Pricing and Revenue Management with Stochastic Demand: Coordinated versus Hierarchical Approaches

Ayse KOCABIYIKOGLU
Ioana POPESCU
Catalina STEFANESCU
2010/83/DS
(Revised version of 2008/47/DS/TOM)
Pricing and Revenue Management with Stochastic Demand:  

Coordinated versus Hierarchical Approaches

Ayşe Kocabiyikoğlu*

Ioana Popescu**

Catalina Stefanescu***

Revised version of 2008/47/DS/TOM

* Assistant Professor at Bilkent University 06800 Bilkent, Ankara, Turkey  
  Email: aysekoca@bilkent.edu.tr

** Associate Professor of Decision Sciences, Booz & Company Professor in Strategic Revenue  
  Management at INSEAD, 1 Ayer Rajah Avenue, Singapore 138676  
  Email: ioana.popescu@insead.edu

*** Associate Professor at European School of Management and Technology, Schlossplatz 1,  
  10178 Berlin, Germany Email: catalina.stefanescu-cuntze@esmt.org

A Working Paper is the author’s intellectual property. It is intended as a means to promote research to interested readers. Its content should not be copied or hosted on any server without written permission from publications.fb@insead.edu

Click here to access the INSEAD Working Paper collection
Pricing and Revenue Management with Stochastic Demand: Coordinated versus Hierarchical Approaches

Ayşe Kocabıyıkoğlu  
Department of Management, Bilkent University, Bilkent, Ankara 06800, Turkey, aysekoca@bilkent.edu.tr

Ioana Popescu  
INSEAD, Decision Sciences Area, 1 Ayer Rajah Avenue, Singapore, ioana.popescu@insead.edu

Catalina Stefanescu  
European School of Management and Technology, Schlossplatz 1, 10178 Berlin, Germany, catalina.stefanescu-cuntze@esmt.org

We investigate the value of coordinating static pricing and revenue management availability decisions under price-sensitive demand uncertainty. We first characterize a general class of stochastic, price-dependent demand models; these lead to several static pricing and revenue management models that can be solved efficiently and admit unique solutions with intuitive sensitivity properties. We then compare the performance of various hierarchical and coordinated models, shedding light on the value of coordinating price and availability decisions as well as of modeling demand uncertainty and resource substitution in pricing and revenue management. Our numerical insights, which are based on industry data, suggest that capturing price-sensitive demand uncertainty in a hierarchical revenue management process can yield significant revenue benefits and thereby achieve most of the potential of a fully coordinated process.

1. Introduction

Revenue management approaches are common in capacity-constrained service industries—including airlines, hotels, car rentals, event ticketing, and TV advertising—where demand is responsive to price changes. However, revenue management models and practice have traditionally focused on capacity allocation decisions and have treated price and demand as exogenous. This focus is partly explained by rigid organizational structures that separate the functions of marketing (including pricing) and operations (revenue management) and also by the technical and operational difficulties inherent in implementing an integrated price–availability decision support system.

Over the past decade, the importance of coordinating decisions on tactical pricing and revenue
management has been widely acknowledged in the revenue management literature (McGill and van Ryzin 1999) and by practitioners (Garrow et al. 2006). In a broad-spectrum review, Fleischmann et al. (2004) observe that “pricing decisions have a direct effect on operations and vice versa. Yet, the systematic integration of operational and marketing insights is in an emerging stage, both in academia and in business practice.” Indeed, a recent survey finds that only 11% of 479 companies practicing revenue management in Europe and North America manage both price and capacity allocation decisions,\(^1\) even as an overwhelming consensus points to price management as having the highest potential for revenue management (Kolisch and Zatta 2010).

The need for a better understanding of the potential benefits and challenges of integrating pricing and revenue management motivates two broad types of research questions. First, from a modeling perspective, what are the challenges and trade-offs entailed by incorporating pricing decisions in a revenue management framework? In particular, what types of demand models lead to well-behaved problems, how should we model price-sensitive demand uncertainty, and when is it actually important to do so? Second, from a benefit assessment perspective, when is it important to integrate pricing and availability decisions, and what is the financial impact of doing so—for example, compared with a sequential (or hierarchical) approach? Our research addresses these issues by incorporating price sensitivity in a framework of static, two-fare class revenue management.

We investigate the effects of incorporating and coordinating pricing decisions in the classical static revenue management (RM) model (Belobaba 1987, Littlewood 1972). This model optimizes the allocation of a limited resource between two customer segments, where higher-paying customers arrive later in the horizon and where prices and demand are exogenously fixed. Considering that RM is a fundamental base model for revenue management and the oldest such model still in practical use, we find it surprising that price sensitivity and optimization have not been previously analyzed in this context. Our paper aims to fill this gap in the literature.

**Contributions.** We extend the standard RM framework by modeling demand as a general stochastic function of price and then jointly optimizing allocation and price for the high-end

\(^1\) Although not necessarily in a coordinated manner.
segment. We also compare the performance of this coordinated model with various hierarchical procedures that first optimize prices (using either deterministic or stochastic heuristics) and then optimize allocation decisions with or without resource substitution. This paper makes the following main contributions, as intimated by the questions raised at the outset.

First, we characterize a general class of stochastic price-sensitive demand models that lead to various coordinated pricing and revenue management models with a unique and computationally efficient solution. We prove unifying demand conditions that are cast in terms of increasing lost sales rate (LSR) elasticity. This property, introduced in Kocabıyıkoğlu and Popescu (2010), is satisfied by a broad class of demand models with increasing failure rate (IFR) risk, including attraction models and additive-multiplicative specifications (e.g., with linear and isoelastic price dependence). These conditions are similar in spirit to the deterministic demand conditions of Ziya et al. (2004) and further enable us to derive sensitivity results that characterize the interaction of price and capacity decisions under demand uncertainty.

Second, we quantify the value of coordinating decisions on pricing and allocation and of modeling price-sensitive demand uncertainty and resource substitution at various stages of the decision process. Using numerical simulations on data from the car rental industry, we find that the revenue gains from coordination can be large (1%–9%) relative to a hierarchical process that sets prices based on deterministic demand models—a common norm in practice. Tighter capacity constraints enhance the value of coordination. However, we show that the benefits of full coordination can be closely replicated by a hierarchical process that sets prices by capturing demand uncertainty (but not resource substitution) and subsequently optimizes booking limits. This suggests that capturing price-sensitive demand uncertainty can be more important than coordination in a static revenue management framework, thereby leveraging our stochastic demand modeling contribution.

These numerical insights have important organizational and implementation consequences in light of the practical challenges posed by coordination, data availability, and forecasting techniques. Moreover, financial consequences can be significant because small positive changes in revenue translate into spectacular profit gains for revenue management industries grappling with high fixed costs.
and extremely thin margins. For example, a 1% increase in revenue would have allowed Avis—which in 2009 posted net profit margins of -1% on revenues of $5 Billion (U.S.)—to break even that year.

**Literature Review.** Our work contributes to the vast literature on revenue management, for which the most comprehensive references to date are the books by Talluri and van Ryzin (2004) and Phillips (2005). Shen and Su (2007) review recent trends in customer behavior modeling in revenue management, Elmaghraby and Keskinocak (2003) focus on dynamic pricing theory and practice, and McGill and van Ryzin (1999) review the earlier revenue management literature.

There is a growing body of work (reviewed by Bitran and Caldentey (2003)) in the revenue management literature that addresses the problem of joint pricing and allocation. Several papers in this area use deterministic demand models to capture complex multiproduct, multiresource, or dynamic environments (e.g., Cote et al. 2003, Kachani and Perakis 2006, Kuyumcu and Garcia-Diaz 2000, Kuyumcu and Popescu 2006). Ziya et al. (2004) analyze demand conditions that ensure regularity in models of deterministic pricing and revenue management.

In contrast, we focus on stochastic demand models: we provide corresponding regularity conditions and assess the value of capturing general, price-sensitive demand uncertainty. Toward this end, we focus on the static, two-fare class capacity allocation model of Littlewood (1972) and Belobaba (1987) and extend it to manage and coordinate pricing decisions. A first step in this direction is due to Weatherford (1997), who investigates numerically the revenue benefits versus the computational effort required to integrate pricing and allocation decisions in a static, single-resource multiproduct environment where additive-linear demand is normally distributed.

A few revenue management papers study joint pricing and allocation problems with aggregate demand uncertainty; they all use additive and/or multiplicative demand forms, which are special cases of our model. Closest to our work is that of Bertsimas and de Boer (2005), who provide regularity conditions for a static model without resource substitution and additive-multiplicative demand (a special case of our model in Section 6) and use that model to devise a heuristic for a multiperiod price–capacity allocation problem. In the context of nonprofit applications, de Vericourt
and Lobo (2009) jointly optimize prices and allocations in a dynamic setting under a multiplicative demand model; their regularity condition for revenue per stage is a special case of our lost sales rate (LSR) elasticity condition. In a dynamic setting with competition, Mookherjee and Friesz (2008) assume increasing price elasticity in a multiplicative demand model with increasing generalized failure rate (IGFR) risk. These papers all rely on static regularity conditions to characterize more complex dynamic problems. Our results extend the static regularity conditions in these papers to more general demand models with and without resource substitution.

Several other approaches are used for modeling price-sensitive demand uncertainty in revenue management. Multiperiod problems capture price-sensitive stochastic demand as a Markov arrival process, which is typically described as Poisson with known price and time-dependent intensity (Feng and Xiao 2006, Gallego and van Ryzin 1994, Maglaras and Meissner 2006). Uncertainty about the arrival rate has been addressed in Bayesian learning frameworks (Aviv and Pazgal 2005, Farias and van Roy 2010) or by using robustness methods (Adida and Perakis 2010, Perakis and Sood 2006).

Finally, our work is also related to a vast operations literature on coordinating pricing and inventory decisions; this literature is reviewed by Chan et al. (2004) and by Fleischmann et al. (2004). An important distinction is that models in this stream focus on storable goods rather than services. Our model can be viewed as an extension of static price-setting newsvendor models (for reviews, see Petruzzi and Dada 1999, Yano and Gilbert 2003). Most of this literature captures price-sensitive demand uncertainty using additive and/or multiplicative models. Closest to our work is the paper of Kocabıyıköglu and Popescu (2010), which uses the concept of increasing LSR elasticity to provide general, unifying demand conditions for the newsvendor pricing problem to be well-behaved. Our results here show that similar demand conditions are sufficient also in revenue management settings that allow multiple products with or without demand and resource substitution.
2. Hierarchical and Coordinated Revenue Management Models

In the standard revenue management model (Belobaba 1987, Littlewood 1972), a monopolistic firm optimizes the allocation of a fixed quantity of a fully flexible resource between two market segments with uncertain, independent demands; the high-price segment arrives after the low-price segment, and prices are predetermined. Yet firms actually have the ability to control prices, which in turn affects demand. In particular, demand for major application areas of revenue management, such as airline travel and car rental, is sensitive to price changes (Talluri and van Ryzin 2004, Chap. 7). To capture price sensitivity, we model demand as a general stochastic function of price, $D(p)$—as further detailed in Section 3—and incorporate this characterization into several pricing and revenue management models, which we describe next.

2.1. Pricing and Revenue Management Models

Let $C$ denote the firm’s capacity and $x$ the protection level for the high-fare class (i.e., the portion of capacity allocated to this class), so that $C - x$ is the booking limit for the low-fare class. The average contribution margins of the high- and low-fare classes are denoted $p$ and $\bar{p}$ (respectively), and the corresponding random demands at these prices are $D(p)$ and $\bar{D}(\bar{p})$, that are assumed to be independent (substitution effects are studied in Section 5.2). Throughout this paper, the parameters pertaining to the low-fare class are denoted by a bar (overline). Table 11 in the Appendix summarizes our notation.

The expected revenue from the low-price product is $\bar{p}\mathbb{E}[\min(\bar{D}(\bar{p}),C - x)]$, since sales to this class are constrained by its demand $\bar{D}(\bar{p})$ and by the booking limit $C - x$. If $y$ units are effectively available to the high-fare class, then the expected revenue from this class is $r(p,y) = p\mathbb{E}[\min(D(p),y)]$. The standard revenue management model allows for resource substitution between classes, so the inventory available to the high-fare class is a priori uncertain and equal to $\max(x,C - \bar{D}(\bar{p}))$; in particular, it may exceed the protection level $x$ if the low-fare demand does not exceed the booking limit—that is, if $\bar{D}(\bar{p}) \leq C - x$. In short, with resource substitution the firm’s expected revenue from the two classes can be expressed as

$$R(\bar{p},p,x) = \bar{p}\mathbb{E} [\min (\bar{D}(\bar{p}),C - x)] + \mathbb{E} [r(p,\max(x,C - \bar{D}(\bar{p})))],$$

(1)
where
\[ r(p, y) = p\mathbb{E}[\min(D(p), y)]. \] (2)

Expectations in (1) and (2) are taken with respect to \( \mathbb{D}(\bar{p}) \) and \( D(p) \), respectively.

