqrt{4\lambda_i^2 - 12\lambda_i + 1}.$$

From inspection of the roots of the different factors, both sides are positive for $\lambda_i \leq 3/2 - \sqrt{2}$, and we can therefore square both sides. Collecting terms, the inequality then simplifies to $64\lambda_i^3 > 0$.

The corollaries are then immediate. Note that they can also be proved directly by a swapping argument: if the monotonicity is violated, the investor can do better equity-wise by swapping $y^*_{1,i}$ and $y^*_{1,j}$ while will still satisfying the problem constraints, thereby contradicting the optimality of y^*_1. The second part of the corollary follows directly from the expression for $y^*_{1,i}$. □

Proof of Result 5.

Proof. Notice that for $\delta = 0$, the two-period problem is identical to the myopic deleveraging problem. Thus, at $\delta = 0$ it must be the case for the optimal solution to satisfy $l_1/e_l = \rho$. Suppose not, *i.e.*, suppose at $\delta = 0$, we have $l_1/e_1 < \rho$. Then the shadow price of the first-period margin constraint is zero, and the problem is equivalent to a problem that maximizes the net equity subject to $-x_0 \leq y_1 \leq 0$. Since we have $y_1 \leq 0$, $\Gamma \geq 0$ and $x_0 \geq 0$, it follows that

$$\begin{aligned} e_1 &= e_0 + x_0'\Gamma y_1 - y_1\left(\Lambda - \frac{1}{2}\Gamma\right)y_1 \\ &\leq e_0 - y_1\left(\Lambda - \frac{1}{2}\Gamma\right)y_1 \\ &\leq 0, \end{aligned}$$

with equality strict for any nonzero y_1 due to $\Lambda - \Gamma/2 \succ 0$. This means $y_1 = 0$ must be optimal, which is a contradiction, as the trader initially satisfied $l_0/e_0 > \rho$.

Now consider the maximum possible δ such that the two-period problem is still feasible. We will consider the problem of finding the maximum such δ and show that at this value, it must be that $l_l/e_l < \rho$. Denote this maximum feasible δ by $\bar{\delta}$; one can see that $\bar{\delta}$ is given by

$$\bar{\delta} = \frac{\rho a_0 - (\rho+1)l_0 + \rho v^\star}{\rho + 1},$$

where $v^\star$ is the optimal value of the (convex) problem

$$\begin{array}{ll} \text{maximize}_{y_1,y_2} & -(p_0 - \rho\Gamma x_0)(y_1+y_2) - [y_1'\ y_2'] \begin{bmatrix} (\rho+1)\left(\Lambda + \frac{1}{2}\Gamma\right) - \rho\Gamma & \frac{1-\rho}{2}\Gamma \\ \frac{1-\rho}{2}\Gamma & (\rho+1)\left(\Lambda + \frac{1}{2}\Gamma\right) - \rho\Gamma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \text{subject to} & \rho a_0 - (\rho+1)l_0 - (p_0 - \rho\Gamma x_0)'y_1 - y_1'\left((\rho+1)\left(\Lambda + \frac{1}{2}\Gamma\right) - \rho\Gamma\right)y_1 \geq 0 \\ & -x_0 \leq y_1 \leq 0 \\ & -x_0 - y_1 \leq y_2 \leq 0. \end{array}$$

We will show that the optimal solution will strictly satisfy the first-period margin constraint, so we will omit this constraint for now as we consider computation of $\bar{\delta}$. Define the following for ease of notation:

$$\begin{aligned} d &\triangleq p_0 - \rho\Gamma x_0 \\ D &\triangleq \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} (\rho+1)\left(\Lambda + \frac{1}{2}\Gamma\right) - \rho\Gamma & \frac{1-\rho}{2}\Gamma \\ \frac{1-\rho}{2}\Gamma & (\rho+1)\left(\Lambda + \frac{1}{2}\Gamma\right) - \rho\Gamma \end{bmatrix}. \end{aligned}$$

Since $d \geq 0$ (by assumption; recall that for any asset i for which $d_i \leq 0$, we can remove it from the original problem without loss of generality) and $D \succ 0$ (strict convexity), it can never be optimal to have $y_1 > 0$ or $y_2 > 0$ above, and therefore we can ignore the non-positivity constraints.