The classical RM model optimizes the protection level \( x \) given fixed prices \( \bar{p} \) and \( p \):

\[ \text{(RM)} \quad R[\text{RM}] = \max_x R(\bar{p}, p, x), \] (3)

where \( R \) denotes revenue. In contrast, a fully coordinated joint pricing and revenue management model, JPRM, simultaneously optimizes both prices and the protection level:

\[ \text{(JPRM)} \quad R[\text{JPRM}] = \max_{\bar{p}, p, x} R(\bar{p}, p, x). \] (4)

Because of the timing of arrivals, decisions regarding the high-end segment can be made later in the horizon—after the low-end price \( \bar{p} \) has been set. This motivates our focus on a semi-coordinated pricing and revenue management (PRM) problem (studied in Section 5), that jointly optimizes the high-end price and allocation:

\[ \text{(PRM)} \quad R[\text{PRM}] = \max_{p, x} R(\bar{p}, p, x). \] (5)

Our focus on optimizing the high-end price reflects actual industry practice: low-end prices are constrained by competitive, historical, or social considerations, whereas the firm enjoys more pricing power and flexibility in the high-end market (see Section 5 for further discussion).

Our results indicate that PRM is generally tractable and leads to good hierarchical approximations of JPRM. In order to devise such approximations, we consider two other models commonly used as benchmarks and heuristics for various revenue management problems: the partitioned allocation (PA) model and the deterministic (DM) model. Unlike previously defined RM-based models, PA and DM do not allow for resource substitution.

The PA model (Belobaba 1987, Bertsimas and de Boer 2005), which is studied further in Section 6, jointly optimizes prices \( p, \bar{p} \) and allocation by dividing capacity into separate blocks of size \( x \) and \( C - x \) that can be sold only to the respective market segments:

\[ \text{(PA)} \quad R[\text{PA}] = \max_{p, \bar{p}, x} V(\bar{p}, p, x), \]
where

\[ V(\bar{p}, p, x) = p \mathbb{E}[\min(D(p), x)] + \bar{p} \mathbb{E}[\min(\bar{D}(\bar{p}), C - x)] \].

The DM model (Bitran and Caldentey 2003, Gallego and van Ryzin 1994) is a certainty-equivalent (or fluid) benchmark that replaces random demands with their means \( \mathbb{E}[D(p)] \) and \( \mathbb{E}[\bar{D}(\bar{p})] \) while optimizing prices \( p, \bar{p} \) subject to capacity constraints:

\[(DM) \quad R[DM] = \max_{p, \bar{p}} p \mathbb{E}[D(p)] + \bar{p} \mathbb{E}[\bar{D}(\bar{p})]
\text{s.t.} \quad \mathbb{E}[D(p)] + \mathbb{E}[\bar{D}(\bar{p})] \leq C.\]

2.2. Hierarchical Pricing and Revenue Management Processes

In practice, most organizations use sequential processes for pricing and revenue management decisions: prices are initially decided by marketing divisions and then used as input to decisions concerning revenue management (allocation or booking limits). Formally, a hierarchical process consists of solving a certain pricing problem \( Y \) and then using its price output as input into the revenue management model \( RM \) to obtain the corresponding booking limit; we refer to this sequential (or hierarchical) model as \( RM-Y \). We further denote by \( PRM-Y \) the model where coordination is achieved by solving \( PRM \) instead of \( RM \) to obtain the high-end price and allocation simultaneously. Specifically:

\[(RM-Y) \quad R[RM-Y] = \max_x R(p^Y, \bar{p}^Y, x),\]

where \( p^Y \) and \( \bar{p}^Y \) optimize \( Y \);

\[(PRM-Y) \quad R[PRM-Y] = \max_{x,p} R(p, \bar{p}^Y, x),\]

where \( \bar{p}^Y \) optimizes \( Y \).

Depending on the industry, several pricing mechanisms \( Y \) are conceivable and used in practice; these include price fixing, cost-plus methods, and matching the competition (Phillips 2005). Unlike revenue management models, the price optimization models typically used in practice are deterministic. This motivates us to use the deterministic model DM as a benchmark pricing heuristic \( Y \) to generate price inputs for various decision processes. We also compare the resulting policies
to those that use the stochastic pricing heuristic PA. Table 1 categorizes the pricing and revenue management processes under consideration in terms of three modeling features: coordinating price–allocation decisions, allowing resource substitution, and capturing demand uncertainty in pricing decisions.

Proposition 1 describes some intuitive relationships among revenues generated by various models, and it formalizes the idea that coordination and resource substitution improve performance. The expected revenue of any stochastic model is bounded by DM; the latter acts as a perfect information model, where decisions are made after all uncertainties are resolved. (All proofs are given in the Appendix.)

**Proposition 1.** For any pricing model \( Y \), \( R_{DM} \geq R_{JPRM} \geq R_{PRM-Y} \geq R_{RM-Y} \). Moreover, \( R_{RM-PA} \geq R_{PA} \).

Two main questions emerge from the description of these models and the relationships among them, questions that reflect the broader issues raised in the Introduction. First, when are these models well-behaved from an optimization standpoint? In particular, under what general demand conditions do they admit unique solutions? Second, which of these policies performs better under what conditions? In particular: what is the value of capturing each of the three modeling features identified in Table 1 (coordination, demand uncertainty, and resource substitution), and when is it important to do so? We address the first question by deriving analytical results in Sections 4–6 that are based on general, unifying demand conditions related to monotone LSR elasticity. We provide preliminary analytical answers to the second question in Proposition 1 and develop them numerically in Section 7.

### 3. Price-Sensitive Stochastic Demand Model

In this section we establish our demand model. Throughout the paper, we use the following general price-sensitive stochastic demand model with density \( f(p,x) \) and cumulative distribution \( F(p,x) \):

\[
D(p) = d(p, Z),
\]

(6)
Table 1  Characteristics of pricing and revenue management models

<table>
<thead>
<tr>
<th></th>
<th>Demand Uncertainty (in pricing)</th>
<th>Resource Substitution</th>
<th>Coordination (price and allocation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>JPRM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>PRM-PA</td>
<td>Yes</td>
<td>Yes</td>
<td>Some</td>
</tr>
<tr>
<td>RM-PA</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>PA</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>PRM-DM</td>
<td>Some</td>
<td>Yes</td>
<td>Some</td>
</tr>
<tr>
<td>RM-DM</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>DM</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

where $d$ is a deterministic demand function and $Z$ is a random variable with finite mean, price-independent density function $\phi$, and cumulative distribution function $\Phi$. The random $Z$ captures the demand risk; in empirical estimation, this can be random noise or an independent variable in a regression model. Conceptually, $Z$ can be any sales driver that is uncertain and not perfectly controlled by the firm; examples include market size, personal disposable income of the target market, brand awareness, and a reference price (see, e.g., Hanssens et al. 2001).

The riskless demand function $d(p, z)$ is decreasing in price $p$, strictly increasing in $z$, and twice differentiable in $p$ and $z$. We assume that the riskless (or pathwise) unconstrained revenue $\pi(p, z) = pd(p, z)$ is strictly concave in $p$ for any realization of $z$; that is, $2d_p(p, z) + pd_{pp}(p, z) < 0$. This assumption is not necessary for all our results, but it simplifies the analysis. In particular, it ensures that the benchmark model DM has a unique solution. Throughout the paper, the terms increasing/decreasing and positive/negative are used in their weak sense, and partial derivatives are denoted by corresponding subscripts.

Kocabıyıköglu and Popescu (2010) define the lost sales rate (LSR) elasticity as the price elasticity of the rate of lost sales. This is the percentage change in the lost sales rate, $q(p, x) = 1 - F(p, x)$, with respect to the percentage change in price for a given protection level $x$.

**Definition 1.** The *LSR elasticity* for a given protection level $x$ and price $p$ is defined as

$$E(p, x) = -\frac{pqq_p(p, x)}{q(p, x)} = \frac{pF_p(p, x)}{1 - F(p, x)}.$$  

(7)
Table 2 Some demand models with increasing LSR elasticity (Kocabıyıkoğlu and Popescu 2010)

<table>
<thead>
<tr>
<th>Demand Model</th>
<th>(d(p,z))</th>
<th>(E(p,x)) ↑ (x)</th>
<th>(E(p,x)) ↑ (p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive-multiplicative</td>
<td>(\alpha(p)z + \beta(p))</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Additive</td>
<td>(z + \beta(p))</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>(\alpha(p)z)</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Additive-linear</td>
<td>(z - bp)</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Multiplicative isoelastic</td>
<td>(ap^{-b}z, b &gt; 1)</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Power</td>
<td>(\frac{a}{e^{bp} - bp})</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Logit</td>
<td>(\frac{a}{1+e^{bp}})</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Exponential</td>
<td>(e^{-bp})</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
<tr>
<td>Log</td>
<td>(\log(z - bp))</td>
<td>Z IFR</td>
<td>Z IFR</td>
</tr>
</tbody>
</table>

The LSR elasticity is a unifying concept for our subsequent analysis. Specifically, our main analytical results hold for demand models with \(E(p,x)\) increasing in \(p\) for all \(x\) or in \(x\) for all \(p\). These properties are guaranteed if, respectively, \(q(p,x)\) is concave in price or submodular (\(q_{px} \leq 0\)). Kocabıyıkoğlu and Popescu (2010) provide milder general conditions for monotone LSR elasticity in terms of the riskless demand \(d\) and increasing (generalized) failure rate (IFR/IFGR) conditions on \(Z\).

Some relevant demand models with increasing LSR elasticity are summarized in Table 2. Additive-multiplicative specifications (Bertsimas and de Boer 2005, Young 1978) characterize most models used in operations, including additive-linear and multiplicative isoelastic models (Petruzzi and Dada 1999). Attraction models, such as power and logit, are the most popular market share models and are becoming increasingly so in revenue management (Agrawal and Ferguson 2007, Phillips 2005).

Not all demand models fit the form (6) studied in this paper. For example, the Poisson model with price-dependent demand rate \(\lambda(p)\), which is commonly used in revenue management (Feng and Xiao 2006, Gallego and van Ryzin 1994) does not fit the \(d(p,Z)\) form. However, its normal approximation \(D(p) = \lambda(p) + \sqrt{\lambda(p)}Z\), with \(Z \sim N(0,1)\) (hence IFR), fits the additive-multiplicative form. The first row of Table 2 indicates that \(E(p,x)\) increases in \(x\); it also increases in \(p\) if \(\lambda(p)\) is concave (or

\[\phi(z)\]
\[1 - \Phi(z)\]
\[z\phi(z)\]
\[1 - \Phi(z)\]

\[\frac{\phi(z)}{1 - \Phi(z)}\]
\[z\phi(z)\]
\[1 - \Phi(z)\]
if $p\lambda'(p)$ is decreasing).

4. Price-Sensitive Revenue Management

In this section we incorporate price-sensitive demand uncertainty into the classical RM model. We present both necessary and sufficient conditions in terms of $\mathcal{E}(p,x)$ for the optimal protection level for the high-end class to decrease in the price for this class. In particular, this holds if the LSR elasticity $\mathcal{E}(p,x)$ is increasing in price. These results provide the basis of the structural properties for coordinated pricing and revenue management described in Section 5.

Throughout this paper, the protection level and price are optimized over (positive) compact intervals $x \in [0,C]$ and $p \in [p_L,p_H]$, where $p_H$ is arbitrary and possibly infinite and where $p_L \geq \bar{p}$. We also set $p_L = \arg \max \{d(p,\Phi^{-1}(1-\bar{p}/p)) \mid p \geq \bar{p}\}$; this assumption is justified analytically at the end of this section and is illustrated with examples in Section 7.3. Our results extend for any subintervals of $P$ and $X$. To simplify notation and without loss of generality, in our structural analysis for both RM and PRM models we set $\bar{p} = 1$ so that the high-end price $p$ can be interpreted as the percentage markup over the low-end price $\bar{p}$. The revenue function (1) then simplifies to $R(p,x)$.