We thus focus on finding the optimal value to the problem

$$\begin{array}{ll} \text{maximize}_{y_1,y_2} & -d'(y_1+y_2) - [y_1' \ y_2']D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \text{subject to} & y_1 + y_2 + x_0 \geq 0. \end{array}$$

Using Lagrange multipliers $\nu \geq 0$ for the inequality constraints, the Lagrangian to this problem is given by

$$\mathcal{L}(y_1, y_2, \nu) = x_0'\nu + (\nu - d)'(y_1 + y_2) - [y_1' \ y_2']D \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and the optimal solution for any ν is given by

$$\begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} = \frac{1}{2}D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix}.$$

If we can find a $\nu \geq 0$ such that the corresponding solution is also feasible, then it must be optimal. Let $H = A + B$ and $\nu = d - Hx_0$. We have

$$\begin{aligned} \nu &= d - (A+B)x_0 \\ &= p_0 - \rho\Gamma x_0 - ((\rho+1)\Lambda + (1-\rho)\Gamma)\, x_0 \\ &\geq p_0 - \rho\Gamma x_0 - (p_0 - \Gamma x_0 + (1-\rho)\Gamma x_0) \\ &= 0, \end{aligned}$$

where in the inequality we are using the condition that the temporary impact component be sufficiently small, *i.e.*, $(\rho+1)\Lambda x_0 \leq p_0 - \Gamma x_0$. Thus, $\nu \geq 0$ for this choice. Moreover,

$$\begin{aligned} \begin{bmatrix} y_1(\nu) \\ y_2(\nu) \end{bmatrix} &= \frac{1}{2}D^{-1} \begin{bmatrix} \nu - d \\ \nu - d \end{bmatrix} \\ &= -\frac{1}{2}D^{-1} \begin{bmatrix} H^{-1}x_0 \\ H^{-1}x_0 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}. \end{aligned}$$

Clearly, splitting up the trade in half over the two periods satisfies the no-short sales constraints. Moreover, by assumption, it strictly satisfies the first-period margin constraint. Thus, this solution is feasible and therefore optimal to the above problem for finding $v^\star$, and hence the maximum level $\bar{\delta}$.

Notice that strict convexity of the objective implies that this is the only optimal solution to this problem; hence, at $\delta = \bar{\delta}$, the trade $(y_1, y_2) = (-x_0/2, -x_0/2)$ is the only feasible solution; since it satisfies the first-period margin constraint strictly, we have found a large enough δ such that the optimal solution to the two-period problem satisfies $l_1/e_l < \rho$.

Thus, at $\delta = 0$ we have $l_1/e_1 = \rho$ and at $\delta = \bar{\delta}$ we have $l_1/e_1 < \rho$; the shadow price z_1 associated with the first-period margin constraint is a continuous and nonincreasing function of δ. This implies that at some $\delta \in (0, \bar{\delta})$, the shadow price goes to zero, proving the threshold property that was claimed. $\square$

Proof of Condition (9).

Proof. Consider two matrices A and B in $\mathbb{R}^{n\times n}$ such that $A = A'$, $B = B'$, $A \succ 0$, and $B \succ 0$. We show that the matrix

$$M = \begin{bmatrix} A & -B \\ -B & A \end{bmatrix}$$

is positive definite if and only if $A \succ B$. Since $A \succ 0$, M is positive definite if and only if its Schur complement is positive definite: $A - BA^{-1}B \succ 0$. Since $A \succ 0$, by change of coordinates this is equivalent to $I - \left(A^{-1/2}BA^{-1/2}\right)^2 \succ 0$. Since $A^{-1/2}BA^{-1/2}$ is symmetrical and therefore has identical left and right eigenvectors, the condition is true if and only if all the eigenvalues of $A^{-1/2}BA^{-1/2}$ satisfy $\lambda^2 < 1$. Since $A^{-1/2}BA^{-1/2}$ is positive definite, this is equivalent to $\lambda < 1$. We conclude that we can write the condition on the Schur complement as $I - A^{-1/2}BA^{-1/2} \succ 0$. By change of coordinates, this is equivalent to $A - B \succ 0$.

Applying this result to the matrix in the quadratic form in (6) leads to the condition

$$\Lambda \succ \frac{\rho - 1}{\rho + 1}\Gamma.$$

By change of coordinates, the matrix in the quadratic form in $\mathbf{E}_\Delta e_2$ is positive definite if

$$\begin{bmatrix} \Lambda - \frac{1}{2}\Gamma & -\sqrt{\pi}\frac{1}{2}\Gamma \\ -\sqrt{\pi}\frac{1}{2}\Gamma & \Lambda - \frac{1}{2}\Gamma \end{bmatrix}$$

is positive definite. Applying the result above to this matrix leads to the condition

$$\Lambda \succ \frac{1 + \sqrt{\pi}}{2}\Gamma.$$

$\square$

Proof of Result 6.

Proof. We prove the result for the interesting Cases in Table 3.2 (Cases 1 and 2); the proof for Cases 3 to 7 follows in similar fashion.

We will show that $y^\star_{1,i} < y^\star_{1,j}$ holds for any such δ and π in the three situations below, which cover all possibilities at the optimal solution:

1. The no-short sales constraints are not binding for either asset i or asset j (i.e., both are in case 1).

2. The no-short sales constraint is binding for asset i but not asset j (asset i is in case 2 and asset j is in case 1; we will argue the intuitive fact that the reverse situation cannot occur).

3. The no-short sales constraints are binding for both assets i and j (both assets are in case 2).

Situation 1: Assume that at the optimal solution, the no-short sales constraints are not binding for either asset i or asset j, and assume that $y^\star_{1,i} \geq y^\star_{1,j}$. We will show that such a solution cannot be optimal. To this end, we will show there exists a feasible direction from $(y^\star_1, y^\star_2)$ in which we can head and strictly improve the objective. This implies that the solution cannot be optimal.

In particular, for some $\epsilon > 0$, let ϵ_1 be the vector with zeros everywhere but $-\epsilon$ in entry i and $+\epsilon$ in entry j. Notice that, for a sufficiently small ϵ, the solution $(y^\star_1 + \epsilon_1, y^\star_2)$ still satisfies the no-short sales constraints.

Now we examine the gradients of the liability amount and the net equity in each period. For the first period, we have

$$\begin{aligned}
\nabla l_1(y_1) &= p_0 + (2\Lambda + \Gamma)y_1 \\
\nabla e_1(y_1) &= \Gamma x_0 - (2\Lambda - \Gamma)y_1,
\end{aligned}$$

and thus, since we have $0 \geq y^\star_{1,i} \geq y^\star_{1,j}$, $\lambda_i < \lambda_j$, and $\gamma_i < \gamma_j$:

$$\epsilon_1' \nabla l_1(y^\star_1)/\epsilon = -(2\lambda_i + \gamma_i)y^\star_{1,i} + (2\lambda_j + \gamma_j)y^\star_{1,j} \leq 0,$$

and (recalling that convexity requires $2\lambda_i - \gamma_i \geq 0$ for all i):

$$\begin{aligned}
\epsilon_1' \nabla e_1(y^\star_1)/\epsilon &= (\gamma_j - \gamma_i)x_0 - (2\lambda_j - \gamma_j)y^\star_{1,j} + (2\lambda_i - \gamma_i)y^\star_{1,i} \\
&\geq (\gamma_j - \gamma_i)x_0 - (2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i))y^\star_{1,j} \\
&\geq \begin{cases} (\gamma_j - \gamma_i)x_0 & \text{if } 2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i) \geq 0 \\ 2(\lambda_j - \lambda_i)x_0 & \text{otherwise,} \end{cases} \\
&> 0.
\end{aligned}$$