We first recall the following well-known characterization of the optimal protection level.

**Lemma 1.** [Littlewood 1972] $R(p,x)$ is quasi-concave in $x$ for any given $p$. The optimal protection level is $x^*(p;C) = \min\{x^*(p),C\}$, where $x^*(p)$ is the unique solution of

$$q(p,x) = 1/p.$$  

(8)

For a given price $p$ and protection level $x$, monotonicity of demand $d$ in $z$ allows us to define $z(p,x)$ uniquely such that $d(p,z(p,x)) = x$. We put $z^*(p) = z(p,x^*(p))$ and, more generally, $f^*(p) = f(p,x)|_{x=x^*(p)}$, the evaluation of any generic function $f$ along the (unique) optimal solution path.

The optimality condition (8) can now be written as $P(Z \geq z^*(p)) = 1/p$ and implies that

$$\frac{\partial z^*(p)}{\partial p} = \frac{1}{p^2 \phi^*(p)} \geq 0;$$

(9)

This is a slight abuse of notation; however, the generic argument of $f^*$ makes the evaluation path unambiguous.
that is, under the optimal allocation policy, \( z^*(p) \) is increasing in \( p \). The behavior of \( z^*(p) \) with respect to \( p \) underscores the generally ambiguous nature of the relationship between \( x^*(p) \) and \( p \). Indeed, by definition, \( x^*(p) = d(p, z^*(p)) \) and so
\[
\frac{\partial x^*(p)}{\partial p} = d_p(p, z^*(p)) + \frac{\partial z^*(p)}{\partial p} d_z(p, z^*(p)).
\]
Since \( d_p \leq 0 \) and \( d_z \geq 0 \), the direction of change in \( x^*(p) \) with respect to \( p \) is not clear.

The next proposition establishes a necessary and sufficient condition for the optimal protection level to decrease in price. It also provides alternative sufficient conditions that require price-increasing LSR elasticity—either for each protection level or along the optimal allocation path \( x^*(p) \).

**Proposition 2.** \( x^*(p) \) is decreasing in \( p \) if and only if \( \mathcal{E}^*(p) \geq 1 \). Moreover, the following alternative conditions are sufficient for \( x^*(p) \) to be decreasing in \( p \): (a) \( \mathcal{E}^*(p) \) is increasing in \( p \); (b) \( \mathcal{E}(p, x) \) is increasing in \( p \) for all \( x \).

The relationship between the optimal protection level and the high-end price is determined by two effects that typically are opposed. On the one hand, a price raise increases the marginal return from protecting more capacity for this class, suggesting higher protection levels. On the other hand, increasing (high-end) prices implies a lower rate of lost sales (due to decreased demand) and hence a decrease in protection levels. Whichever effect dominates will determine the direction of change in \( x^*(p) \). When the rate of lost sales is elastic with respect to changes in price (along the optimal path; i.e., when \( \mathcal{E}^*(p) \geq 1 \)), the decrease in the rate of lost sales dominates the increase in marginal return and so leads to lower protection levels. In contrast, when demand is not a function of price, price changes have no impact on the rate of lost sales; hence the optimal protection level increases in \( p \).

The proof of Proposition 2 relies on the lower bound \( p_L \) on price defined at the beginning of this section. We argue that this bound is necessary for the result. Consider a maximal domain \( P = [p_L, p_H] \) for \( p_L \geq 1 \), where \( x^*(p) \) is monotone. By Proposition 2, this implies that \( \mathcal{E}^*(p) \geq 1 \) for all \( p \in P \). So by continuity, either \( p_L = \bar{p} = 1 \) or \( \mathcal{E}^*(p_L) = 1 \); otherwise \( p_L \) could be decreased, which would contradict the maximality of \( P \). This is precisely what the definition of \( p_L \) ensures, as follows.
Lemma 2. \( E^*(p_L) = 1 \) or \( p_L = \bar{p} = 1 \).

To illustrate the conditions presented in this section, Table 3 provides closed-form expressions for \( E(p,x) \), \( E^*(p) \), and \( p_L \) for the additive-linear and the multiplicative isoelastic demand models with uniform \((0, 1)\) or mean-1 exponential risk \( Z \); both distributions are IFR. It is easy to verify that \( E(p,x) \) is increasing and that \( E^*(p) \geq 1 \) if and only if \( p \geq p_L \). Moreover, if \( b > 1 \) (i.e., if demand is sufficiently price sensitive), then \( p_L = \bar{p} = 1 \) for additive models and so the lower bound \( p_L \) is unrestrictive.

### 5. Coordination with Resource Substitution

In this section we investigate the PRM model, which optimizes expected revenue \( R(p,x) \) as a function of high-end price \( p \) and allocation \( x \). This model corresponds to a setting in which price and availability decisions for the high-end segment are managed jointly (e.g., in a centralized environment) after the low-end price has been set.

In many revenue management settings, such as concerts and sporting events, low-end prices are maintained for brand image and for historical, fairness, or social considerations while high-end prices are actively managed. In other settings—such as airlines, hotels, car rentals, and advertising—the low-end market is highly competitive and with little degree of pricing power relative to the high-end segment (Zhang and Kallesen 2008). In fact, the first revenue management initiative, the American Airlines “Ultimate Super Saver” program, was purposely designed to conditionally match low-fare competitor People Express on the low-end segment while reserving capacity for higher-margin sales. Major airlines continue to offer low-fare products on a limited basis in order to compete against low-cost airlines such as Southwest, Ryanair, and EasyJet. In the high-end market, however, airline price dispersion is extremely high (up to 700%, Donofrio 2002) and competition less stark, suggesting that price is an important profit lever. Motivated by these

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( a - bp + Z )</th>
<th>( ap^{-b}Z )</th>
<th>( a - bp + Z )</th>
<th>( ap^{-b}Z )</th>
<th>( p_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>( \frac{bp}{1-x+a-bp} )</td>
<td>( \frac{b}{ap^{-b}} )</td>
<td>( b(p - 1) )</td>
<td>( \max \left( 1, \frac{1}{b} \right) )</td>
<td>( (1 + \frac{1}{b}) )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( \frac{bp}{b} )</td>
<td>( \frac{b}{ap^{-b}} )</td>
<td>( bp )</td>
<td>( \max(1, \frac{1}{b}) )</td>
<td>( e^{1/b} )</td>
</tr>
</tbody>
</table>
examples, in this section we focus on optimizing allocation and pricing decisions for the high-end segment.

We begin by presenting alternative conditions for the PRM model to admit a unique optimal price and allocation solution \((p^{**}, x^{**})\). A sufficient condition is concavity of the expected revenue function in price along the optimal protection level path, \(R^*(p) = R(p, x^*(p))\). This is guaranteed by an LSR elasticity bound of \(1/2\) along the optimal allocation path \(x^*(p)\) or, alternatively, by price monotonicity of LSR elasticity either everywhere or along the optimal allocation path \(x^*(p)\).

**Proposition 3.** The pricing and revenue management (PRM) model has a unique optimal price–allocation solution \((p^{**}, x^{**})\) if one of the following conditions holds:

(a) \(\mathcal{E}^*(p) \geq 1/2\) for all \(p\)—in particular, \(\mathcal{E}(p, x) \geq 1/2\) for all \(p, x\); or

(b) \(\mathcal{E}^*(p)\) is increasing in \(p\); or

(c) \(\mathcal{E}(p, x)\) is increasing in \(p\) for all \(x\).

Moreover, the revenue corresponding to the optimal protection level, \(R^*(p)\) is concave in \(p\).

This result allows us to solve PRM as a one-dimensional concave optimization problem. In some cases, the global, respectively pathwise lower bounds of \(1/2\) are not only sufficient but also necessary for concavity of the revenue function. For example, if \(d\) is linear in \(p\) (i.e., if \(d(p, z) = \delta(z)p + \gamma(z)\)), then \(\mathcal{E} \geq 1/2\) is necessary and sufficient for the joint concavity of \(R(p, x)\) and \(\mathcal{E}^* \geq 1/2\) is necessary and sufficient for the concavity of \(R^*(p)\). Therefore, no weaker constant bound can be expected to hold for all demand functions.

**5.1. Sensitivity Results**

This section presents sensitivity results for the optimal revenue, high-end price, and protection level obtained from the PRM model. We first state some preliminary results.

**Lemma 3.** The revenue function \(R(p, x)\) is concave in \(p\).

This lemma allows us to uniquely define the optimal high-end price corresponding to a given protection level \(x\), \(p^*(x; C) = \arg \max_p R(p, x; C)\), as well as to investigate its sensitivity with respect to changes in allocation \(x\) and capacity \(C\). These results require a uniform lower bound of 1 on the
LSR elasticity along the optimal price path, $E^*(x) \geq 1$, a condition that is not directly implied by monotone elasticity but can be translated into bounds on the protection level.

**Proposition 4.** (a) The optimal high-end price $p^*(x;C)$ is decreasing in $x$ if and only if $E^*(x) \geq 1$.

(b) If $E^*(x) \geq 1$ and $E(p,x)$ is increasing in $x$, then $p^*(x;C)$ decreases with capacity $C$.

The first part provides a counterpart to Proposition 2, which characterized monotonicity of the optimal protection level $x^*(p)$ in a sequential revenue management process. In such a setting, Littlewood’s rule (8) implies that, for a given price $p$, the optimal protection level $x^*(p;C)$ is independent of (or equal to) capacity. However, this no longer holds when price and allocation decisions are made jointly. Our next result characterizes the effect of capacity on the optimal coordinated price–allocation solution.

**Proposition 5.** Assume that $E(p,x)$ is increasing in $p$ and $x$.

(a) The optimal high-end price $p^{**}(C)$ for the PRM model decreases with capacity $C$.

(b) The optimal protection level $x^{**}(C)$ for the PRM model increases with capacity $C$.

We conclude this section by studying the sensitivity of the optimal revenue obtained from the PRM model, $R^*(C) = R(p^{**},x^{**};C)$, with respect to capacity.

**Proposition 6.** The optimal revenue, $R^*(C)$, from the PRM model increases with capacity $C$; whereas the optimal revenue per unit of capacity, $R^*(C)/C$, decreases with $C$.

In sum, we have shown that firms with larger capacity should expect more revenue but lower revenue rates—for example, lower RAS (revenue per available seat) for airlines and lower REVPAR (revenue per available room) for hotels. Therefore, such firms should set higher protection levels and lower high-end prices if LSR elasticity is increasing in price and quantity.

### 5.2. Extension: Substitution Effects

The model described so far implicitly assumes that the market is perfectly segmented into low- and high-fare customers. Traditionally, airlines have achieved this segmentation by designing product
fences (restrictions) such as booking more than 14 days prior to departure or staying over a Saturday night. These restrictions allowed airlines to charge prices up to 7 times higher for the greater flexibility offered (Donofrio 2002). However, in other practical settings (e.g., event ticketing) where perfect segmentation is more difficult to achieve, firms offer comparable products and consumers make choices based on price and product characteristics.

This section captures such substitution effects by modeling demand for each product as a function of the other product’s price. Keeping the low-end price fixed (\( \bar{p} = 1 \), without loss of generality), we model low-fare demand as \( \bar{D}(p) = \bar{d}(p, \bar{Z}) \), where \( \bar{Z} \) is independent of \( Z \) and \( \bar{d}(p, z) \) is increasing in \( p \); this captures the substitution effect. Here the joint price–allocation problem is formulated as

\[
\max_{p,x} R(p, x) = \mathbb{E} \left[ \min \left( \bar{D}(p), C - x \right) \right] + \mathbb{E} \left[ r \left( p, \max \left( x, C - \bar{D}(p) \right) \right) \right],
\]

where the revenue \( r(p, y) \) from the high-end class is given by (2).

**Proposition 7.** Assume that \( \bar{d}_{pp} \leq 0 \). Then (10) has a unique price–allocation solution if either of the following conditions holds: (a) \( \mathcal{E}(p, x) \) is increasing in \( p \); or (b) \( \mathcal{E}^*(p) \) is increasing in \( p \).