We also note that

$$\begin{aligned}
\nabla l_2(y_1, y_2) &= \begin{bmatrix} p_0 + (2\Lambda + \Gamma)y_1 + \Gamma y_2 \\ p_0 + \Gamma y_1 + (2\Lambda + \Gamma)y_2 \end{bmatrix} \\
\nabla e_2(y_1, y_2) &= \begin{bmatrix} \Gamma x_0 - (2\Lambda - \Gamma)y_1 + \Gamma y_2 \\ \Gamma x_0 + \Gamma y_1 - (2\Lambda - \Gamma)y_2 \end{bmatrix}
\end{aligned}$$

We now distinguish several sub-scenarios, and show how we can find an ϵ_2 in each case such that $[\epsilon_1' \ \epsilon_2'] \nabla l_2(y_1^\star, y_2^\star) \le 0$ and $[\epsilon_1' \ \epsilon_2'] \nabla e_2(y_1^\star, y_2^\star) \ge 0$ in each case.

First, if $\epsilon_1' \Gamma(y_1^\star + y_2^\star) \le 0$ and $\epsilon_1' \Gamma(x_0 + y_1^\star + y_2^\star) \ge 0$, then $[\epsilon_1' \ 0] \nabla l_2(y_1^\star, y_2^\star) \le 0$ and $[\epsilon_1' \ 0] \nabla e_2(y_1^\star, y_2^\star) \ge 0$.

Second, if $\epsilon_1' \Gamma(y_1^\star + y_2^\star) \le 0$ but $\epsilon_1' \Gamma(x_0 + y_1^\star + y_2^\star) < 0$, then let $\epsilon_2 = \epsilon_1$. Note that $\epsilon_1' \Gamma(x_0 + y_1^\star + y_2^\star) < 0$ requires $y_{1,j}^\star + y_{2,j}^\star < y_{1,i}^\star + y_{2,i}^\star$, so $\epsilon_1' \Lambda(y_1^\star + y_2^\star) \le 0$ and $\epsilon_1' \Gamma(y_1^\star + y_2^\star) \le 0$, and hence $[\epsilon_1' \ \epsilon_2'] \nabla l_2(y_1^\star, y_2^\star) \le 0$. In addition, we have

$$\begin{aligned}
[\epsilon_1' \ \epsilon_2'] \nabla e_2(y_1^\star, y_2^\star)/\epsilon &= 2(\epsilon_1' \Gamma x_0 - \epsilon_1'(2\Lambda - \Gamma)(y_1^\star + y_2^\star))/\epsilon \\
&= 2((\gamma_j - \gamma_i)x_0 - (2\lambda_j - \gamma_j)(y_{1,j}^\star + y_{2,j}^\star) + (2\lambda_i - \gamma_i)(y_{1,i}^\star + y_{2,i}^\star)) \\
&\ge 2((\gamma_j - \gamma_i)x_0 - ((2\lambda_j - \lambda_i) - (\gamma_j - \gamma_i))(y_{1,j}^\star + y_{2,j}^\star)) \\
&\ge \begin{cases} 2(\gamma_j - \gamma_i)x_0 & \text{if } 2(\lambda_j - \lambda_i) - (\gamma_j - \gamma_i) \ge 0 \\ 4(\lambda_j - \lambda_i)x_0 & \text{otherwise,} \end{cases} \\
&> 0.
\end{aligned}$$