The additional assumption of diminishing marginal impact of substitute prices on low-fare demand holds, for example, for additive-linear demand models \( D(p) = Z - b_1 p \) and \( \bar{D}(p) = \bar{Z} + b_2 p \) as well as for isoelastic multiplicative demand models \( D(p) = p^{-b_1} Z \) and \( \bar{D}(p) = p^{b_2} \bar{Z} \), where \( b_i \geq 0 \) for \( i = 1, 2 \). Table 2 indicates that for these models, \( \mathcal{E}(p, x) \) is increasing in \( p \) if \( Z \) is IGFR.

### 6. Coordination without Resource Substitution

In this section we investigate the partitioned allocation (PA) model where capacity is divided into blocks that can be sold only to a designated market segment. In contrast with RM and PRM models, this model jointly optimizes prices and allocations for both classes but allows no resource substitution; that is, capacity that is not utilized by one class is not made available to other (higher-price) classes.

The PA model can be relevant, for instance, when deciding prices and seat categories for concerts and sport events or for establishing aircraft capacity dedicated for different classes of seating (i.e.,
business versus coach) and the corresponding prices. Because of its tractability, this model is also useful as an approximation for more complex revenue management models. For example, Belobaba (1987) uses this problem as a base case for the RM model (with resource substitution) in an airline context, and Bertsimas and de Boer (2005) use an additive-multiplicative version as an approximation for a multiperiod problem.

The objective of the PA model given in Section 2.1 can be written as

$$V(p, \bar{p}, x) = r(p, x) + \bar{r}(\bar{p}, x),$$

(11)

where $r(p, x) = pE[\min(D(p), x)]$ and $\bar{r}(\bar{p}, x) = \bar{p}E[\min(\bar{D}(\bar{p}), C - x)]$ denote the expected revenue for each product. The stochastic price-dependent demands for each product class, $D(p) = d(p, Z)$ and $\bar{D}(\bar{p}) = \bar{d}(\bar{p}, \bar{Z})$, follow the specifications of Section 3. We make no assumptions regarding dependence of the risk variables $Z$ and $\bar{Z}$ and thus allow correlated demands for the two products.

We first provide conditions for the PA model to admit a unique solution and then compare it with the PRM model solution.

The next result shows that the PA model has a unique solution and can be solved efficiently if the LSR elasticities $\mathcal{E}(p, x)$ and $\bar{\mathcal{E}}(\bar{p}, x)$ for the respective demand classes are increasing in $x$. These conditions also ensure monotonicity of the optimal price for each product in its own allocation, paralleling our previous results for RM models.

**Proposition 8.** Assume that $\mathcal{E}(p, x)$ and $\bar{\mathcal{E}}(\bar{p}, x)$ are increasing in $x$ for all $p$ and for all $\bar{p}$, respectively.

(a) Under model PA, the optimal price for each product (keeping all other variables constant) is decreasing in its own allocation and is independent of the other product’s price.

(b) The PA model has a unique optimal solution $(x^{PA}, p^{PA}, \bar{p}^{PA})$, which satisfies

$$\int_0^x q(p, v)(1 - \mathcal{E}(p, v)) \, dv = 0, \quad \int_0^{C - x} \bar{q}(\bar{p}, v)(1 - \bar{\mathcal{E}}(\bar{p}, v)) \, dv = 0,$$

(12)

For the low-fare demand, $\bar{D}(\bar{p}), \bar{\mathcal{E}}(\bar{p}, x) = -\bar{p}\bar{q}(\bar{p}, x)/\bar{q}(\bar{p}, x)$. 

\[\text{For the low-fare demand, } \bar{D}(\bar{p}), \bar{\mathcal{E}}(\bar{p}, x) = -\bar{p}\bar{q}(\bar{p}, x)/\bar{q}(\bar{p}, x).]
and the marginal revenue condition

\[ pq(p, x) = \bar{p}q(\bar{p}, x). \]  

(13)

We now provide local comparisons of the price and allocation decisions produced by the PRM and PA models. For the same price levels, we show that the PA model allocates more capacity to the high-fare class than the protection level set by the PRM model. This is because the PA model limits sales to the high-fare class whereas the PRM model does not. Moreover, for the same allocation and low-end price, the high-end price set by the PA model exceeds that set by PRM provided that LSR elasticity is increasing in \( x \). Hence resource substitution leads to lower prices for the high-end segment. These results are summarized as follows.

**Proposition 9.** (a) The PA model allocates a greater portion of capacity to the high-fare class than the protection level set by the PRM model for the same price levels \( p \) and \( \bar{p} \).

(b) Assume that \( \mathcal{E}(p, x) \) is increasing in \( x \). Then the optimal high-end price set by the PA model is higher than the optimal high-end price set by the PRM model for the same allocation \( x \) and low-end price \( \bar{p} \).

We reiterate that our main results for the PA model rely on LSR elasticity \( \mathcal{E}(p, x) \) increasing in \( x \). Table 2 suggests that, for commonly used demand models, this condition is milder than (and implied by) the requirement that \( \mathcal{E}(p, x) \) be increasing in \( p \), which guarantees the results for models with resource substitution (RM and PRM).

7. **Performance Assessment: Numerical Insights**

In this section we provide numerical analysis on industry data in order to evaluate the performance of various hierarchical and coordinated models for pricing and revenue management, as summarized in Table 1. We quantify the benefits of coordinating pricing and allocation decisions, of allowing for resource substitution, and of incorporating demand uncertainty in pricing decisions. We evaluate how these benefits vary with respect to capacity levels and demand parameters such as location, scale, and variability. We first describe the data set and then use it to fit a stochastic, price-sensitive demand model, as detailed in Section 3.
7.1. Demand Model for Car Rental Bookings

The booking data set that we analyze here comes from Avis car rental. The data consist of car rentals at major airports in four European countries (France, Germany, Italy, and Spain) between 1 January 2008 and 31 March 2008. Avis rents cars to both corporate and individual customers. We consider in the analysis only rentals by individual customers, since corporate rentals are less price sensitive and have different demand patterns. The data set contains the length of rental and the price (in euros) per rental day quoted at the time of booking. The cars are categorized into different groups depending on vehicle characteristics, and different price ranges are applicable to each group. We analyze the data in the high-price and low-price groups separately.

Figure 1 plots the aggregated demand in all countries versus price per day for the high-price vehicle group. As expected, price has a negative impact on demand. The plots of demand versus price per day when data is stratified by country (not reported here) all show similar patterns, implying that the relationship between price and demand has similar characteristics in the stratified as in the aggregate data. The pattern in Figure 1 suggests that an additive or multiplicative model would be appropriate for capturing the relationship between price and demand. We shall thus fit the additive-linear model $D(p) = a - bp + Z$ and the multiplicative demand model $D(p) = ap^{-b}Z$.

We establish the most appropriate functional form for the relationship between price and demand by comparing the adjusted $R^2$ for the additive and multiplicative demand models. Table 4 reports
the adjusted $R^2$ for both models, as well as the estimated parameters, with data stratified by country. The additive specification generally provides a better fit than the multiplicative one, although the difference is marginal in some cases. The parameter estimates vary slightly across countries. The standard errors of the parameters (reported in parentheses) imply that the estimated negative impact of price on demand is highly statistically significant, which is consistent with prior expectations and with Figure 1.

As a second step, we determine the best-fitting distribution for the random component $Z$ under the additive demand model by analyzing the residuals from the least-squares fit. Figure 2 gives the probability plots of the residuals under four distributional assumptions—normal, exponential, gamma, and uniform. It is apparent from Figure 2 that the normal distribution is the most appropriate parametric model for $Z$. Maximum likelihood estimation leads to a mean of 0 and standard deviation $\sigma = 2.45$ for the normally distributed $Z$.

A similar analysis of the low-price car group reveals that the average rental price per day ranges between €28.45 for Italy and €33.64 for France; the demand risk $\bar{Z}$ from the additive model has a normal distribution with mean 0 and standard deviation $\bar{\sigma} = 11.43$.

This analysis motivates us to focus the presentation of our numerical experiments on an additive-linear demand model: $D(p) = a - bp + Z$ and $\bar{D}(p) = \bar{a} - \bar{b}p + \bar{Z}$; we assume that demand risks $Z$ and $\bar{Z}$ have independent normal distributions with mean 0 and standard deviations $\sigma$ and $\bar{\sigma}$,
Figure 2  Probability plots of $Z$ for high-price car rentals and the additive demand model under four distributional assumptions (normal, exponential, gamma, and uniform); the normal distribution gives the best fit.

respectively. Under this model, LSR elasticity is increasing in both $p$ and $x$ (see Table 2) and $\pi(p, z)$ is strictly concave in $p$. For this data set, the lower bounds on price $p_L$ introduced in Section 4 are practically unconstraining, as illustrated in Section 7.3. According to the results reported in previous sections, this ensures that all models summarized in Table 1 can be solved efficiently and admit a unique solution. We find the best JPRM solution via a search policy; preliminary analysis suggests that the general demand conditions used in this paper may not be sufficient to optimize prices and the protection level simultaneously.

For purposes of illustration, we present numerical results for specific demand models and parameters. Our extensive numerical experiments with a wide range of parameters suggest that these insights are robust. Robustness tests are reported in Section 7.3, where we also assess the impact on
optimal revenue of assuming different distributions (normal, gamma, or uniform) for the demand risks $Z$ and $\bar{Z}$.

7.2. Numerical Results: Benefits Assessment

Proposition 1 provided a partial ordering of various pricing and revenue management policies introduced in Section 2. This section uses numerical experiments to quantify the magnitude of performance gaps between various policies and thereby to assess the benefits of coordination and of modeling price-sensitive demand uncertainty and resource substitution.

In order to evaluate the benefits of modeling demand uncertainty, we consider as a benchmark a deterministic, coordinated pricing and revenue management policy that uses average demand data to set prices based on the DM model and then sets the protection level $x$ for high-end demand equal to the expected demand at that price. Specifically:

\[(DMX)\quad \mathcal{R}[DMX] = R(p^{DM}, p^{DM}, x), \quad \text{where} \quad x = \mathbb{E}[D(p^{DM})].\]

This implies that $\mathcal{R}[DMX] \leq \mathcal{R}[RM-DM]$. To simplify the presentation, throughout this section we report the increases in revenue obtained from each policy relative to the DMX benchmark policy. The relative improvement of policy $X$ over $Y$ is denoted $\Delta(X, Y) = (\mathcal{R}[X] - \mathcal{R}[Y]) / \mathcal{R}[Y]$.

Next we investigate how the improvements in revenue are affected by load in the market, demand slope, and variability.

7.2.1. Load in the Market. In revenue management, load is typically defined ex post as the ratio of expected demand to capacity. Because expected demand is a function of selling prices that are not determined a priori, we define here ex ante load as $L = (\mathbb{E}[D(p^o)] + \mathbb{E}[\bar{D}(\bar{p}^o)]) / C = \frac{1}{2}(a + \bar{a}) / C$; this definition is based on expected demand at the (mid-range) unconstrained revenue-optimizing prices $p^o = a/2b$ and $\bar{p}^o = \bar{a}/2\bar{b}$.

The results in this section are obtained by varying the capacity $C$, which leads to ex ante load factors $L$ varying between 0.8 and 5. We emphasize that $L$ is different from (and typically much larger than) the actual load that materializes in the market, which depends on demand realizations.
and on the firm’s price and allocation policies. In fact, as long as \( L \geq 1 \), the expected load at DM prices is unity: \( L(DM) = \frac{E[D(p^DM)] + E[D(\bar{p}^DM)]}{C} = 1 \). The upper limit \( L = 5 \) is set so that both markets are still served under the fluid DM model. This suggests that pricing and revenue management decisions are particularly relevant in this range.

The demand parameters for the high- and low-demand classes are set to \( a = 30 \) and \( b = 0.25 \), respectively, \( \bar{a} = 80 \) and \( \bar{b} = 2.00 \). The demand risks \( Z \) and \( \bar{Z} \) have independent normal distributions with mean zero and standard deviations \( \sigma = 2 \) and \( \bar{\sigma} = 12 \), respectively. This choice of parameters is motivated by the analysis of the French bookings of Avis car rentals in the previous section because, among all countries, the additive demand model has the highest explanatory power for the French market data.

Figure 3 plots on the right-hand side the percentage increase in revenues relative to the DMX policy, for each of the policies under consideration, as a function of capacity reflected in the ex ante load factor \( L \). In order to reflect the additional value of protecting capacity, in the left-hand side we illustrate the improvements relative to a more naive pricing policy without revenue management, P-DM, which sets DM prices but not protection level:

\[
(P-DM) \quad R[P-DM] = R(\bar{p}^DM, p^DM, x = 0).
\]

The PRM-PA and RM-PA heuristics perform similarly and are almost equal to the upper bound.
Table 5  Relative policy benefits (percentages) for varying levels of ex ante load

<table>
<thead>
<tr>
<th>L</th>
<th>1</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ( PRM-DM, RM-DM)</td>
<td>0.40</td>
<td>0.01</td>
<td>0.30</td>
<td>0.75</td>
<td>1.24</td>
<td>1.77</td>
<td>2.28</td>
<td>2.76</td>
<td>2.91</td>
</tr>
<tr>
<td>Δ( RM-PA, RM-DM)</td>
<td>2.19</td>
<td>0.22</td>
<td>2.50</td>
<td>5.78</td>
<td>9.47</td>
<td>13.43</td>
<td>17.66</td>
<td>22.37</td>
<td>25.88</td>
</tr>
<tr>
<td>Δ( RM-PA, PA)</td>
<td>0.59</td>
<td>0.02</td>
<td>0.58</td>
<td>0.45</td>
<td>0.32</td>
<td>0.19</td>
<td>0.07</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Δ( JPRM, RM-PA)</td>
<td>0.00</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

of JPRM. They are followed closely by PA, whereas PRM-DM and RM-DM show smaller revenue improvements. The absolute revenue per capacity (which, for conciseness, we do not report here) increases with load for all policies, in line with the results from Proposition 6.

Pricing and revenue management decisions are relevant when capacity is constraining ($L > 1$) yet large enough to serve both segments ($L \leq 5$ in our case). Under this scenario, the overall revenue gains from capturing demand uncertainty exceed 10%; this figure is higher for tighter capacity (albeit nonmonotonically). Relative to a hierarchical model with deterministic prices (RM-DM), the value of full coordination (JPRM) typically exceeds 2% and can reach up to 25%. Relative to RM-PA, however, the value of coordination is much lower (less than 0.03%; see Table 5), suggesting that most revenue benefits actually come from adjusting prices to reflect price-sensitive demand uncertainty.

7.2.2. Demand Slope. With an additive model $D(p) = a - bp + Z$, price influences the mean of the demand distribution without affecting the variability. Here we investigate the impact of changes in the high-end and low-end demand slopes on the relative performance of these policies.

First, we vary the market parameter $b$ between 0.10 and 0.35 while keeping the ex ante load factor $L$ and the other demand parameters constant in order to control for the effects of capacity or demand variability in the market ($a = 30, \bar{a} = 80, \bar{b} = 2, \sigma = 2, \bar{\sigma} = 12, L = 2$). Then we fix $b = 0.2$ and vary the low-end slope $\bar{b}$ between 0.7 and 2.2 while keeping the other parameters constant.

Figure 4 plots the percentage increase in optimal revenues over the benchmark DMX policy, for each of the policies that we consider, as functions of the demand slopes $b$ and $\bar{b}$. Although the absolute expected revenues for all policies (not illustrated here) decrease with both demand slopes, the trend in improvements relative to DMX depends on whether PA or DM policies are used for
pricing. Figure 4 confirms that the PRM-PA and RM-PA heuristics perform best and are very close to the upper bound of JPRM. Their difference relative to PA shows that the benefit of allowing for resource substitution is significant (on the order of 1%-2%; see Table 6). The value of coordination relative to a hierarchical model with DM prices is on the order of 1%-3%.

### 7.2.3. Demand Variability.

We next investigate the improvement in optimal revenue with respect to high-end and low-end demand variability as measured by the coefficients of variation (CV). The results are obtained by separately varying the values of the standard deviations $\sigma$ and $\bar{\sigma}$ of $Z$ and $\bar{Z}$ so that the coefficients of variation of the base demand $D(p^o)$ and $\bar{D}(\bar{p}^o)$ ($\text{CV} = a/2\sigma$, $\text{CV} = \bar{a}/2\bar{\sigma}$) range between 0.1 and 0.6. This corresponds to a range of (1.5, 9.0) for $\sigma$ and of (4.0, 24.0) for $\bar{\sigma}$. Consistently with the values in the rest of this section, when $\sigma$ varies we fix $\bar{\sigma} = 12$ and, conversely, when $\bar{\sigma}$ varies we fix $\sigma = 2$. The other parameters of the demand models are held constant, as in Section 7.2.2 ($a = 30$, $b = 0.25$, $\bar{a} = 80$, $\bar{b} = 2.0$), and the ex ante load is $L = 2$. 

---

**Figure 4  Revenue versus high-end and low-end demand slopes**

**Table 6  Relative policy benefits (percentages) for varying high-end and low-end demand slopes**

<table>
<thead>
<tr>
<th></th>
<th>$b$</th>
<th>$\bar{b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$(PRM-DM, RM-DM)</td>
<td>0.06 0.13 0.21 0.30 0.40 0.49</td>
<td>0.89 0.60 0.41 0.30 0.22 0.13</td>
</tr>
<tr>
<td>$\Delta$(RM-PA, RM-DM)</td>
<td>1.38 1.87 2.24 2.50 2.67 2.78</td>
<td>2.72 2.84 2.69 2.50 2.31 2.13</td>
</tr>
<tr>
<td>$\Delta$(RM-PA, PA)</td>
<td>1.13 1.43 1.57 1.60 1.53 1.38</td>
<td>0.19 1.18 1.51 1.61 1.60 1.60</td>
</tr>
<tr>
<td>$\Delta$(JPRM, RM-PA)</td>
<td>0.01 0.01 0.02 0.03 0.05 0.05</td>
<td>0.09 0.06 0.03 0.03 0.02 0.03</td>
</tr>
</tbody>
</table>
Figure 5 plots the relative increase in revenues over the benchmark DMX as a function of the CVs of the high-end and low-end demand. The percentage improvement over DMX decreases as high-end demand becomes more variable (i.e., as the system becomes more difficult to control); low-end demand variability has the opposite effect. As before, the PRM-PA and RM-PA perform better than the other models and are close to the upper bound of JPRM. When high-end demand is more uncertain, PA underperforms RM-DM, which illustrates the importance of capturing resource substitution. The relative benefit of capturing price-sensitive demand uncertainty is again prevalent: it increases as low-end demand becomes more variable and high-end demand becomes less variable (i.e., as the differentiation between segments increases). We emphasize that these are relative measures; the absolute expected revenues from all policies (not reported here) decrease with variability in both demands.
Table 8  Policy performance ordering and percentage revenue gaps based on numerical analysis

<table>
<thead>
<tr>
<th></th>
<th>DMX</th>
<th>RM-DM</th>
<th>PRM-DM</th>
<th>RM-PA</th>
<th>PRM-PA</th>
<th>JPRM</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2%–20%]</td>
<td>[0%–0.5%]</td>
<td>[1%–9.5%]</td>
<td>[0%–0.01%]</td>
<td>[0%–0.05%]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.3. Summary and Robustness Tests

Table 8 summarizes our main findings from the numerical experiments; it also complements our theoretical bounds in Proposition 1 by ordering the performance of the policies described in Section 2.2. Our numerical analysis suggests that the value of an integrated pricing and revenue management system is significant and is enhanced by tighter capacity. The greatest benefits come from two separate factors: (i) incorporating uncertainty (i.e., using PA rather than DM) in pricing; and (ii) setting booking limits with resource substitution (i.e., using RM or PRM). The hierarchical processes PRM-PA and RM-PA closely match the performance of a fully coordinated JPRM system.

These findings are replicated in experiments where for all policies the low-end price is exogenously fixed to values in a range suggested by the data. As remarked previously, fixing low prices is motivated by image, competition, and/or other factors. In this case, PRM models outperform hierarchical RM-based models and hierarchical models using PA prices outperform those using DM prices; performance gaps are of the same order of magnitude as those presented in Table 8.

Extensive numerical simulations with a wide range of parameters and distribution classes indicate that these results are robust. In particular: starting with parameter values inspired by the results from the Avis data, we investigated ranges of (10, 50) for \( a \), (0.02, 0.80) for \( b \), (40, 120) for \( \bar{a} \), (0.40, 3.00) for \( \bar{b} \), (0.4, 5.0) for \( L \), (0.5, 11.0) for \( \sigma \), and (2.00, 20.00) for \( \bar{\sigma} \).

To conclude our numerical experiments, we assess the sensitivity of the optimal revenue to assumptions about the distribution of demand risk. Motivated by our analysis of car rental data, the numerical experiments have so far focused on the case when the demand risks \( Z \) and \( \bar{Z} \) have normal distributions. We turn now to study the impact on optimal revenue of incorrectly assuming a certain distribution (i.e., when a different distribution fits the data better). How does the potential revenue loss compare with the revenue impact of using different pricing policies with the correct
demand distribution? Is the robustness of the demand assumption more or less important than the choice of pricing policy?

To answer these questions, we focus on three distributions (normal, gamma, and uniform) for the demand risks $Z$ and $\bar{Z}$. For consistency, we set the demand model parameters to values similar to those used in the rest of this section; thus, we put $a = 30$, $\bar{a} = 80$, $b = 0.15$, $\bar{b} = 2.00$, and $L = 2$. We then generate three data sets of 100 demand realizations each while assuming normal, gamma, or uniform distributions for the demand risks $Z$ and $\bar{Z}$. The normal distributions have mean 0 and standard deviations $\sigma = 2.00$ and $\bar{\sigma} = 12.00$; the gamma distributions have scale parameters $c = 1.30$ and $\bar{c} = 2.30$ and shape parameters $d = 1.30$ and $\bar{d} = 2.30$; and the uniform distributions are defined on the intervals $(-h, h)$ and $(-\bar{h}, \bar{h})$, where $h = 2.50$ and $\bar{h} = 6.00$.

For each of the three demand data sets and each pricing policy, we compute the optimal revenues separately under the assumption that the demand risks have normal, gamma, or uniform distributions. Table 9 gives the optimal revenues for all policies under all nine combinations of true and assumed demand models. In the six cases when a demand model other than the true one is assumed for pricing, the table also reports (in parentheses) the robustness “gap”. This is computed as the percentage difference between the expected revenues under the assumed demand model and the true one for a given pricing policy.

The robustness gaps are small unless the uniform distribution is incorrectly assumed for the demand risks. Assuming a normal distribution when the true model is not normal has a small negative impact on optimal revenue across all pricing policies; in most cases, the robustness gaps for the normal distribution are less than 1%. This suggests that, if a normal distribution is assumed for the demand risks, then the potential violation of this assumption is less critical to optimal revenues than is the choice of pricing policy.