Finally, consider the case $\epsilon_1' \Gamma(y_1^\star + y_2^\star) \ge 0$. Then let $\epsilon_2 = -\epsilon_1$ and note that

$$\begin{aligned}
\begin{bmatrix} \epsilon_1' & \epsilon_2' \end{bmatrix} \nabla l_2(y_1^\star, y_2^\star)/\epsilon &= 2\epsilon_1' \Lambda(y_1^\star + y_2^\star)/\epsilon \\
\begin{bmatrix} \epsilon_1' & \epsilon_2' \end{bmatrix} \nabla e_2(y_1^\star, y_2^\star)/\epsilon &= -2\epsilon_1 \Lambda(y_1^\star + y_2^\star)/\epsilon,
\end{aligned}$$

and we therefore need to show that $\epsilon_1' \Lambda(y_1^\star + y_2^\star) \le 0$. We claim this is always true under the given conditions. Note that $\epsilon_1' \Gamma(y_1^\star + y_2^\star) \ge 0$ requires $y_{2,i}^\star \le (\gamma_j/\gamma_i) y_{2,j}^\star$. Therefore,

$$\begin{aligned}
\epsilon_1' \Lambda(y_1^\star + y_2^\star)/\epsilon &= (\lambda_j y_{1,j}^\star - \lambda_i y_{1,i}^\star) + (\lambda_j y_{2,j}^\star - \lambda_i y_{2,i}^\star) \\
&\le \lambda_j y_{2,j}^\star - \lambda_i y_{2,i}^\star \\
&\le \lambda_j y_{2,j}^\star - \frac{\lambda_i \gamma_j}{\gamma_i} y_{2,j}^\star \\
&< \lambda_j y_{2,j}^\star - \lambda_j y_{2,j}^\star \\
&= 0,
\end{aligned}$$

where in the first inequality, we use $0 \geq y^\star_{1,i} \geq y^\star_{1,j}$ and $\lambda_j > \lambda_i$, in the second inequality we use $y^\star_{2,i} \leq (\gamma_j/\gamma_i)y^\star_{2,j}$ and in the third inequality we use $\lambda_i/\gamma_i > \lambda_j/\gamma_j$.

Therefore, in each circumstance we have constructed a feasible direction for which l_1 and l_2 are no larger, e_2 is no smaller, and e_1 is strictly larger. Since $\pi < 1$, this means we have found a direction that still satisfies all the problem constraints (no-short sales and margin constraints) with strictly larger objective, contradicting the optimality of $(y^\star_1, y^\star_2)$.

Situation 2: Assume that at the optimal solution, the no-short sales constraints are binding for asset i but not binding for asset j (using an argument similar to the one above, we can argue that $y^\star_{1,i} + y^\star_{2,i} < y^\star_{1,j} + y^\star_{2,j}$ must hold, meaning it can never be the case that the constraints are binding in the reverse direction). Note that the no-short sales constraint is binding for asset i, so the above argument no longer applies (as we cannot sell any more of asset i).

For the next two cases, the following will be of use.

Lemma 2. *Let $r_{z_1,z_2} : \mathbb{R}^2_+ \to \mathbb{R}$ be the family of functions*

$$r_{z_1,z_2}(\lambda,\gamma) \quad \triangleq \quad \frac{\rho(1-\pi+z_1)\gamma - 2(\pi\rho + (1+\rho)z_2)\lambda - z_1\frac{p_0}{x_0}}{(\rho(\pi-1)+z_1(1-\rho))\gamma + 2(\rho(1+\pi)+(z_1+2z_2)(1+\rho))\lambda},$$

parameterized by $(z_1, z_2) \in \mathbb{R}^2_+$, and where $\pi \in [0,1)$, $\rho \geq 0$, and the denominator is strictly positive. If $\lambda_i < \lambda_j$, $\gamma_i < \gamma_j$, and $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then for any $0 \leq z_1 < \infty$, $0 \leq z_2 < \infty$, $r_{z_1,z_2}(\lambda_i,\gamma_i) < r_{z_1,z_2}(\lambda_j,\gamma_j)$.