Finally, we conclude by evaluating the lower bound $p_L$ on high-end prices (introduced in Section 4) for the demand models presented here and for a wider range of low-end prices $\bar{p}$ than would be suggested by the data (cf. Section 7.1) and models; recall that the linear model implies $\bar{p} \leq \bar{a}/\bar{b} = 40$. The values for $p_L$ reported in Table 10 are consistently close to the low-end prices $\bar{p}$. 
Table 9  Robustness tests: Revenues under different true and assumed demand models; percentage robustness gaps in parentheses

<table>
<thead>
<tr>
<th>True model</th>
<th>Fitted model</th>
<th>JPRM</th>
<th>PRM-PA</th>
<th>RM-PA</th>
<th>PA</th>
<th>PRM-DM</th>
<th>RM-DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>Normal</td>
<td>1829.90</td>
<td>1829.81</td>
<td>1829.81</td>
<td>1824.90</td>
<td>1806.05</td>
<td>1804.21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>1810.03</td>
<td>1809.13</td>
<td>1809.12</td>
<td>1801.22</td>
<td>1771.08</td>
<td>1767.38</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.09)</td>
<td>(1.13)</td>
<td>(1.13)</td>
<td>(1.30)</td>
<td>(1.94)</td>
<td>(2.04)</td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>1763.41</td>
<td>1732.69</td>
<td>1726.85</td>
<td>1752.74</td>
<td>1413.43</td>
<td>1206.79</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.63)</td>
<td>(5.31)</td>
<td>(5.63)</td>
<td>(3.95)</td>
<td>(21.74)</td>
<td>(33.11)</td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>2042.93</td>
<td>2042.48</td>
<td>2042.38</td>
<td>2038.20</td>
<td>2040.00</td>
<td>2038.86</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.30)</td>
<td>(0.27)</td>
<td>(0.28)</td>
<td>(0.22)</td>
<td>(0.14)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>2049.00</td>
<td>2047.91</td>
<td>2047.61</td>
<td>2044.00</td>
<td>2044.50</td>
<td>2041.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>1895.30</td>
<td>1894.74</td>
<td>1894.70</td>
<td>1887.89</td>
<td>1838.10</td>
<td>1820.99</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(7.50)</td>
<td>(7.48)</td>
<td>(7.47)</td>
<td>(7.64)</td>
<td>(10.10)</td>
<td>(10.81)</td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>1857.17</td>
<td>1857.15</td>
<td>1857.10</td>
<td>1853.80</td>
<td>1844.78</td>
<td>1836.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.58)</td>
<td>(0.56)</td>
<td>(0.55)</td>
<td>(0.53)</td>
<td>(1.08)</td>
<td>(1.44)</td>
<td></td>
</tr>
<tr>
<td>Gamma</td>
<td>1848.35</td>
<td>1847.25</td>
<td>1846.84</td>
<td>1842.43</td>
<td>1840.18</td>
<td>1836.87</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.05)</td>
<td>(1.09)</td>
<td>(1.10)</td>
<td>(1.14)</td>
<td>(1.32)</td>
<td>(1.45)</td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>1867.93</td>
<td>1867.57</td>
<td>1867.46</td>
<td>1863.59</td>
<td>1864.85</td>
<td>1863.84</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10  Lower bound on price $p_L$ for different demand distributions

<table>
<thead>
<tr>
<th>p</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>28</td>
<td>33</td>
<td>38</td>
<td>42</td>
<td>47</td>
<td>52</td>
<td>57</td>
<td>62</td>
<td>67</td>
</tr>
<tr>
<td>Gamma</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>45</td>
<td>50</td>
<td>55</td>
<td>60</td>
</tr>
<tr>
<td>Uniform</td>
<td>26</td>
<td>29</td>
<td>32</td>
<td>35</td>
<td>40</td>
<td>45</td>
<td>50</td>
<td>55</td>
<td>60</td>
</tr>
</tbody>
</table>

which suggests that the technical assumption $p \geq p_L$ for optimizing PRM is practically unrestricted. The largest difference for normally distributed $Z$ amounts to a markup of approximately 1.4 on the low-fare price; for $Z$ with a gamma distribution we have $p_L = \bar{p}$, which in effect imposes no additional constraints on the high-fare price.

8. Summary and Conclusions

This paper investigated the effects of characterizing stochastic price-sensitive demand and coordinating price–allocation decisions in a framework of static, two-fare class revenue management. The first part of the paper considered various pricing and revenue management models and characterized the associated demand conditions for these to admit unique solutions with natural sensitivity
properties. The second part of the paper employed these models within several different hierarchical and simultaneous revenue management processes in order to assess numerically the benefits of integrating pricing and revenue management decisions. In this section we summarize our main findings and insights along these two dimensions.

First, from a methodological perspective, we have identified a broad class of demand models for which the coordinated pricing and revenue management models PRM and PA can be solved efficiently and admit unique solutions. A unifying sufficient condition is that LSR elasticity be increasing in both \( x \) and \( p \), a condition that is satisfied by a wide variety of demand models (as summarized in Table 2). This condition also led to sensitivity results—for instance, that the optimal protection level decreases in price in sequential decision processes and that the joint optimal price–allocation solution is monotonic with respect to capacity.

Second, in terms of assessing benefits, our numerical experiments on data from the car rental industry demonstrated the value of coordinating price–allocation decisions and of capturing price-sensitive demand uncertainty and resource substitution. Specifically, we found that hierarchical models that use prices obtained from the PA model yield revenues significantly higher than those using DM prices. The benefits of coordinated versus hierarchical approaches appeared to be higher for prices that were set based on deterministic models (DM), for larger ex ante market load, and for greater differentiation between segments. Resource substitution yields considerable increases in revenue, especially when high-end demand is more uncertain. Overall, our numerical findings suggested that capturing price-sensitive demand uncertainty in a hierarchical revenue management process can have significant revenue benefits and can deliver most of the value potential of a fully coordinated process. These revenue figures may have a large impact on profitability: given the industry’s notoriously thin margins, a 1% increase in revenue could actually double profits.

Our model and results have several limitations. First, our current analytical results do not optimize the low-end price, a modeling choice motivated by the practical considerations discussed in the Introduction and in Section 5. From a technical standpoint, preliminary analysis suggests that the general demand conditions used in this paper may need to be strengthened in order for
prices and protection level to be optimized simultaneously—in other words, for the JPRM model to be well-behaved. Our numerical experiments, however, indicate that the practical benefits of such full coordination are negligible relative to a sequential policy that models demand uncertainty in pricing: the difference in performance between JPRM and RM-PA is less than 0.06%.

These results pertain to a static, two-fare class monopolistic model. Such a model is limited in that it ignores multidimensional, dynamic, and competitive aspects of the revenue management problem. Nevertheless, as suggested in Bitran and Caldentey (2003, p. 224), simple static models can serve as good sources of approximation for more realistic and more complex problems. We expect that similar demand conditions involving increasing LSR elasticity can be useful for modeling stochastic price-sensitive demand in other revenue management settings—for example, in extending the current analytical results to multiple customer classes and to dynamic and competitive frameworks. It would be interesting to explore the relative benefits of capturing various modeling features (e.g., coordination, demand uncertainty, resource substitution) as well as the robustness of our numerical insights in these contexts.

Acknowledgments

We thank James Dana, Guillermo Gallego, Guillaume Roels, Kalyan Talluri, and Garrett van Ryzin for insightful comments on an earlier version of this paper. We are grateful to the pricing group at Avis Europe for providing us with the pricing data used in the paper.

References


Appendix: Notation and Proofs

Notation. Here we review briefly some of the notation used throughout the paper, most of which is summarized in Table 11. Functions related to low-fare demand are denoted with an overline and defined with respect to $C - x$. We use $f^*(p) = f(p, x)|_{x = x^*(p)}$ to denote the evaluation of any generic function $f$ along the optimal solution path $x^*(p)$, and we use $f'_*(p) = f_x(p, x)|_{x = x^*(p)}$ to denote the derivative of $f(p, x)$ with respect to $x$ evaluated at the optimal quantity. In this notation, which is used throughout the paper, the derivative always precedes functional evaluation. We employ the standard joint expectation notation: $E[A; B] = E[A|B]P(B)$.

Proof of Proposition 1. Clearly, $\mathcal{R}[JPRM] = \max_{\bar{p}, p, x} \mathcal{R}(\bar{p}, p, x) \geq \mathcal{R}[PRM-Y] = \max_{p, x} \mathcal{R}(p^*, p, x) \geq \max_{\bar{p}^*, p^*, x} \mathcal{R}(\bar{p}^*, p^*, x) = \mathcal{R}[RM-Y]$. To show that $\mathcal{R}[JPRM] \leq \mathcal{R}[DM]$, denote the sample path revenue for a given policy and demand realization as follows:

$$\hat{R}(p, \bar{p}, x, d, \bar{d}) = \bar{p}[\min(\bar{d}, C - x)] + p[\min(d, \max(C - \bar{d}, x))]$$(14)

We use $q$ and $\bar{q}$ to denote (respectively) the number of units actually sold to high- and low-fare customers in (14), so that $\bar{q} = \min(\bar{d}, C - x)$ and $q = \min(d, \max(C - \bar{d}, x)) = \min(c, C - \bar{q})$; hence we obtain

$$\hat{R}(p, \bar{p}, x, d, \bar{d}) = \max_{q, \bar{q}} pq + \bar{p}\bar{q}$$

s.t. $q \leq d, \bar{q} \leq \bar{d}$

$q + \bar{q} \leq C$.

The right-hand side is a linear program, so $\hat{R}$ is concave in $d$ and $\bar{d}$. This allows us to apply Jensen’s inequality to show that $\mathcal{R}[JPRM] = \max_{p, p, x} E[\hat{R}(p, \bar{p}, x, D(p), \bar{D}(\bar{p}))] \leq \max_{p, \bar{p}, x} \hat{R}(p, \bar{p}, x, E[D(p)], E[\bar{D}(\bar{p})]) = \mathcal{R}[DM]$. 

relies on the following lemma.

Because $d$ from (8) and (16) we obtain

density and cdf of lost sale rate (LSR); high-fare demand survival function
low-fare stochastic demand

high-end and low-end price (usually $\bar{p}$)
capacity
lower bound on high-end price
high-fare stochastic demand function
random high-fare demand component
inverse of the deterministic demand $d(p, z) = x$

riskless profit

high-fare stochastic demand function
density and cdf of $D(p)$
density and cdf of $Z$

low-fare stochastic demand
survival function of $\bar{D}$ relative to $C - x$
density and cdf of $\bar{D}$ defined with respect to $C - x$
event that high-fare demand does not exceed $x$
event that low-fare demand does not exceed $C - x$
LSR elasticity (elasticity of the rate of lost sales)

To show $R[RM-PA] \geq R[PA] = r(p^{PA}, x^{PA}) + \bar{r}(\bar{p}^{PA}, x^{PA})$, we use that $r(p, x)$ is increasing in $x$ to argue that

$$R[PA] \leq E[r(p^{PA}, \max(C - \bar{D}(p^{PA}), x^{PA}))] + \bar{r}(\bar{p}^{PA}, x^{PA}) = R(p^{PA}, \bar{p}^{PA}, x^{PA})$$

$$\leq \max_x R(p^{PA}, \bar{p}^{PA}, x) = R[RM-PA]. \quad \square$$

**Proof of Proposition 2.** We first prove the necessary and sufficient condition. Let $f^*_p(p) = f^*_p(p, x)|_{x=x^*(p)}$, the derivative of the generic function $f(p, x)$ with respect to $x$ evaluated at the optimal quantity. (Recall that in this paper the derivative always precedes functional evaluation.) Uniqueness of $x^*(p)$ and the implicit function theorem imply $\frac{\partial x^*(p)}{\partial p} = -\frac{R_{xx}(p)}{R_{xx}(p)}$. Writing $\bar{q}(x) = 1 - \bar{F}(x) = P(\bar{D} \geq C - x)$, we calculate

$$R_s(p, x) = \bar{q}(x)[1 - p q(p, x)]$$

(15)

$$R_{xx}(p, x) = \bar{f}(x)[pq(p, x) - 1] - \bar{q}(x)p f(p, x),$$

(16)

$$R_{zp}(p, x) = \bar{q}(x)[q(p, x) + pd_p(p, z(p, x)) f(p, x)] = \bar{q}(x)q(p, x)(1 - E(p, x)).$$

(17)

From (8) and (16) we obtain $R_{xx}^*(p) = -\bar{q}^*(x)p f^*(p)$, so

$$\frac{\partial x^*(p)}{\partial p} = -\frac{R_{zp}^*(p)}{R_{xx}^*(p)} = d^*_p(p)\left[1 - \frac{1}{E^*(p)}\right].$$

(18)

Because $d(p, z)$ is decreasing in $p$, (18) is negative whenever $E^*(p) \geq 1$.

We next prove the sufficient conditions. The proof of part (a) follows from Lemma 2. The proof of part (b) relies on the following lemma.
LEMMA 4. For all \( p \geq p_L \), we have \( p_L q(p_L, x^*(p)) \geq 1 \).