Proof. We will fix a λ and a γ as well as all parameters in the function. The claim holds if and only if $\epsilon'\nabla r_{z_1,z_2}(\lambda,\gamma) > 0$ for all vectors $\epsilon = [\epsilon_\lambda \ \epsilon_\gamma]' \in \mathbb{R}^2_+$ with $\epsilon_\lambda \leq (\lambda/\gamma)\epsilon_\gamma$ (*i.e.*, moving in any directions such that λ and γ do not decrease, nor does their ratio γ/λ, must increase $r_{z_1,z_2}(\lambda,\gamma)$). After some algebra, we arrive at

$$\nabla r_{z_1,z_2}(\lambda,\gamma) \quad \propto \quad \begin{bmatrix} BD\gamma - A\left(C\gamma - \frac{z_1 p_0}{x_0}\right) \\ AC\lambda - D\left(B\lambda - \frac{z_1 p_0}{x_0}\right) \end{bmatrix},$$

where

$$\begin{aligned} A &\triangleq \rho(1+\pi) + (z_1+2z_2)(1+\rho) \\ B &\triangleq \rho\pi + (1+\rho)z_2 \\ C &\triangleq \rho(1-\pi+z_1) \\ D &\triangleq \rho(1-\pi) + z_1(1-\rho). \end{aligned}$$

To verify that the gradient condition holds over all such ϵ, we need only check the extreme rays of the set, which are given by $\epsilon_1 = [0\ 1]'$ and $\epsilon_2 = [(\lambda/\gamma)\ 1]'$. This is clearly true for ϵ_1, as $A > 0$, $C > 0$, $A\lambda \geq B\lambda - z_1 p_0/(x_0)$, and $C \geq D$. For ϵ_2, we have

$$\begin{aligned}
\epsilon_2' \nabla r_{z_1,z_2}(\lambda,\gamma) &= \frac{\lambda}{\gamma}\left(DB\gamma - A\left(C\gamma - \frac{z_1 p_0}{x_0}\right)\right) + \left(AC\lambda - D\left(B\lambda - \frac{z_1 p_0}{x_0}\right)\right) \\
&= \frac{\lambda}{\gamma}(BD - AC)\gamma + (AC - BD)\lambda + (A\lambda + B\gamma)\frac{z_1 p_0}{x_0} \\
&= (A\lambda + B\gamma)\frac{z_1 p_0}{x_0} \\
&\geq 0,
\end{aligned}$$

where we are using $A \geq 0$, $B \geq 0$. □

Now note that in this case, $y^\star_{1,i} = r_{z_1,z_2}(\lambda_i, \gamma_i)$, with (z_1, z_2) the optimal Lagrange multipliers, because at the optimal solution the no-short sales constraint is active for asset i ($r_{z_1,z_2}(\lambda_i,\gamma_i)$ is the form of the optimal solution, as discussed earlier, for asset i in this case). We will argue that for asset j, which satisfies $y^\star_{1,j} + y^\star_{2,j} > -x_0$ by assumption, we have $y^\star_{1,j} > r_{z_1,z_2}(\lambda_j, \gamma_j)$. Since Lemma 2 implies that $r_{z_1,z_2}(\lambda_j,\gamma_j) > r_{z_1,z_2}(\lambda_i,\gamma_i) = y^\star_{1,i}$, this will establish the result.

We can interpret $r_{z_1,z_2}(\lambda_j,\gamma_j)$ as the optimal first-period trade for asset j (with the Lagrange multipliers fixed at (z_1, z_2)) if we are forcing asset j to satisfy the no-short sales constraint tightly (*i.e.*, forcing $y_{1,j} + y_{2,j} = -x_0$. Put another way, $r_{z_1,z_2}(\lambda_j,\gamma_j)$ is the optimal solution y to the problem

$$\begin{array}{ll}
\text{maximize} & -\begin{bmatrix} y - y^\star_{1,j} \\ -x - y^\star_{2,j} - y \end{bmatrix}' \begin{bmatrix} f_j & h_j \\ h_j & g_j \end{bmatrix} \begin{bmatrix} y - y^\star_{1,j} \\ -x - y^\star_{2,j} - y \end{bmatrix} \\
\text{subject to} & -x \leq y \leq 0,
\end{array}$$