**Proof.** By definition (using \( \bar{p} = 1 \)), \( p_L = \arg \max \{ d(p, \Phi^{-1}(1-1/p) \mid p \geq 1 \} = \arg \max \{ d(p, z^*(p)) \mid p \geq 1 \} = \arg \max \{ x^*(p) \mid p \geq 1 \} \). Hence, \( x^*(p) \leq x^*(p_L) \) for all \( p \geq p_L \). It follows that, for all \( p \geq p_L \), \( q(p_L, x^*(p)) \geq q(p_L, x^*(p_L)) = 1/p_L \). □

We now proceed to prove part (b) of the sufficient conditions. The marginal revenue condition (8) states that the following expression, evaluated at \( x^*(p) \), is equal to 0:

\[
pq(p, x) - 1 = \int_{p_L}^p \frac{\partial}{\partial v}(vq(v, x) - 1)dv + p_L q(p_L, x) - 1 \\
= \int_{p_L}^p q(v, x)(1 - \mathcal{E}(v, x))dv + (p_L q(p_L, x) - 1).
\]

From Lemma 4 it follows that the second term, evaluated at \( x^*(p) \), is nonnegative. Hence the first term must be nonpositive at \( x^*(p) \); that is, \( \int_{p_L}^p Q(v, x^*(p))dv \leq 0 \), where \( Q(p, x) = q(p, x)(1 - \mathcal{E}(p, x)) \). Since \( \mathcal{E}(p, x) \) increases in \( p \) and since \( q(p, x) \geq 0 \), it follows that \( Q(p, x) \) crosses 0 at most once (and from above) as \( p \) increases. Therefore \( Q(p, x^*(p)) \leq 0 \); that is, \( \mathcal{E}^*(p) \geq 1 \). The rest follows from the necessary and sufficient condition proved previously. □

**Proof of Lemma 2.** Rewriting the LSR elasticity definition (7) in terms of \( Z \), we obtain:

\[
\mathcal{E}(p, x) = -\frac{\phi(z(p, x))}{1 - \Phi(z(p, x))} \frac{p d_z(p, z(p, x))}{d_z(p, z(p, x))}.
\]

Using the critical fractile condition \( z^*(p) = \Phi^{-1}(1-1/p) \), we can now write \( \mathcal{E}(p_L, x^*(p_L)) \) as:

\[
\mathcal{E}^*(p_L) = -\frac{\phi^*(p_L)}{1 - \Phi^*(p_L)} \frac{p_L d_z^*(p_L)}{d_z^*(p_L)} = -\phi^*(p_L) \frac{p_L^2}{\Phi^{-1}(1 - 1/p_L)} = 1,
\]

where the last equality follows because, by definition, if \( p_L > 1 \) then \( p_L \) solves:

\[
p_L^2 \phi \left( \Phi^{-1} \left( 1 - \frac{1}{p_L} \right) \right) = -\frac{d_z}{d_p} \left( \frac{p_L \Phi^{-1} \left( 1 - \frac{1}{p_L} \right)}{\Phi^{-1} \left( 1 - \frac{1}{p_L} \right)} \right).
\]

**Proof of Proposition 3.** (a) To show that \( R^*(p) \) is concave in \( p \), we first show that \( r^*_p(p) \) is decreasing in \( p \) if \( \mathcal{E}^*(p) \geq 1/2 \). Let \( \Omega = \Omega(p, x) = (D(p) \leq x) \) denote the event that high-fare demand falls short of the protection level \( x \), so \( q(p, x) = 1 - F(p, x) = P(D(p) \geq x) = 1 - P(\Omega) \). We then have:

\[
r(p, x) = pE[\min(D(p), x)] = px(1 - P(\Omega)) + E[p, Z; \Omega] P(\Omega) = pxq(p, x) + E[p, Z; \Omega],
\]

where, in the standard notation, \( E[A; B] = E[A | B] P(B) \). Differentiating and using (8), we obtain:

\[
r^*_p(p) = x^*(p) q^*(p) + \mathbb{E}[\pi_p(p, Z; \Omega^*)] = \frac{x^*(p)}{p} + \mathbb{E}[\pi_p(p, Z; \Omega^*)]
\]
where \( \Omega^* = \Omega(p, x^*(p)) \). Differentiating with respect to \( p \) then yields

\[
\frac{\partial r^*_p(p)}{\partial p} = \frac{1}{p} \frac{\partial x^*(p)}{\partial p} - \frac{x^*(p)}{p^2} + \mathbb{E}[\pi_{pp}(p, Z); \Omega^*] + (d^*(p) + pd^*_p(p)) \phi^*(p) \frac{\partial z^*(p)}{\partial p}
\]

\[
= \frac{d^*_p(p)}{p} \left( 1 - \frac{1}{\mathbb{E}^*(p)} \right) - \frac{d^*(p)}{p^2} + \mathbb{E}[\pi_{pp}(p, Z); \Omega^*] + (d^*(p) + pd^*_p(p)) \phi^*(p) \frac{\partial z^*(p)}{\partial p}
\]

where the second inequality is obtained by using (18) and \( d^*(p) = d(p, z^*(p)) = x^*(p) \) and the third equality by using the expression for \( \frac{\partial z^*(p)}{\partial p} \) obtained in (9). Canceling and regrouping terms, we have

\[
\frac{\partial r^*_p(p)}{\partial p} = \frac{d^*_p(p)}{p} \left( 2 - \frac{1}{\mathbb{E}^*(p)} \right) + \mathbb{E}[\pi_{pp}(p, Z); \Omega^*]. \tag{21}
\]

This expression is negative because \( \pi \) is concave, \( d^*_p(p) \leq 0 \), and \( \mathbb{E}^*(p) \geq 1/2 \); hence \( r^*_p(p) \) is decreasing in \( p \).

We next establish concavity of \( R^*(p) \). By the envelope theorem,

\[
\frac{\partial R^*(p)}{\partial p} = R^*_p(p) = R^*(p, x)|_{x = x^*(p)} = \bar{q}^*(p)r^*_p(p) + \mathbb{E}[r_p(p, C - \bar{D}); \Omega^*],
\]

where \( \Omega(x) = (\bar{D} \leq C - x) \) denotes the event that low-fare demand does not exceed the booking limit. To show that this equation is decreasing in \( p \), we differentiate with respect to \( p \) and obtain

\[
\frac{\partial R^*_p(p)}{\partial p} = \frac{\partial q^*(p)}{\partial p}r^*_p(p) + \bar{q}^*(p)\frac{\partial r^*_p(p)}{\partial p} + \mathbb{E}[r_{pp}(p, C - \bar{D}); \Omega^*] - r^*_p(p) \bar{f}^*(p) \frac{\partial x^*(p)}{\partial p}. \tag{22}
\]

Because \( \frac{\partial q^*(p)}{\partial p} = \bar{f}^*(p)\frac{\partial x^*(p)}{\partial p} \), the first and the last (boundary condition) terms cancel. The second term is negative because \( r^*_p(p) \) is decreasing in \( p \) if \( \mathbb{E}^*(p) \geq 1/2 \), and the third is negative by concavity of \( \pi \). This concludes the proof of part (a).

For part (b), we observe that by Proposition 2, the monotonicity of \( \mathbb{E}^*(p) \) in \( p \) guarantees the elasticity bound required by (a). For part (c), Proposition 2 shows that the monotonicity of LSR elasticity in \( p \) implies \( \mathbb{E}^*(p) \geq 1 \). This guarantees the elasticity bound required by (a), which completes the proof. \( \square \)

**Proof of Lemma 3.** By (1) and (2) we can write

\[
R(p, x) = \mathbb{E} \left[ \min (\bar{D}, C - x) \right] + p\mathbb{E} \left[ \min(p, C - x) \max \left( x, (C - \bar{D}) \right) \right] = \mathbb{E} \left[ \min (\bar{D}, C - x) \right] + \mathbb{E} \left[ \min(p, p, z) \max \left( x, (C - \bar{D}) \right) \right].
\]

The first term is not a function of \( p \). Since the minimum of two concave functions is concave, the concavity of \( \pi(p, z) \) in \( p \) implies that the second term, and hence \( R(p, x) \), is concave in \( p \). \( \square \)
Proof of Proposition 4. (a) By the implicit function theorem, 
\[
\frac{\partial p^*(x; C)}{\partial x} = -R_{pp}(p, x; C) \quad \text{This expression is negative whenever } \mathcal{E}^*(x) \geq 1 \text{ because, by (17),}
\]
\[
R_{xp}(p^*(x; C), x) = \dot{q}(x)q(p^*(x; C), x)(1 - \mathcal{E}^*(x)) \leq 0.
\]
(b) Similarly, 
\[
\frac{\partial p^*(x; C)}{\partial C} = -R_{pp}(p, x; C) \quad \text{so it is sufficient to show that } R_{pc}(p^*(x; C), x; C) \leq 0.
\]
The derivative of \( R_p(p, x; C) = \mathbb{E}[r_p(p, \max(x, C - \bar{D}))] = \dot{q}(x)r_p(p, x) + \mathbb{E}[r_p(p, C - \bar{D}); \bar{\Omega}] \) with respect to \( C \) is
\[
R_{pc}(p, x; C) = \mathbb{E}[r_{pc}(p, C - \bar{D}); \bar{\Omega}] = \mathbb{E}[(1 - \mathcal{E}(p, C - \bar{D}))q(p, C - \bar{D}); \bar{\Omega}].
\] 
(23)
Evaluating (23) at \( p = p^*(x; C) \), we obtain
\[
R_{pc}(p^*(x; C), x; C) = \mathbb{E}[(1 - \mathcal{E}(p^*(x; C), C - \bar{D}))q(p^*(x; C), C - \bar{D}); \bar{\Omega}] \leq 0
\]
because \( \mathcal{E}(p, x) \) increases in \( x \leq C - \bar{D} \) on \( \bar{\Omega} \), \( \mathcal{E}(p^*(x; C), x) = \mathcal{E}^*(x) \geq 1, \) and \( q \geq 0. \)

Proof of Proposition 5. (a) By definition, \( p^*(C) = \max_p R(p, x^*(p; C); C) = \max_p R^*(p; C) \); hence, by Topkis’s theorem (Topkis 1998, Thm. 2.8.2) it is enough to show that \( R^*(p; C) \) is submodular in \( (p, C) \). By the envelope theorem, 
\[
\frac{\partial R^*(p; C)}{\partial p} = R_p(p, x^*(p; C); C).
\]
From Littlewood’s rule (8) it follows that the optimal protection level for a given price, \( x^*(p; C) \equiv x^*(p) \), is independent of capacity. This allows us to write
\[
\frac{\partial}{\partial C} \frac{\partial R^*(p; C)}{\partial p} = \frac{\partial}{\partial C} R_p(p, x^*(p; C); C) = R_{pc}(p, x^*(p; C); C).
\]
From (23) we have
\[
R_{pc}(p, x^*(p; C)) = \mathbb{E}[r_{pc}(p, C - \bar{D}; \bar{\Omega}) = \mathbb{E}[q(p, C - \bar{D})(1 - \mathcal{E}(p, C - \bar{D})); \bar{\Omega}]
\]
\[
\leq (1 - \mathcal{E}(p, x))\mathbb{E}[q(p, C - \bar{D})]; \bar{\Omega}]
\] 
(24)
(25)
because \( \mathcal{E}(p, x) \) is increasing in \( x \) and \( \bar{\Omega} = (C - \bar{D} \geq x) \). We thus obtain
\[
R_{pc}(p, x^*(p; C)) \leq (1 - \mathcal{E}(p, x^*(p; C))\mathbb{E}[q(p, C - \bar{D}); \bar{\Omega}] \leq 0
\]
whenever \( \mathcal{E}^*(p) = \mathcal{E}(p, x^*(p)) \geq 1 \), which follows since (by Proposition 2) \( \mathcal{E}(p, x) \) is increasing in \( p \).
(b) By Littlewood’s rule (8), \( x^*(C) = x^*(p^*(C); C) = x^*(p^*(C)) \); that is, the optimal protection level for a given price is affected by changes in \( C \) only through \( p^*(C) \). Thus we obtain
\[
\frac{\partial x^*(C)}{\partial C} = \frac{\partial x^*(p^*(C))}{\partial C} = \frac{\partial x^*(p^*(C))}{\partial p} \frac{\partial p^*(C)}{\partial C}.
\]
The second term is negative from part (a). The first term is negative if $\mathcal{E}^*(p) \geq 1$—or in particular, if $\mathcal{E}(p, x)$ is increasing in $p$ (by Proposition 2). \qed