where $f_j > g_j \geq h_j$. Assuming the inequalities are inactive at the optimal solution (it is easy to verify that $y = 0$ cannot be optimal, and if $-x_0$ is optimal, then $y = r_{z_1,z_2}(\lambda_j,\gamma_j) = -x_0 < y^\star_{1,j}$, so we are done), the optimal solution y must satisfy the first-order condition:

$$\begin{aligned}
(f_j - h_j)(y - y^\star_{1,j}) - (g_j - h_j)(-x - y^\star_{2,j} - y) &= 0 \\
&\Updownarrow \\
(f_j + g_j - 2h_j)y &= (f_j - h_j)y^\star_{1,j} + (g_j - h_j)(-x - y^\star_{2,j}).
\end{aligned}$$

Since $y^\star_{1,j} + y^\star_{2,j} > -x_0$ by assumption in this case, we have

$$(f_j + g_j - 2h_j)y \quad < \quad (f_j + g_j - 2h_j)y^\star_{1,j},$$

and since $f_j > g_j \geq h_j$, this implies the optimal y must satisfy $y < y^\star_{1,j}$, which gives us the result.

Situation 3: Assume that at the optimal solution, the no-short sales constraints are binding for both assets i and j. In this case, we have $y^\star_{1,i} = r_{z_1,z_2}(\lambda_i, \gamma_i)$ and $y^\star_{1,j} = r_{z_1,z_2}(\lambda_j, \gamma_j)$. The result now follows by Lemma 2. $\square$

Proof of Result 7.

Proof. Throughout the proof, we will use the notation (δ) to denote that a parameter in question (*e.g.*, optimal solution, shadow price, etc.) is a function of the second-period shock size, $\delta \geq 0$, which will be varying. We will show that under the given conditions we can find a $\hat{\delta}$ with $z_1(\hat{\delta}) = 0$, the no-short sales constraints active for both asset i and asset j and $z_2(\hat{\delta})$ finite but arbitrarily large. We start with the following lemma.

Lemma 3. *Let $r_{z_1,z_2} : \mathbb{R}^2_+ \to \mathbb{R}$ be the family of functions as described in Lemma 2. Then if $\gamma_i/\lambda_i < \gamma_j/\lambda_j$, then there exists a $0 \leq z_2 < \infty$ such that $r_{0,z_2}(\lambda_i, \gamma_i) < r_{0,z_2}(\lambda_j, \gamma_j)$.*

Proof. Note that when $z_1 = 0$, we can express the function in question as

$$r_{0,z_2}(\lambda, \gamma) \quad = \quad \frac{\rho\left(1 - \pi - 2\pi\left(\frac{\lambda}{\gamma}\right)\right) - 2(1+\rho)\left(\frac{\lambda}{\gamma}\right) z_2}{\rho\left(\pi - 1 + 2(1+\pi)\left(\frac{\lambda}{\gamma}\right)\right) + 4(1+\rho)\left(\frac{\lambda}{\gamma}\right) z_2}.$$

Denoting λ/γ by σ, note that we can write the functions as

$$\begin{aligned} r_{0,z_2}(\lambda_i, \gamma_i) &= \frac{a_i - b_i z_2}{c_i + 2b_i z_2} \\ r_{0,z_2}(\lambda_j, \gamma_j) &= \frac{a_j - b_j z_2}{c_j + 2b_j z_2}, \end{aligned}$$

where

$$\begin{aligned} a_i &= \rho(1 - \pi - 2\pi\sigma_i) \\ b_i &= 2(1+\rho)\sigma_i \\ c_i &= \rho(\pi - 1 + 2(1+\pi)\sigma_i), \end{aligned}$$

and analogously for (a_j, b_j, c_j). Since the denominators are both positive, one can verify that there exists a $z_2 \geq 0$ such that $r_{0,z_2}(\lambda_i, \gamma_i) < r_{0,z_2}(\lambda_j, \gamma_j)$ if