**Proof of Proposition 6.** The first part is obvious. For the second part, the sign of the derivative
\[
\frac{\partial}{\partial C} \left( \frac{R^*(C)}{C} \right) = \frac{CR^*_C(C) - R^*(C)}{C^2}
\]
is determined by the right-hand side’s numerator. To show that it is negative, we calculate each term separately. We write the optimal revenue as
\[
R^*(C) = E \left[ \min \left( \bar{D}, C - x^{**} \right) \right] + E \left[ r(p^{**}, \max \left( x^{**}, C - \bar{D} \right)) \right]
\]
\[
= \left[ \bar{q}(x)(C - x) + E \left[ \bar{D}; \bar{\Omega} \right] + \bar{q}(x)pq(p, x) + \bar{q}(x)E \left[ \pi(p, Z); \Omega \right] \right]
\]
\[
+ E \left[ p(C - \bar{D})q(p, C - \bar{D}; \bar{\Omega}) + E \left[ \pi(p, Z); D \leq C - \bar{D}; \bar{\Omega} \right] \right]_{x = x^{**}, p = p^{**}}.
\]
By the envelope theorem,
\[
R^*_C(C) = \frac{\partial R(p, x; C)}{\partial C} \bigg|_{x = x^{**}, p = p^{**}} = \bar{q}(x^{**}) + p^{**}E[q(p^{**}, C - \bar{D}); \bar{\Omega}].
\]
Using (26) and (27), we can write
\[
CR^*_C(C) - R^*(C) = \left[ -\bar{q}(x)pq(p, x) - 1 \right] - \bar{q}(x)E \left[ \pi(p, Z); \Omega \right]
\]
\[
+ E \left[ p(q(p, C - \bar{D}) - 1) \bar{D}; \bar{\Omega} \right] - E \left[ \pi(p, Z); D \leq C - \bar{D}; \bar{\Omega} \right] \bigg|_{x = x^{**}, p = p^{**}}.
\]
The first term is equal to 0, because $p^{**}q(p^{**}, x^{**}) = 1$. Negativity of the third term follows because $q(p^{**}, C - \bar{D}) \leq 1/p^{**}$ on $\bar{\Omega} = (C - \bar{D} \geq x)$. The second and fourth terms are negative because $\pi(p, z) \geq 0$. It follows that $CR^*_C(C) - R^*(C) \leq 0$, concluding the proof. \qed

**Proof of Proposition 7.** Lemma 1 ensures the existence of a unique optimal protection level $x^*(p)$. Hence, concavity of $R^*(p)$ is sufficient for the existence of a unique optimum. Denote low-fare revenue by $\bar{r}(p, x) = E[\min(C - x, \bar{D}(p))]$, and let $\bar{C}(p) = C - \bar{D}(p)$ be the uncertain excess capacity after all low-fare demand has been served. From the envelope theorem it follows that
\[
\frac{\partial R^*(p)}{\partial p} = \bar{r}^*_p(p) = \bar{r}^*_p(p) + \bar{q}^*(p)r^*_p(p) + E[r_p(p, \bar{C}(p)); \bar{\Omega}^*],
\]
where $\bar{\Omega}^* = \bar{\Omega}^*(p) = (\bar{D}(p) \leq C - x^*(p)) = \bar{C}(p) \geq x^*(p)$ denotes the event that the low-fare demand does not exceed the booking limit. To show that (28) is decreasing, we first calculate the derivative of each term with respect to $p$. Given $\bar{r}^*_p(p) = E[d_p(p, \bar{Z}); \bar{\Omega}^*]$, we obtain the derivative of the first term as
\[
\frac{\partial \bar{r}^*_p(p)}{\partial p} = E[d_{pp}(p, \bar{Z}); \bar{\Omega}^*] - \bar{d}^*_p(p)f^*(p)\frac{\partial x^*(p)}{\partial p}.
\]
The derivative of the second term in (28) can be written as
\[
\frac{\partial}{\partial p} q^*(p) r^*_p(p) = q^*(p) \frac{\partial r^*_p(p)}{\partial p} + \frac{\partial q^*(p)}{\partial p} r^*_p(p). (30)
\]
Finally, we calculate the derivative of the third term in (28). Letting \(\tilde{\pi}(p, \tilde{z}) = p \tilde{d}(p, \tilde{z})\), we first note that
\[
\mathbb{E}[r_p(p, \tilde{C}(p)); \tilde{\Omega}^*] = \mathbb{E}[\pi_p(p, \tilde{z}); D(p) \leq \tilde{C}(p); \tilde{\Omega}^*] + \mathbb{E}[(C - \tilde{\pi}_p(p, \tilde{z})) q(p, \tilde{C}(p)); \tilde{\Omega}^*]. (31)
\]
Taking the derivative of (31) with respect to \(p\) then yields
\[
\frac{\partial}{\partial p} \mathbb{E}[r_p(p, \tilde{C}(p)); \tilde{\Omega}^*] = \mathbb{E}[\pi_p(p, \tilde{z}); D(p) \leq \tilde{C}(p); \tilde{\Omega}^*] \\
- \mathbb{E}[\pi_p(p, \tilde{z}) q(p, \tilde{C}(p)); \tilde{\Omega}^*] \\
- \mathbb{E}[(d_p(p, z(p, \tilde{C}(p))) + \tilde{d}_p(p, \tilde{Z}))^2 f(p, \tilde{C}(p)); \tilde{\Omega}^*] \\
- f^*(p) r^*_p(p) \\
+ f^*(p) d^*_p(p) q^*(p) \frac{\partial q^*(p)}{\partial p}. (32)
\]
Summing up (29), (30), and (32), we find that because \(\frac{\partial q^*(p)}{\partial p} = \tilde{f}^*(p)\), the last term of (30) and the fourth term of (32) add up to 0. Similarly, \(p^* q^*(p) = 1\) implies that the last terms of (29) and (32) sum to 0. Therefore,
\[
\frac{\partial R^*_p(p)}{\partial p} = \mathbb{E}[\pi_p(p, \tilde{z}); D(p) \leq \tilde{C}(p); \tilde{\Omega}^*] \\
- 2\mathbb{E}[\tilde{d}_p(p, \tilde{z}) q(p, \tilde{C}(p)); \tilde{\Omega}^*] \\
- \mathbb{E}[\tilde{d}_p(p, \tilde{z}) (pq(p, \tilde{C}(p)) - 1); \tilde{\Omega}^*] \\
- \mathbb{E}[(d_p(p, z(p, \tilde{C}(p))) + \tilde{d}_p(p, \tilde{Z}))^2 f(p, \tilde{C}(p)); \tilde{\Omega}^*] \\
+ q^*(p) \frac{\partial q^*(p)}{\partial p}. (33)
\]
The first term is negative by concavity of \(\pi(p, z)\) and the second because \(\tilde{d}_p(p, \tilde{z}) \geq 0\). Negativity of the third term follows because \(\tilde{d}_p(p, \tilde{z}) \leq 0\) and \(q(p, \tilde{C}(p)) \leq q^*(p) = 1/\tilde{p}\) on \(\tilde{\Omega}^* = (\tilde{C}(p) \geq x^*(p))\). The fourth term is obviously negative. From the proof of Proposition 3(a), \(r^*_p(p)\) is decreasing in \(p\) if \(\mathcal{E}^*(p) \geq 1/2\), which entails the negativity of the last term. It follows that \(R^*_p(p)\) is decreasing, so \(R^*(p)\) is concave in \(p\). □

**Proof of Proposition 8.** (a) By (11), \(p^{PA}(x) = \arg\max_p V(p, \bar{p}, x) = \arg\max_p r(p, x)\) is independent of \(\bar{p}\). Optimality of \(p^{PA}(x)\) and the implicit function theorem imply that \(p^{PA}(x)\) is decreasing in \(x\) whenever \(\mathcal{E}(p^{PA}(x), x) \geq 1\), because
\[
V_{xp}(p, \bar{p}, x)|_{p=p^{PA}(x)} = r_{xp}(p, x)|_{p=p^{PA}(x)} = q(p^{PA}(x), x)(1 - \mathcal{E}(p^{PA}(x), x)) \leq 0.
\]
It remains to show that $E(p_{PA}(x), x) \geq 1$ if $E(p, x)$ is increasing in $x$. Write $r_p(p, x) = \int_0^s Q(p, v) \, dv$, where $Q(p, x) = q(p, x)(1 - E(p, x))$. Since $E(p, x)$ is increasing in $x$ and $q(p, x) \geq 0$, it follows that $Q(p, x)$ crosses 0 at most once and from above. Therefore, the first-order condition $\int_0^s Q(p_{PA}(x), v) \, dv = 0$ implies $Q(p_{PA}(x), x) \leq 0$ (i.e., $E(p_{PA}(x), x) \geq 1$), which concludes the proof. The monotonicity of $p_{PA}(x)$ in $x$ follows by a similar argument.

(b) By the envelope theorem we have

$$\frac{\partial^2 V(p_{PA}(x), \bar{p}_{PA}(x), x)}{\partial x^2} = r_{xx}(p, x) + \bar{r}_{xx}(\bar{p}, x) - \frac{r_{xp}(p, x)}{\bar{r}_{xp}(\bar{p}, x)} - \frac{r_{xx}(\bar{p}, x)}{\bar{r}_{xx}(\bar{p}, x)}.$$  

The second-order derivatives are $r_{xx}(p, x) = -pf(p, x), \bar{r}_{xx}(\bar{p}, x) = -\bar{p}\bar{f}(\bar{p}, x)$, and

$$r_{pp}(p, x) = E[\pi_{pp}(p, Z); \Omega] - pf(p, x)d_x^2(p, z(p, x)),$$

$$\bar{r}_{pp}(\bar{p}, x) = E[\bar{\pi}_{pp}(\bar{p}, Z); \Omega] - \bar{p}\bar{f}(p, x)d_x^2(p, \bar{z}(\bar{p}, x)).$$

Therefore, we can write

$$r_{xx}(p, x)r_{pp}(p, x) - r_{xp}(p, x) = -pf(p, x)E[\pi_{pp}(p, Z); \Omega] + q(p, x)^2(2E(p, x) - 1).$$

The first term is positive by concavity of $\pi$. The second term, evaluated at $p = p_{PA}(x)$, is positive when $E(p_{PA}(x), x) \geq 1/2$, which is guaranteed by the monotonicity of elasticity in $x$ from part (a). Similarly, we can show that $\bar{r}_{xx}(\bar{p}, x)\bar{r}_{pp}(\bar{p}, x) - \bar{r}_{xp}(\bar{p}, x)\leq 0$ when $\bar{\pi}$ is concave and $\bar{E}(\bar{p}_{PA}(x), x) \geq 1/2$. Finally, since $r_{pp}, \bar{r}_{pp} \leq 0$, it follows that $\frac{\partial^2 V(p_{PA}(x), \bar{p}_{PA}(x), x)}{\partial x^2} \leq 0$ and so PA has a unique optimal solution. \qed

**Proof of Proposition 9.** (a) Recall that $x^\ast$ denotes the optimal protection level set by the PRM model, and that $x^{PA}$ denotes the high-fare allocation set by the PA model. Using $pq(p, x^\ast) = \bar{p}$ from Littlewood’s rule (8) in the marginal revenue condition (13), we obtain $V_x(p, \bar{p}, x)|x=x^\ast = pq(p, x^\ast) - \bar{p}q(\bar{p}, x^\ast) = \bar{p}\bar{F}(\bar{p}, x^\ast) \geq 0$. Because $V_x(p, \bar{p}, x) = -pf(p, x) - \bar{p}\bar{f}(\bar{p}, x) \leq 0$, this implies $x^{PA}(p, \bar{p}) \geq x^\ast(p, \bar{p})$.

(b) With $p^\ast$ and $p_{PA}$ the optimal prices from the PRM and PA models, respectively, we write

$$R_p(p, x)|_{p=p_{PA}} = \bar{q}(x)r_p(p_{PA}, x) + E[r_p(p_{PA}, C - \bar{D}); \bar{\Omega}].$$

From the optimality of $p_{PA}$ it follows that $r_p(p_{PA}, x) = 0$. For the second term, note that if $E(p, x)$ is increasing in $x$, then $r_{pu}(p_{PA}(x), x) \leq 0$ or $r_p(p_{PA}(x), x)$ is decreasing in $x$ (Proposition 8). Since on $\Omega$ we have $C - \bar{D} \geq x$, this implies $E[r_p(p_{PA}, C - \bar{D}); \bar{\Omega}] \leq r_p(p_{PA}, x) = 0$. It then follows that $R_p(p, x)|_{p=p_{PA}} \leq 0$ and $p_{PA}(x, \bar{p}) \geq p^\ast(x, \bar{p})$. \qed
Europe Campus
Boulevard de Constance
77305 Fontainebleau Cedex, France
Tel: +33 (0)1 60 72 40 00
Fax: +33 (0)1 60 74 55 00/01

Asia Campus
1 Ayer Rajah Avenue, Singapore 138676
Tel: +65 67 99 53 88
Fax: +65 67 99 53 99

Abu Dhabi Campus
Muroor Road - Street No 4
P.O. Box 48049
Abu Dhabi, United Arab Emirates
Tel: +971 2 651 5200
Fax: +971 2 443 9461

www.insead.edu