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) \quad < \quad 0$$

holds. Some simple algebra shows that

$$2(a_i b_j - a_j b_i) - (b_i c_j - b_j c_i) \quad = \quad 2\rho(1+\rho)(1-\pi)(\sigma_j - \sigma_i),$$

and since $\rho > 0$, $\pi \in [0, 1)$, $\sigma_j < \sigma_i$ implies the result. □

Now consider the problem of finding the maximum possible $\bar{\delta} > 0$ such that the two-period problem is still feasible. Following the proof of Result 5, we find that such a $\bar{\delta}$ corresponds to the trade $y_1 = -x_0/2$, $y_2 = -x_0/2$, i.e., splitting up all assets equally across the two periods. Since the objective function for computing this $\bar{\delta}$ is strictly convex, the solution $y_1 = -x_0/2$ and $y_2 = -x_0/2$ can be the only solution that satisfies the second-period margin constraint at $\bar{\delta}$, and it must be that $\bar{\delta} > 0$ (since the trade $y_1 = -x_0/2$, $y_2 = 0$ satisfies the constraints for the problem $\delta = 0$ and the objective function for computing $\bar{\delta}$ is strictly convex). So, the feasible set to the original problem with $\delta = \bar{\delta}$ is a singleton at which the no-short sales constraints are obviously tight. By assumption, the first-period margin constraint is strictly satisfied, and hence $z_1(\bar{\delta}) = 0$. Moreover, for any $\delta > \bar{\delta}$, the problem is infeasible, and therefore we must have $z_2(\bar{\delta}) = +\infty$.

To complete the proof, we need to argue that we can find a small enough perturbation, $\epsilon > 0$, such that at $\bar{\delta} - \epsilon$, it is still optimal to have both box constraints active, $z_1(\bar{\delta} - \epsilon) = 0$, and $z_2(\bar{\delta} - \epsilon)$ is finite but arbitrarily large. If we can do this, we will have found a δ for which the first-period margin constraint is inactive, assets i and j are both tight on the no-short sales constraint, and $z_2(\delta)$ can be made as large as desired; since $y^\star_{1,i}(\delta) = r_{0,z_2(\delta)}(\lambda_i, \gamma_i)$ and $y^\star_{1,j}(\delta) = r_{0,z_2(\delta)}(\lambda_j, \gamma_j)$, the result will then follow by Lemma 3.

First, notice that the feasible set at $\delta = \bar{\delta}$ is a singleton, as argued above. For any $\epsilon > 0$, we are enlarging a single ellipsoid, and the feasible set must therefore still be compact (closed and bounded). The singleton at $\bar{\delta}$ is strictly contained inside the first-period margin ellipsoid. Since the feasible set shrinks to a singleton strictly contained in the first-period margin ellipsoid as $\epsilon \to 0$, we can find a sufficiently small $\epsilon_1 > 0$ such that the feasible set is still strictly contained inside the first-period margin constraint for all $\epsilon \in [0, \epsilon_1]$.

Now consider how z_2 varies with δ. z_2 is an optimal Lagrange multiplier; it is obtained by minimizing a rational function and therefore $z_2(\delta)$ is continuous, and $z_2(\bar{\delta}) = \infty$. Since it is a continuous function, we can find an $\epsilon_2 > 0$ such that $z_2(\bar{\delta} - \epsilon_2)$ is finite but arbitrarily large. Since it can be arbitrarily large, we can make it large enough such that the no-short sales constraints for both assets i and j must still be active for any $\epsilon \in [0, \epsilon_2]$.

Now take $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$. We have $z_1(\bar{\delta} - \epsilon) = 0$, $z_2(\bar{\delta} - \epsilon) < \infty$, and both no-short sales constraints active at $\bar{\delta} - \epsilon$. The proof is complete. □

Proof of Result 8.

Proof. (a) follows directly by Result 6. (b) follows by applying Result 7 to construct such a δ for each pair of assets in $\{1, \ldots, m\}$, then taking the maximum over all such δ. $\square$

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